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# ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF SOME SYSTEMS OF QUASILINEAR DIFFERENTIAL EQUATIONS

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Abstract. We investigate the asymptotic behaviour of solutions of some systems of differential equations appearing in studies of $n$th order quasilinear differential equations.

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*Keywords:* quasilinear differential equations, asymptotic representation of the solution

## 1. Introduction

Let us consider the system of differential equations

$$\frac{dy}{dx} = [W(x) + R(x)]y + F(x, y), \tag{1.1}$$

where $W, R \in C\left([x_0, +\infty[\,; \mathbb{R}^{n\times n}\right)$, $F \in C([a, +\infty[\times\mathbb{R}^n; \mathbb{R}^n)$,

$$W(x) = \operatorname{diag}\left[W_1(x), \dots, W_s(x)\right]$$

with

$$W_i(x) = \begin{pmatrix} \omega_{i1}(x)I_2 & I_{i2}(x) & O_2 & \dots & O_2 & O_2 \\ O_2 & \omega_{i2}(x)I_2 & I_{i3}(x) & \dots & O_2 & O_2 \\ O_2 & O_2 & \omega_{i3}(x)I_2 & \dots & O_2 & O_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O_2 & O_2 & O_2 & \dots & \omega_{in_i-1}(x)I_2 & I_{in_i}(x) \\ O_2 & O_2 & O_2 & \dots & O_2 & \omega_{in_i}(x)I_2 \end{pmatrix} \tag{1.2}$$

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for $i = 1, \ldots, r$,

$$W_i(x) = \begin{pmatrix} \omega_{i1}(x) & 1 & 0 & \ldots & 0 & 0 \\ 0 & \omega_{i2}(x) & 1 & \ldots & 0 & 0 \\ 0 & 0 & \omega_{i3}(x) & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \omega_{in_i-1}(x) & 1 \\ 0 & 0 & 0 & \ldots & 0 & \omega_{in_i}(x) \end{pmatrix} \tag{1.3}$$

for $i = r+1, \ldots, s$.

Here, $r \in \{0, \ldots, s\}$ and $\sum_{i=1}^{r} 2n_i + \sum_{i=r+1}^{s} n_i = n$,[1]

$$I_{ik}(x) = \begin{pmatrix} l_{ik}^{(2)}(x) & -l_{ik}^{(1)}(x) \\ l_{ik}^{(1)}(x) & l_{ik}^{(2)}(x) \end{pmatrix} \qquad (k = 2, \ldots, n_i,\ \ i \in \{1, \ldots, r\}),$$

$I_2$ is the identity matrix, $O_2$ is the zero matrix of the second order; $\omega_{ik}$, $l_{ik}^{(1)}$, $l_{ik}^{(2)} \in C([x_0, +\infty[; \mathbb{R})$ are so that

$$\lim_{x\to+\infty} l_{ik}^{(1)}(x) = 0, \qquad \lim_{x\to+\infty} l_{ik}^{(2)}(x) = 1 \qquad (k = 2, \ldots, n_i,\ \ i \in \{1, \ldots, r\}),$$

$$\lim_{x\to+\infty} \omega_{ik}(x) = \omega_i^0 = \text{const} \qquad (k = 1, \ldots, n_i,\ \ i \in \{1, \ldots, r\}).$$

We can reduce the quasilinear equation

$$u^{(n)} = \sum_{k=0}^{n-1} [p_{0k}(t) + p_{1k}(t)]u^{(k)} + q(t, u, u', \ldots, u^{(n-1)}) \tag{1.4}$$

to form (1.1) by means of some real transformation if $q : [a, +\infty[\times \mathbb{R}^n \to \mathbb{R}$, $p_{1k} : [a, +\infty[\to \mathbb{R}$ $(k = 0, \ldots, n-1)$ are "small" in some integral sense for continuous functions, $p_{0k} : [a, +\infty[\to \mathbb{R}$ $(k = 0, \ldots, n-1)$ are continuously differentiable functions such that there are $\varphi, \psi \in C^2([a, +\infty[\,;\ ]0, +\infty[)$ for which

$$\lim_{t\to+\infty} a_k(t) = a_{0k} = const, \quad \lim_{t\to+\infty} b_k(t) = b_{0k} = const \qquad (k = 0, \ldots, n-1),$$

where

$$a_k(t) = \varphi^{-1}(t)\psi^{-1-k}(t)\left(\varphi(t)\psi^k(t)\right)', \quad b_k(t) = \psi^{k-n}(t)p_{0k}(t) \ \ (k = 0, \ldots, n-1).$$

Then, in system (1.1), each block (1.2) corresponds to complex conjugate roots of multiplicity $n_i$ of the algebraic equation

$$\prod_{j=0}^{n-1}(\lambda + a_{0j}) = \sum_{k=1}^{n-1} b_{0k} \prod_{j=0}^{k-1}(\lambda + a_{0j}) + b_{00} \tag{1.5}$$

and each block (1.3) corresponds to a real root of (1.5) of multiplicity $n_i$.

[1]In case $r = 0$, the matrix $W$ consists of the blocks $W_i$ of type (1.3), whereas for $r = s$ the matrix $W$ consists of the blocks $W_i$ of type (1.2). If $k > m$, we assume that $\sum_{j=k}^{m}[\ldots] = 0$.

In this paper, we obtain asymptotic representations for solutions of the system of quasilinear differential equations of form (1.1). I. T. Kiguradze's results [1] on the asymptotic behaviour of solutions of (1.4), obtained for the case of simple roots of (1.5), are extended here to the case where multiple roots of (1.5) may exist. By virtue of the structure of equation (1.4) and that of the transformation reducing this equation to system (1.1), asymptotic formulas are obtained for solutions of system (1.1) with some consideration for small additive perturbations.

Many papers (see, e.g., [2–6]) are devoted to the study of the asymptotic equivalence between solutions of a quasilinear system and those of the corresponding linear system. Note that the results presented in this paper cannot be obtained by allying the results from [2–6].

## 2. Basic assumptions on the principal part of the abridged system (1.1)

Let us consider the following auxiliary linear system of differential equations corresponding to system (1.1):

$$\frac{du}{dx} = W(x)u. \tag{2.1}$$

In order to obtain an asymptotic representation of the fundamental system of solutions of (2.1), we need some notation.

We write

$$N_1 = 0, \qquad N_i = \sum_{k=1}^{i-1} m_k \qquad (i = 2, \ldots, s),$$

where

$$m_i = \begin{cases} 2n_i & \text{if } i \in \{1, \ldots, r\}, \\ n_i & \text{if } i \in \{r+1, \ldots, s\}. \end{cases}$$

We introduce two sets

$$J_0 = \{(i, j): \; j = 1, \ldots, n_i, \; 1 \le i \le s\}, \qquad J = \{(i, k): \; k = 1, \ldots, m_i, \; 1 \le i \le s\}$$

and denote by $e_{pv}$, where $(p, v) \in J$, the $n$-dimensional vector whose $N_p + v$th component equals 1, whereas the others are equal to zero.

Setting, for every $i \in \{1, \ldots, s\}$,

$$d_{im}(x) = \exp \int_a^x (\omega_{im}(s) - \omega_{im-1}(s))\, ds \qquad (m = 2, \ldots, n_i),$$

we define the functions $B^i_{mk}$ $(1 \le m \le k \le n_i,\ 1 \le i \le s)$ and $B^{i\mu}_{mk}$ $(1 \le m \le k \le n_i,$ $1 \le i \le r;\ \mu = 1, 2)$ by the following recurrence relations:

$$B^i_{kk}(x) = 1, \quad B^{i1}_{kk}(x) = 0, \quad B^{i2}_{kk}(x) = 1 \quad (1 \le k \le n_i),$$

$$B^i_{mk}(x) = \int_{\beta^i_{mk}}^x B^i_{m+1k}(t) d_{im+1}(t)dt, \quad (1 \le m < k \le n_i),$$

$$B^{i\mu}_{mk}(x) = \int_{\alpha^{i\mu}_{mk}}^x \left[B^{i\mu}_{m+1k}(t) l^{(2)}_{im+1}(t) + (-1)^{3-\mu} B^{i\,3-\mu}_{m+1k}(t) l^{(1)}_{im+1}(t)\right] d_{im+1}(t)dt$$
$$(1 \le m < k \le n_i),$$

where

$$\beta^i_{mk} = \begin{cases} a & \text{if } B^i_{mk0} = +\infty, \\ +\infty & \text{if } B^i_{mk0} < +\infty, \end{cases} \qquad B^i_{mk0} = \int_a^{+\infty} \left| B^i_{m+1k}(t) \right| d_{im+1}(t)dt,$$

$$\alpha^{i\mu}_{mk} = \begin{cases} a & \text{if } B^{i\mu}_{mk0} = +\infty, \\ +\infty & \text{if } B^{i\mu}_{mk0} < +\infty, \end{cases}$$

$$B^{i\mu}_{mk0} = \int_a^{+\infty} \left| B^{i\mu}_{m+1k}(t) l^{(2)}_{im+1}(t) + (-1)^{3-\mu} B^{i\,3-\mu}_{m+1k}(t) l^{(1)}_{im+1}(t) \right| d_{im+1}(t)\, dt.$$

Using these functions, we also introduce the functions $D^i_{kj}$ $(1 \le k \le j \le n_i,\ i = r+1, \dots, s)$ and $D^{i\mu}_{kj}$ $(1 \le k \le j \le n_i,\ i = 1, \dots, r;\ \mu = 1, 2)$ by the recurrence relations

$$D^i_{kk}(x) = 1 \quad (k = 1, \dots, n_i),$$

$$D^i_{kj}(x) = -\sum_{m=k+1}^{j} B^i_{km}(x) D^i_{mj}(x) \quad (1 \le k < j \le n_i);$$

$$D^{i1}_{kk}(x) = 1, \quad D^{i2}_{kk}(x) = 0 \quad (k = 1, \dots, n_i),$$

$$D^{i\mu}_{kj}(x) = -\sum_{m=k+1}^{j} \left(B^{i2}_{km}(x) D^{i\mu}_{mj}(x) + (-1)^\mu B^{i1}_{km}(x) D^{i\,3-\mu}_{mj}(x)\right)$$
$$(1 \le k < j \le n_i;\ \mu = 1, 2).$$

Integrating the system "upside down," we obtain

$$u_{N_i+2k-1}(x) = \exp\int_a^x \omega_{ik}(s)ds \sum_{j=k}^{n_i} \left(C_{N_i+2j-1}B^{i2}_{kj}(x) - C_{N_i+2j}B^{i1}_{kj}(x)\right),$$

$$u_{N_i+2k}(x) = \exp\int_a^x \omega_{ik}(s)ds \sum_{j=k}^{n_i} \left(C_{N_i+2j-1}B^{i1}_{kj}(x) + C_{N_i+2j}B^{i2}_{kj}(x)\right)$$

$$(k = 1,\dots,n_i,\ \ i = 1,\dots,r);$$

$$u_{N_i+k}(x) = \exp\int_a^x \omega_{ik}(s)ds \sum_{j=k}^{n_i} C_{N_i+j}B^{i}_{kj}(x) \quad (k = 1,\dots,n_i,\ i = r+1,\dots,s),$$

where $(C_m)_{m=1}^n = C$ is a constant vector. Setting $C = e_{pv}$ $(v = 1,\dots,m_p;\ p = 1,\dots,s)$ we find for (2.1) a fundamental matrix of solutions of the form $\Phi(x) = \operatorname{diag}[\Phi_1(x),\dots,\Phi_s(x)]$, where

$$\Phi_i(x) = \begin{pmatrix} \Phi^i_{11}(x) & \Phi^i_{12}(x) & \dots & \Phi^i_{1n_i}(x) \\ O^i & \Phi^i_{22}(x) & \dots & \Phi^i_{2n_i}(x) \\ \vdots & \vdots & \ddots & \vdots \\ O^i & O^i & \dots & \Phi^i_{n_in_i}(x) \end{pmatrix}$$

with

$$O^i = \begin{cases} O_2 & \text{for} \quad i \in \{1,\dots,r\}, \\ 0 & \text{for} \quad i \in \{r+1,\dots,s\} \end{cases}$$

and

$$\Phi^i_{kj}(x) = \begin{cases} \begin{pmatrix} B^{i2}_{kj}(x) & -B^{i1}_{kj}(x) \\ B^{i1}_{kj}(x) & B^{i2}_{kj}(x) \end{pmatrix} \exp\int_a^x \omega_{ik}(s)ds & \text{if } 1 \le k \le j \le n_i,\ i = 1,\dots,r \\ B^i_{kj}(x)\exp\int_a^x \omega_{ik}(s)ds & \text{if } 1 \le k \le j \le n_i,\ i = r+1,\dots,s. \end{cases}$$

It is not difficult to see that the inverse matrix for $\Phi(x)$ is given by the equality $\Phi^{-1}(x) = \operatorname{diag}\left[\Phi_1^{-1}(x),\dots,\Phi_s^{-1}(x)\right]$, where

$$\Phi_i^{-1}(x) = \begin{pmatrix} (\Phi_i^{-1}(x))_{11} & (\Phi_i^{-1}(x))_{12} & \dots & (\Phi_i^{-1}(x))_{1n_i} \\ O^i & (\Phi_i^{-1}(x))_{22} & \dots & (\Phi_i^{-1}(x))_{2n_i} \\ \vdots & \vdots & \ddots & \vdots \\ O^i & O^i & \dots & (\Phi_i^{-1}(x))_{n_in_i} \end{pmatrix} \quad (i = 1,\dots,s)$$

with

$$
(\Phi_i^{-1}(x))_{kj} = \begin{cases} \begin{pmatrix} D_{kj}^{i1}(x) & -D_{kj}^{i2}(x) \\ D_{kj}^{i2}(x) & D_{kj}^{i1}(x) \end{pmatrix} \exp\Big(-\int_a^x \omega_{ij}(s)ds\Big), \\ \qquad \text{if } 1 \le k \le j \le n_i,\ i = 1, \dots, r, \\ D_{kj}^{i}(x) \exp\Big(-\int_a^x \omega_{ij}(s)ds\Big), \\ \qquad \text{if } 1 \le k \le j \le n_i,\ i = r+1, \dots, s. \end{cases}
$$

In what follows, we assume that the following condition $(S_B)$ is satisfied for system (1.1):

$(S_B)$ *For every* $i \in \{1, \dots, s\}$,

$$B_{1m}^i(x) \neq 0 \quad (m = 1, \dots, n_i)$$

*in some neighbourhood of* $+\infty$ *and, moreover, for arbitrary* $j$ *and* $m$ *such that* $1 \le j \le m \le n_i$, *the relations*

$$\limsup_{x \to +\infty} \left| \frac{B_{1j}^i(x) B_{jm}^{i\mu}(x)}{B_{1m}^i(x)} \right| < +\infty \quad \textit{if } i \in \{1, \dots, r\},\ \ \mu = 1, 2,$$

$$\limsup_{x \to +\infty} \left| \frac{B_{1j}^i(x) B_{jm}^{i}(x)}{B_{1m}^i(x)} \right| < +\infty \quad \textit{if } i \in \{r+1, \dots, s\}$$

*are true.*

**Lemma 1.** *If condition* $(S_B)$ *is satisfied, then there are constants* $c_1 \ge 1$ *and* $x_1 \ge x_0$ *such that for all* $i \in \{1, \dots, s\}$ *and all natural* $k$ *and* $j$ *with* $1 \le k \le j \le n_i$, *the estimates*

$$
\begin{aligned}
&\sum_{m=k}^{j} \left| B_{1m}^i(x) D_{mj}^{i\mu}(x) \right| \le (2c_1)^{j-k} \left| B_{1j}^i(x) \right| \quad \textit{if} \quad i \in \{1, \dots, r\}\ (\mu = 1, 2), \\
&\sum_{m=k}^{j} \left| B_{1m}^i(x) D_{mj}^{i}(x) \right| \le (2c_1)^{j-k} \left| B_{1j}^i(x) \right| \quad \textit{if} \quad i \in \{r+1, \dots, s\}
\end{aligned}
\tag{2.2$_i$}
$$

*are valid for* $x \ge x_1$.

Lemma 1 is proved similarly to Lemma 1 from [7].

*Proof of Lemma 1.* In view of condition $(S_B)$, there are constants $x_1 \ge x_0$ and $c_1 \ge 1$ such that the relations

$$\left| \frac{B_{1j}^i(x) B_{jm}^{i\mu}(x)}{B_{1m}^i(x)} \right| \le \frac{c_1}{2} \qquad (i = 1, \dots, r;\ \ \mu = 1, 2),$$

$$\left| \frac{B_{1j}^i(x) B_{jm}^{i}(x)}{B_{1m}^i(x)} \right| \le c_1 \qquad (i = r+1, \dots, s)$$

are satisfied.

Let us show that ($2.2_i$) is true for $x_1 \geq x_0$. We fix an arbitrary number $i \in \{1, \ldots, s\}$ and prove ($2.2_i$) by induction in $j-k$ $(0 \leq j-k \leq n_i - k)$.

Indeed, let $i \in \{1, \ldots, r\}$. Since $D^{i1}_{kk}(x) = 1$ and $D^{i2}_{kk}(x) = 0$ for $k = 1, \ldots, n_i$, we see that, for all $j, k \in \{1, \ldots, n_i\}$ such that $j-k = 0$, estimate ($2.2_i$) is true.

Let us suppose that ($2.2_i$) holds for all $j, k \in \{1, \ldots, n_i\}$ that satisfy the condition $j-k = l < n_i - k$. We shall show that ($2.2_i$) is valid for arbitrary $j, k \in \{1, \ldots, n_i\}$ satisfying the condition $j-k = l+1 < n_i - k$. By the definition of the functions $D^{i\mu}_{kj}$, $1 \leq k \leq j \leq n_i$, $i = 1, \ldots, r$, $\mu = 1, 2$, using the assumption of the induction and taking condition ($S_B$) into account, for $j-k = l+1$, we obtain

$$\sum_{m=k}^{j} \left|B^i_{1m}(x)D^{i1}_{mj}(x)\right| = \left|B^i_{1k}(x)D^{i1}_{kj}(x)\right| + \sum_{m=k+1}^{j} \left|B^i_{1m}(x)D^{i1}_{mj}(x)\right| \leq$$

$$\leq \left|B^i_{1k}(x) \sum_{m=k+1}^{j} B^{i1}_{km}(x)D^{i2}_{mj}(x) - B^{i2}_{km}(x)D^{i1}_{mj}(x)\right| + (2c_1)^l \left|B^i_{1j}(x)\right| \leq$$

$$\leq \left|B^i_{1k}(x) \sum_{m=k+1}^{j} B^{i1}_{km}(x)D^{i2}_{mj}(x)\right| + \left|B^i_{1k}(x) \sum_{m=k+1}^{j} B^{i2}_{km}(x)D^{i1}_{mj}(x)\right| +$$

$$+ (2c_1)^l \left|B^i_{1j}(x)\right| \leq \frac{c_1}{2} \sum_{m=k+1}^{j} \left|B^i_{1m}(x)D^{i2}_{mj}(x)\right| + \frac{c_1}{2} \sum_{m=k+1}^{j} \left|B^i_{1m}(x)D^{i1}_{mj}(x)\right| +$$

$$+ (2c_1)^l \left|B^i_{1j}(x)\right| \leq 2\frac{c_1}{2}(2c_1)^l \left|B^i_{1j}(x)\right| + (2c_1)^l \left|B^i_{1j}(x)\right| \leq (2c_1)^{l+1} \left|B^i_{1j}(x)\right|$$

for $x \geq x_1$. Similarly,

$$\sum_{m=k}^{j} \left|B^i_{1m}(x)D^{i2}_{mj}(x)\right| \leq (2c_1)^{j-k} \left|B^i_{1j}(x)\right|$$

for $x \geq x_1$.

If $i \in \{r+1, \ldots, s\}$, then

$$\sum_{m=k}^{j} \left|B^i_{1m}(x)D^i_{mj}(x)\right| = \left|B^i_{1k}(x)D^i_{kj}(x)\right| + \sum_{m=k+1}^{j} \left|B^i_{1m}(x)D^i_{mj}(x)\right| \leq$$

$$\leq \left|B^i_{1k}(x) \sum_{m=k+1}^{j} B^{i1}_{km}(x)D^i_{mj}(x)\right| + (2c_1)^l \left|B^i_{1j}(x)\right| \leq$$

$$\leq c_1 \sum_{m=k+1}^{j} \left|B^i_{1m}(x)D^i_{mj}(x)\right| + (2c_1)^l \left|B^i_{1j}(x)\right| \leq c_1(2c_1)^l \left|B^i_{1j}(x)\right| +$$

$$+ (2c_1)^l \left|B^i_{1j}(x)\right| \leq (2c_1)^{l+1} \left|B^i_{1j}(x)\right|$$

for $x \geq x_1$.

The lemma is proved. □

## 3. Main results

In order to formulate the theorems, besides the notation introduced in Section 2, we also need some other definitions.

For an arbitrary pair $(i,k) \in J_0$, we write

$$H_{i2k-1}(x) = H_{i2k}(x) = B^i_{1k}(x) \exp \int_a^x (\omega_{i1}(t) - \omega_{ik}(t))\, dt \qquad \text{if} \quad i \in \{1, \dots, r\},$$

and

$$H_{ik}(x) = B^i_{1k}(x) \exp \int_a^x (\omega_{i1}(t) - \omega_{ik}(t))\, dt \qquad \text{if} \quad i \in \{r+1, \dots, s\}.$$

We also define the matrix

$$\Delta(x) = \operatorname{diag}\left[\Delta_1(x), \dots, \Delta_s(x)\right],$$

where $\Delta_i(x) = \operatorname{diag}\left[H_{i1}(x), \dots, H_{im_i}(x)\right]$ $(i = 1, \dots, s)$, and denote by $\mathbb{R}^n_b$ the following set:

$$\mathbb{R}^n_b = \left\{ \left(z_j\right)_{j=1}^n \in \mathbb{R}^n : |z_1| \leq b, \dots, |z_n| \leq b \right\}.$$

**Definition 1.** Let $\gamma = (\gamma_i)_{i=1}^s$ be an $n$-dimensional vector function with the continuous components $\gamma_i : [a, +\infty] \to ]0, +\infty[$ $(i = 1, \dots, s)$.

We shall say that a system of continuous functions $\mu_{im} :]a, +\infty[\to \mathbb{R}$ $(m = 1, \dots, n_i$, $i = 1, \dots, s)$ *satisfies the condition* $(M - L)_{pq}$ (where $(p,q) \in J_0$) *on the interval* $[x_0, +\infty[\subset]a, +\infty[$ with *the weight coefficient* $\gamma$ if, for arbitrary $(i,m) \in J_0$, either

$$\inf \left\{ \int_\tau^x [\mu_{pq}(s) - \mu_{im}(s)] ds - \ln \frac{\gamma_i(x)}{\gamma_i(\tau)} : x \geq \tau \geq x_0 \right\} > -\infty$$

and

$$\lim_{x \to +\infty} \left[ \int_{x_0}^x [\mu_{pq}(s) - \mu_{im}(s)] ds - \ln \frac{\gamma_i(x)}{\gamma_i(x_0)} \right] = +\infty$$

or

$$\sup \left\{ \int_\tau^x [\mu_{pq}(s) - \mu_{im}(s)] ds - \ln \frac{\gamma_i(x)}{\gamma_i(\tau)} : x \geq \tau \geq x_0 \right\} < +\infty.$$

If condition $(M - L)_{pq}$ holds with the weight coefficient $\gamma = (\gamma_i)_{i=1}^s$, we use the diagonal matrix

$$\Gamma(x) = \operatorname{diag}\left[\gamma_1(x) I_{m_1}, \dots, \gamma_s(x) I_{m_s}\right],$$

where $I_{m_i}$ $(i = 1, \dots, s)$ is the identity matrix of order $m_i$.

**Theorem 1.** *Let condition* $(S_B)$ *be satisfied and, for a certain pair* $(p,v) \in J$ *and some* $n$*-dimensional vector function* $\gamma = (\gamma_i)_{i=1}^s$ *with continuous components* $\gamma_i : [a, +\infty[\to]0, +\infty[$ $(i = 1, \dots, s)$, *the following conditions hold:*

(i) *The system of functions*

$$\mu_{ik}(x) = \omega_{i1}(x) + \frac{\left(B^i_{1k}(x)\right)'}{B^i_{1k}(x)} \qquad (k = 1, \dots, n_i, \quad 1 \le i \le s)$$

*satisfies the condition* $(M - L)_{pq}$*, where* $q$ *is defined by the equality*[2]

$$q = \begin{cases} \left[\frac{\nu+1}{2}\right] & \text{if } \; p \in \{1, \dots, r\} \\ \nu & \text{if } \; p \in \{r+1, \dots, s\} \end{cases}$$

*on the interval* $[x_0, +\infty[\, \subset\, ]a, +\infty[$ *with the weight coefficient* $\gamma$*;*

(ii) *For every* $(i, k) \in J$ *and all* $m \in \{1, \dots, \nu\}$*, one has*

$$\int_{x_0}^{+\infty} \gamma_i(x) \left| \frac{H_{ik}(x)}{H_{pm}(x)} r_{N_i+kN_p+m}(x) \right| dx < +\infty;$$

(iii) *For arbitrary* $(i, k), (l, m) \in J$

$$\int_{x_0}^{+\infty} \frac{\gamma_i(x)}{\gamma_l(x)} \left| \frac{H_{ik}(x)}{H_{lm}(x)} r_{N_i+kN_l+m}(x) \right| dx < +\infty;$$

(iv) *There is a number* $b \in ]0, +\infty[$ *and there exists a continuous function*

$$\phi : [a, +\infty[ \to \mathbb{R}_+$$

*such that, for all* $(i, k) \in J$ *and all* $z \in \mathbb{R}^n_b$*, the relations*

$$\left| F_{N_i+k}(x, \Delta^{-1}(x)\Gamma^{-1}(x)\xi_{pq}(x)z + \Phi(x)e_{p\nu} \right| \le \phi(x)$$

*and*

$$\int_{x_0}^{+\infty} \gamma_i(x)\phi(x) \left| \frac{H_{ik}(x)}{\xi_{pq}(x)} \right| dx < +\infty$$

*are true.*

*Then system (1) has at least one solution* $y_{p\nu}(x)$ *that admits the following asymptotic representation for* $x \to +\infty$*:*

$$y_{p\nu}(x) = \Phi(x)e_{p\nu} + \Delta^{-1}(x)\Gamma^{-1}(x)\overline{o}\left(\xi_{pq}(x)\right),$$

*where*

$$\xi_{pq}(x) = B^p_{1q}(x) \exp \int_a^x \omega_{p1}(s) ds.$$

*Proof.* Given the numbers $(p, q) \in J_0$ and the $n$-dimensional vector function $\gamma$, we introduce the auxiliary functions

$$g_{im}(x, \tau) = \frac{\gamma_i(x) B^i_{1q}(x) B^p_{1q}(\tau)}{\gamma_i(\tau) B^i_{1q}(\tau) B^p_{1q}(x)} \exp \int_\tau^x \left(\omega_{i1}(s) - \omega_{p1}(s)\right) ds,$$

---

[2]Here, [. . . ] denotes the integer part of the expression.

$m = 1, \dots, n_i, \quad i = 1, \dots, s$, and

$$f_{pv}(x) = \frac{\Delta(x)\Phi(x)}{\xi_{pq}(x)} e_{pv}.$$

Due to the structure of the matrices $\Delta$ and $\Phi$, the components of the vector $f_{pv}(x) = \big((f_{pv})_k\big)_{k=1}^n$ are determined as follows: for all $\eta \in \{1, \dots, q\}$,

$$\big(f_{p2q-1}\big)_{N_p+2\eta-1}(x) = \frac{B^p_{1\eta}(x) B^{p2}_{\eta q}(x)}{B^p_{1q}(x)},$$

$$\big(f_{p2q-1}\big)_{N_p+2\eta}(x) = \frac{B^p_{1\eta}(x) B^{p1}_{\eta q}(x)}{B^p_{1q}(x)},$$

$$\big(f_{p2q}\big)_{N_p+2\eta-1}(x) = -\frac{B^p_{1\eta}(x) B^{p1}_{\eta q}(x)}{B^p_{1q}(x)},$$

$$\big(f_{p2q}\big)_{N_p+2\eta}(x) = \frac{B^p_{1\eta}(x) B^{p2}_{\eta q}(x)}{B^p_{1q}(x)},$$

$$\big(f_{p2q-1}\big)_j(x) = 0, \quad \big(f_{p2q}\big)_j(x) = 0 \quad (j \neq N_p + 1, \dots, N_p + v)$$

if $p \in \{1, \dots, r\}$ and

$$\big(f_{pq}\big)_{N_p+\eta}(x) = \frac{B^p_{1\eta}(x) B^p_{\eta q}(x)}{B^p_{1q}(x)},$$

$$\big(f_{pq}\big)_j(x) = 0 \qquad (j \neq N_p + 1, \dots, N_p + q)$$

if $p \in \{r + 1, \dots, s\}$.

In view of condition (i) of the theorem, the set $J_0$ is decomposed as the union of two disjoint subsets $J_{1pq}$ and $J_{2pq}$,

$$J_0 = J_{1pq} \bigcup J_{2pq},$$

such that

$$\lim_{x \to +\infty} g_{im}(x, \tau) = 0 \quad \text{for all } \tau \geq x_0$$

and

$$\eta_{im} = \sup\{|g_{im}(x, \tau)| : x \geq \tau \geq x_0\} < +\infty$$

when $(i, m) \in J_{1pq}$, and

$$\eta_{im} = \sup\{|g_{im}(x, \tau)| : \tau \geq x \geq x_0\} < +\infty$$

when $(i, m) \in J_{2pq}$.

According to this decomposition of the set $J_0$, the matrix $\Phi$ can be represented as the sum of two diagonal matrices

$$\Phi^{(\rho)}(x) = \operatorname{diag}\left[\Phi_1^{(\rho)}(x), \dots, \Phi_s^{(\rho)}(x)\right] \qquad (\rho = 1, 2),$$

where $\Phi_i^{(\rho)}(x)$ $(i \in \{1, \dots, s\},\ \rho = 1, 2)$ are the $m_i$th order matrices with the elements

$$\left(\Phi_i^{(\rho)}(x)\right)_{jm} = \begin{cases} (\Phi_i(x))_{jm} & \text{if } (i, m) \in J_{\rho pq}, \\ 0 & \text{if } (i, m) \notin J_{\rho pq} \end{cases}.$$

We consider

$$\alpha_\rho = \begin{cases} x_2 & \text{for } \rho = 1, \\ +\infty & \text{for } \rho = 2. \end{cases}$$

We shall show that, for a sufficiently large number $x_2 \geq x_1$, where $x_1$ is as in Lemma 1, the system of integral equations

$$\begin{aligned} z(x) &= \sum_{\rho=1}^{2} \int_{a_\rho}^{x} K^\rho(x, \tau) \Big[ G(\tau) \left\{ z(\tau) + \Gamma(\tau) f_{pv}(\tau) \right\} + \\ &+ \frac{1}{\xi_{pq}(\tau)} \Gamma(\tau)\Delta(\tau) F\left(\tau, \Delta^{-1}(\tau)\Gamma^{-1}(\tau)\xi_{pq}(\tau) z(\tau) + \Phi(\tau) e_{pv}\right) \Big] \, d\tau, \end{aligned} \tag{3.2}$$

where

$$K^\rho(x, \tau) = \frac{\xi_{pq}(\tau)}{\xi_{pq}(x)} \Gamma(x)\Delta(x)\Phi^{(\rho)}(x)\Phi^{-1}(\tau)\Delta^{-1}(\tau)\Gamma^{-1}(\tau),$$

$$G(\tau) = \Gamma(\tau)\Delta(\tau)R(\tau)\Delta^{-1}(\tau)\Gamma^{-1}(\tau),$$

has at least one solution $z = (z_k)_{k=1}^n \in C\left([x_2, +\infty[;\ \mathbb{R}_b^n\right)$ vanishing when $x \to +\infty$.

In this system,

$$K^{(\rho)}(x, \tau) = \operatorname{diag}\left[K_1^{(\rho)}(x, \tau), \dots, K_s^{(\rho)}(x, \tau)\right] \qquad (\rho = 1, 2),$$

$$K_i^\rho(x, \tau) = \frac{\xi_{pq}(\tau)\gamma(x)}{\xi_{pq}(x)\gamma(\tau)} \Delta_i(x)\Phi_i^{(\rho)}(x)\Phi_i^{-1}(\tau)\Delta_i^{-1}(\tau).$$

Here, $K_i^\rho$ consists of the blocks

$$\left(K_i^\rho(x, \tau)\right)_{kj} = \begin{cases} \begin{pmatrix} \left(K_i^\rho(x, \tau)\right)_{kj}^{(1)} & -\left(K_i^\rho(x, \tau)\right)_{kj}^{(2)} \\ \left(K_i^\rho(x, \tau)\right)_{kj}^{(2)} & \left(K_i^\rho(x, \tau)\right)_{kj}^{(1)} \end{pmatrix} & \text{if } 1 \leq k \leq j \leq n_i, \\ O_2 & \text{if } 1 \leq j < k \leq n_i, \end{cases}$$

where

$$\left(K_i^{(\rho)}(x,\tau)\right)_{kj}^{(\mu)} = \sum_{\substack{m=k \\ (i,m)\in J_{\rho pq}}}^{j} g_{im}(x,\tau)\left(\frac{B_{1k}^i(x)B_{km}^{i\,3-\mu}(x)}{B_{1m}^i(x)}\,\frac{B_{1m}^i(\tau)D_{mj}^{i1}(\tau)}{B_{1j}^i(\tau)} + \right.$$

$$\left. +(-1)^{\mu}\ \frac{B_{1k}^i(x)B_{km}^{i\,\mu}(x)}{B_{1m}^i(x)}\,\frac{B_{1m}^i(\tau)D_{mj}^{i2}(\tau)}{B_{1j}^i(\tau)}\right) \qquad (\mu = 1, 2)$$

for $i = 1, \dots, r$, and, for $i = r+1, \dots, s$, it consists of the blocks

$$\left(K_i^{\rho}(x,\tau)\right)_{kj} = \begin{cases} \displaystyle\sum_{\substack{m=k \\ (i,m)\in J_{\rho pq}}}^{j} g_{im}(x,\tau)\frac{B_{1k}^i(x)B_{km}^i(x)}{B_{1m}^i(x)}\,\frac{B_{1m}^i(\tau)D_{mj}^i(\tau)}{B_{1j}^i(\tau)} & \\ & \text{if } 1 \le k \le j \le n_i \\ 0 & \text{if } 1 \le j < k \le n_i. \end{cases}$$

The matrix $G$ is given by

$$G(\tau) = \begin{pmatrix} G_{11}(\tau) & \dots & G_{1s} \\ \vdots & \ddots & \vdots \\ G_{s1}(\tau) & \dots & G_{ss} \end{pmatrix},$$

where $G_{il}$ is the block of dimension $m_i \times m_l$ having the elements

$$(G_{il}(\tau))_{kj} = \frac{\gamma_i(\tau)}{\gamma_l(\tau)}\frac{H_{ik}(\tau)}{H_{lj}(\tau)} r_{N_i+kN_l+j}(\tau), \qquad (i,k),\ (l,j) \in J.$$

Considering the representations above, we rewrite (3.2) as

$$z_k(x) = \int_{x_2}^{x} \psi_{1k}(\tau, x, z(\tau))\,d\tau + \int_{x}^{+\infty} \psi_{2k}(\tau, x, z(\tau))\,d\tau \quad (k = 1, \dots, n), \qquad (3.3)$$

where the functions $\psi_{\rho k}$ ($\rho = 1, 2,\ k = 1, \dots, n$) are defined as follows: for all $\nu \in \{1, \dots, n_\eta\}$, we put

$$\begin{aligned}
\psi_{\rho N_\eta+2\nu-1}(\tau, x, z) = & \sum_{i=1}^{s} \sum_{m=1}^{m_i} \sum_{l=1}^{s} \sum_{j=\nu}^{n_\eta} \Big\{ \left(K_\eta^{(\rho)}(x,\tau)\right)_{\nu j}^{(1)} \left(G_{\eta l}(\tau)\right)_{2j-1\,m} - \\
& - \left(K_\eta^{(\rho)}(x,\tau)\right)_{\nu j}^{(2)} \left(G_{\eta l}(\tau)\right)_{2j\,m} \Big\} z_{N_i+m} + \\
& + \sum_{m=1}^{\nu} \sum_{j=\nu}^{n_\eta} \Big\{ \left(K_\eta^{(\rho)}(x,\tau)\right)_{\nu j}^{(1)} \left(G_{\eta p}(\tau)\right)_{2j-1\,m} - \\
& - \left(K_\eta^{(\rho)}(x,\tau)\right)_{\nu j}^{(2)} \left(G_{\eta p}(\tau)\right)_{2j\,m} \Big\} \gamma_p(\tau) \left(f_{p\nu}(\tau)\right)_{N_p+m} + \\
& + \sum_{j=\nu}^{n_\eta} \frac{\gamma_\eta(\tau)}{\xi_{pq}(\tau)} \Big\{ \left(K_\eta^{(\rho)}(x,\tau)\right)_{\nu j}^{(1)} H_{\eta\,2j-1}(\tau) \widetilde{F}_{N_\eta+2j-1}(\tau, z) - \\
& - \left(K_\eta^{(\rho)}(x,\tau)\right)_{\nu j}^{(2)} H_{\eta\,2j}(\tau) \widetilde{F}_{N_\eta+2j}(\tau, z) \Big\}
\end{aligned}$$

and

$$\begin{aligned}
\psi_{\rho N_\eta+2\nu}(\tau, x, z) = & \sum_{i=1}^{s} \sum_{m=1}^{m_i} \sum_{l=1}^{s} \sum_{j=\nu}^{n_\eta} \Big\{ \left(K_\eta^{(\rho)}(x,\tau)\right)_{\nu j}^{(2)} \left(G_{\eta l}(\tau)\right)_{2j-1\,m} + \\
& + \left(K_\eta^{(\rho)}(x,\tau)\right)_{\nu j}^{(1)} \left(G_{\eta l}(\tau)\right)_{2j\,m} \Big\} z_{N_i+m} + \\
& + \sum_{m=1}^{\nu} \sum_{j=\nu}^{n_\eta} \Big\{ \left(K_\eta^{(\rho)}(x,\tau)\right)_{\nu j}^{(2)} \left(G_{\eta p}(\tau)\right)_{2j-1\,m} + \\
& + \left(K_\eta^{(\rho)}(x,\tau)\right)_{\nu j}^{(1)} \left(G_{\eta p}(\tau)\right)_{2j\,m} \Big\} \gamma_p(\tau) \left(f_{p\nu}(\tau)\right)_{N_p+m} + \\
& + \sum_{j=\nu}^{n_\eta} \frac{\gamma_\eta(\tau)}{\xi_{pq}(\tau)} \Big\{ \left(K_\eta^{(\rho)}(x,\tau)\right)_{\nu j}^{(2)} H_{\eta\,2j-1}(\tau) \widetilde{F}_{N_\eta+2j-1}(\tau, z) + \\
& + \left(K_\eta^{(\rho)}(x,\tau)\right)_{\nu j}^{(1)} H_{\eta\,2j}(\tau) \widetilde{F}_{N_\eta+2j}(\tau, z) \Big\}
\end{aligned}$$

when $\eta \in \{1, \dots, r\}$, and we put

$$\psi_{\rho N_\eta+\nu}(\tau,x,z) = \sum_{i=1}^{s}\sum_{m=1}^{m_i}\sum_{l=1}^{s}\sum_{j=\nu}^{n_\eta}\Big(K_\eta^{(\rho)}(x,\tau)\Big)_{\nu j}\Big(G_{\eta l}(\tau)\Big)_{jm} z_{N_i+m}+$$
$$+\sum_{m=1}^{\nu}\sum_{j=\nu}^{n_\eta}\Big(K_\eta^{(\rho)}(x,\tau)\Big)_{\nu j}\Big(G_{\eta p}(\tau)\Big)_{jm}\gamma_p(\tau)\Big(f_{p\nu}(\tau)\Big)_{N_p+m}+$$
$$+\sum_{j=\nu}^{n_\eta}\frac{\gamma_\eta(\tau)}{\xi_{pq}(\tau)}\Big(K_\eta^{(\rho)}(x,\tau)\Big)_{\nu j} H_{\eta j}(\tau)\widetilde{F}_{N_\eta+j}(\tau,z)$$

if $\eta \in \{r+1, \dots, s\}$. Here,

$$\widetilde{F}_k(\tau,z) = F_k\Big(\tau, \Delta^{-1}(\tau)\Gamma^{-1}(\tau)\xi_{pq}(\tau)z + \Phi(\tau)e_{p\nu}\Big) \quad (k=1,\dots,n).$$

Let us choose $c_1$ as in Lemma 1 and introduce the constant

$$c = \sum_{i=1}^{r}(2c_1)^{n_i}{n_i}^2\eta_i + \sum_{i=r+1}^{s}c_1(2c_1)^{n_i-1}{n_i}^2\eta,$$

where

$$\eta_i = max\{\eta_{im} : 1 \le m \le n_i\} \qquad (i=1,\dots,s).$$

We set

$$\zeta(x,\tau) = \frac{c_1}{c}\left\{\sum_{i=1}^{r}\sum_{k=1}^{n_i}\sum_{j=k}^{n_i}\sum_{\substack{m=k\\(i,m)\in J_{1pq}}}^{j}|g_{im}(x,\tau)|\left(\left|\frac{B^i_{1m}(\tau)D^{i1}_{mj}(\tau)}{B^i_{1j}(\tau)}\right| + \left|\frac{B^i_{1m}(\tau)D^{i2}_{mj}(\tau)}{B^i_{1j}(\tau)}\right|\right)+\right.$$
$$\left.+\sum_{i=r+1}^{s}\sum_{k=1}^{n_i}\sum_{j=k}^{n_i}\sum_{\substack{m=k\\(i,m)\in J_{1pq}}}^{j}|g_{im}(x,\tau)|\left|\frac{B^i_{1m}(\tau)D^{i1}_{mj}(\tau)}{B^i_{1j}(\tau)}\right|\right\}$$

if $J_{1pq} \ne \oslash$, and $\zeta(x,\tau) = 0$ for $J_{1pq} = \oslash$.

Furthermore, we set

$$g(\tau) \quad = c\left\{\sum_{\eta=1}^{s}\sum_{\nu=1}^{n_\eta}\left(b\sum_{i=1}^{s}\sum_{m=1}^{m_i}\sum_{l=1}^{s}\sum_{j=\nu_\eta}^{m_\eta}\left|\Big(G_{\eta l}(\tau)\Big)_{jm}\right|+\right.\right.$$
$$\left.\left.+\ d_p\gamma_p(\tau)\sum_{m=1}^{\nu}\sum_{j=\nu_\eta}^{m_\eta}\left|\Big(G_{\eta p}(\tau)\Big)_{jm}\right| + \sum_{j=\nu_\eta}^{m_\eta}\gamma_\eta(\tau)\phi(\tau)\left|\frac{H_{\eta j}(\tau)}{\xi_{pq}(\tau)}\right|\right)\right\},$$

where

$$\nu_\eta = \begin{cases} 2\nu-1, & \text{if } \eta \in \{1,\dots,r\}, \\ \nu, & \text{if } \eta \in \{r+1,\dots,s\}, \end{cases} \qquad d_p = \begin{cases} \frac{c_1}{2} & \text{for } p \in \{1,\dots,r\}, \\ c_1 & \text{for } p \in \{r+1,\dots,s\}. \end{cases}$$

Taking conditions (ii)–(iv) into account, we can choose the number $x_2 \geq x_1$ so large that the inequality

$$\int_{x_2}^{+\infty} g(\tau)d\tau \leq b$$

be satisfied. Note that, by virtue of condition $(S_B)$ and relation (3.1), for all $z \in \mathbb{R}_b^n$, the following estimates are true:

$$|\psi_{1k}(\tau, x, z)| \leq \zeta(x, \tau)g(\tau) \qquad \text{for} \quad x_2 \leq \tau \leq x \ \ (k = 1, \dots, n),$$
$$|\psi_{2k}(\tau, x, z)| \leq g(\tau) \qquad \text{for} \ \ x_2 \leq x \leq \tau \ \ (k = 1, \dots, n).$$

Therefore, by using (3.1), we get

$$\zeta(x, \tau) \leq 1 \quad \text{for} \quad x \geq \tau \geq x_2$$

and

$$\lim_{x \to +\infty} \zeta(x, \tau) = 0 \qquad \text{for} \quad \tau \geq x_2.$$

Thus, for the chosen $x_2 \geq x_1$, system (3.3) satisfies all the conditions of Lemma 7.6 in the book [1] (Chapter II, §7, p. 201) and, therefore, it has at least one solution $z_{pv}(x)$ vanishing as $x \to +\infty$.

Setting

$$z_{pv}(x) = \frac{\Gamma(x)\Delta(x)y_{pv}(x) - \Gamma(x)\Delta(x)\Phi(x)e_{pv}}{\xi_{pq}(x)}$$

in (3.2), we note that $y_{pv}$ is a solution of system (1.1) for which the asymptotic representation

$$y_{pv}(x) = \Phi(x)e_{pv} + \Delta^{-1}(x)\Gamma^{-1}(x)\overline{o}\left(\xi_{pq}(x)\right)$$

is valid as $x \to +\infty$.

The theorem is proved. □

For some fixed pair $(p, l) \in J$, let us now consider the system of $n$-dimensional vector functions

$$u_{N_p+j}(x) = \Phi(x)e_{pj} \qquad (j = 1, \dots, l).$$

Using these vector functions, we consider the functions

$$Y_{pl}(x) = \sum_{i=1}^{l} C_i u_{N_p+i}(x),$$

where $C_1, \dots, C_l$ are real constants.

Our question now is in which cases system (1.1) has a solution that is asymptotically close, as $x \to +\infty$, to $Y_{pl}(x)$.

In order to study this problem, we formulate the following.

**Definition 2.** We say that a system of continuous functions $q_{im} : [a,+\infty[\times[a,+\infty[ \to \mathbb{R}$ $(m = 1,\dots,n_i,\ \ i = 1,\dots,s)$ *satisfies condition* $(M - L)$ *on* $[x_0,+\infty[\times[x_0,+\infty[$ if, for arbitrary $(i,m) \in J_0$, either

$$\lim_{x\to+\infty} q_{im}(x,\tau) = 0 \qquad (\tau \geq x_0)$$

and (3.4)

$$\theta_{im} = \sup\{q_{im}(x,\tau) : x \geq \tau \geq x_0\} < +\infty,$$

or

$$\theta_{im} = \sup\{q_{im}(x,\tau) : \tau \geq x \geq x_0\} < +\infty. \tag{3.5}$$

**Theorem 2.** *Let condition* $(S_B)$ *be satisfied and, moreover,*

$$\int_{x_0}^{+\infty} \left| \frac{H_{i\nu}(x)}{H_{jm}(x)} r_{N_i+\nu N_j+m}(x) \right| dx < +\infty$$

*for all* $(i,\nu)$ *and* $(j,m)$ *in* $J$*. Furthermore, assume that, for some fixed pair* $(p,k) \in J_0$*, there exists a function* $U_{pl} \in C([a,+\infty[\,;\,]0,+\infty[)$*, where*

$$l = \begin{cases} 2k & \text{if} \quad p \in \{1,\dots,r\} \\ k & \text{if} \quad p \in \{r+1,\dots,s\} \end{cases}$$

*such that, for each* $j \in \{1,\dots,l\}$*,*

$$U_{pl}(x) = \underline{O}\left( \sum_{i=N_p+1}^{N_p+j} \left| (u_{N_p+j}(x))_i \right| \right)$$

*as* $x \to +\infty$*, and the following conditions hold:*

(i) *The system*

$$q_{im}(x,\tau) = \frac{U_{pl}(\tau)B^i_{1m}(x)}{U_{pl}(x)B^i_{1m}(\tau)} \exp \int_\tau^x \omega_{i1}(s)\,ds \qquad ((i,m) \in J_0)$$

*satisfies condition* $(M - L)$ *on* $[x_0,+\infty[\times[x_0,+\infty[$;

(ii) *For all* $(\eta, j) \in J$ *and arbitrary* $m, \xi$ *with* $1 \leq m \leq k$ *and* $m \leq \xi \leq k$*,*

$$\int_{x_0}^{+\infty} \frac{1}{U_{pl}(x)} \left| \frac{H_{\eta j}(x)}{H_{p\,2m-\rho}(x)} r_{N_\eta+jN_p+2m-\rho}(x) B^{p\mu}_{m\xi}(x) \right| \exp \int_a^x \omega_{pm}(s)ds \;\; dx < +\infty$$

*if* $p \in \{1,\dots,r\}$ $(\rho = 0,1\ ,\ \ \mu = 1,2)$ *and*

$$\int_{x_0}^{+\infty} \frac{1}{U_{pl}(x)} \left| \frac{H_{\eta j}(x)}{H_{p\,m}(x)} r_{N_\eta+jN_p+m}(x) B^{p}_{m\xi}(x) \right| \exp \int_a^x \omega_{pm}(s)ds \;\; dx < +\infty$$

*when* $p \in \{r+1,\dots,s\}$*;*

(iii) *There is a number* $b \in ]0, +\infty[$ *and there is a continuous function*

$$\phi : [a, +\infty[\to \mathbb{R}_+$$

*such that, for every* $(i, \nu) \in J$ *and arbitrary* $z \in \mathbb{R}^n_b$,

$$\left|F_{N_i+\nu}(x, \Delta^{-1}(x)U_{pl}(x)z + Y_{pl}(x))\right| \leq \phi(x)$$

*and*

$$\int_{x_0}^{+\infty} \frac{\phi(x)}{U_{pl}(x)} |H_{i\nu}(x)| \, dx < +\infty.$$

*Then system* (1.1) *has at least one solution* $y_{pl}(x)$ *that satisfies the asymptotic relation*

$$y_{pl}(x) = Y_{pl}(x) + \Delta^{-1}(x)\bar{o}\left(U_{pl}(x)\right)$$

*when* $x \to +\infty$.

*Proof.* By the definition of the vector function $Y_{pl}(x) = \left((Y_{pl})_j\right)_{j=1}^n$, its components are determined as follows: for every $\xi \in \{1, \ldots, k\}$, we have

$$\left(Y_{pl}\right)_{N_p+2\xi-1}(x) = \sum_{\nu=\xi}^{k} \left[C_{2\nu-1}B^{p2}_{\xi\nu}(x) - C_{2\nu}B^{p1}_{\xi\nu}(x)\right] \exp \int_a^x \omega_{p\xi}(s)ds,$$

$$\left(Y_{pl}\right)_{N_p+2\xi}(x) = \sum_{\nu=\xi}^{k} \left[C_{2\nu-1}B^{p1}_{\xi\nu}(x) - C_{2\nu}B^{p2}_{\xi\nu}(x)\right] \exp \int_a^x \omega_{p\xi}(s)ds,$$

$$\left(Y_{pl}\right)_j(x) = 0 \qquad (j \neq N_p + 1, \ldots, N_p + l)$$

when $p \in \{1, \ldots, r\}$, and

$$\left(Y_{pl}\right)_{N_p+\xi}(x) = \sum_{\nu=\xi}^{k} C_\nu B^{p}_{\xi\nu}(x) \exp \int_a^x \omega_{p\xi}(s)ds,$$

$$\left(Y_{pl}\right)_j(x) = 0 \qquad (j \neq N_p + 1, \ldots, N_p + l)$$

when $p \in \{r + 1, \ldots, s\}$.

Under condition (i) of the theorem, the set $J_0$ decomposes as the union of disjoint subsets $J_{1pl}$ and $J_{2pl}$,

$$J_0 = J_{1pl} \bigcup J_{2pl},$$

such that if $(i, m) \in J_{1pl}$, then estimate (3.4) is true and if $(i, m) \in J_{2pl}$, then relation (3.5) is correct. Therefore,

$$\Phi(x) = \Phi^{(1)}(x) + \Phi^{(2)}(x)$$

with

$$\Phi^{(\rho)}(x) = \operatorname{diag}\left[\Phi^{(\rho)}_1(x), \ldots, \Phi^{(\rho)}_s(x)\right] \qquad (\rho = 1, 2),$$

where $\Phi_i^{(\rho)}(x)$ $(i \in \{1, \dots, s\};\ \ \rho = 1, 2)$ are $m_i$th order matrices with the elements

$$\left(\Phi_i^{(\rho)}(x)\right)_{jm} = \begin{cases} (\Phi_i(x))_{jm} & \text{for } (i, m) \in J_{\rho pl} \\ 0 & \text{for } (i, m) \notin J_{\rho pl}. \end{cases}$$

Let us show that, for a sufficiently large number $x_3 \geq x_2$ (where $x_2$ is as in Theorem 1), the system of integral equations

$$\begin{aligned} z(x) &= \sum_{\rho=1}^{2} \int_{a_\rho}^{x} Q^\rho(x, \tau) \left[ G(\tau) \left\{ z(\tau) + \frac{1}{U_{pl}(\tau)} \right\} + \right. \\ &\left. + \frac{1}{U_{pl}(\tau)} \Delta(\tau) F\left(\tau, \Delta^{-1}(\tau) U_{pl}(\tau) z(\tau) + Y_{pl}(\tau)\right) \right] d\tau \end{aligned} \tag{3.5}$$

with

$$Q^\rho(x, \tau) = \frac{U_{pl}(\tau)}{U_{pl}(x)} \Delta(x) \Phi^{(\rho)}(x) \Phi^{-1}(\tau) \Delta^{-1}(\tau),$$
$$G(\tau) = \Delta(\tau) R(\tau) \Delta^{-1}(\tau),$$

and

$$a_\rho = \begin{cases} x_3 & \text{if } \rho = 1 \\ +\infty & \text{if } \rho = 2, \end{cases}$$

has at least one solution $z = (z_k)_{k=1}^n \in C\left([x_3, +\infty[;\ \mathbb{R}_b^n\right)$ vanishing when $x \to +\infty$.

Here,

$$Q^{(\rho)}(x, \tau) = \operatorname{diag}\left[Q_1^{(\rho)}(x, \tau), \dots, Q_s^{(\rho)}(x, \tau)\right] \quad (\rho = 1, 2),$$

$$Q_i^\rho(x, \tau) = \tfrac{U_{pl}(\tau)}{U_{pl}(x)} \Delta_i(x) \Phi_i^{(\rho)}(x) \Phi_i^{-1}(\tau) \Delta_i^{-1}(\tau) \qquad (i = 1, \dots, s;\ \ \rho = 1, 2),$$

For $i = 1, \dots, r$, the matrices $Q_i^\rho$ consist of the blocks

$$\left(Q_i^\rho(x, \tau)\right)_{\nu j} = \begin{cases} \begin{pmatrix} \left(Q_i^\rho(x, \tau)\right)_{\nu j}^{(1)} & -\left(Q_i^\rho(x, \tau)\right)_{\nu j}^{(2)} \\ \left(Q_i^\rho(x, \tau)\right)_{\nu j}^{(2)} & \left(Q_i^\rho(x, \tau)\right)_{\nu j}^{(1)} \end{pmatrix} & \text{if } 1 \leq \nu \leq j \leq n_i \\ O_2 & \text{if } 1 \leq j < \nu \leq n_i, \end{cases}$$

where

$$\left(Q_i^{(\rho)}(x,\tau)\right)_{kj}^{(\mu)} = \sum_{\substack{m=k\\(i,m)\in J_{\rho pq}}}^{j} q_{im}(x,\tau)\left(\frac{B^i_{1k}(x)B^{i\,3-\mu}_{km}(x)}{B^i_{1m}(x)}\frac{B^i_{1m}(\tau)D^{i1}_{mj}(\tau)}{B^i_{1j}(\tau)} + \right.$$

$$\left. +(-1)^{\mu}\ \frac{B^i_{1k}(x)B^{i\,\mu}_{km}(x)}{B^i_{1m}(x)}\frac{B^i_{1m}(\tau)D^{i2}_{mj}(\tau)}{B^i_{1j}(\tau)}\right) \qquad (\mu=1,2),$$

and, for $i = r+1,\dots,s$, they consist of the blocks

$$\left(Q_i^{\rho}(x,\tau)\right)_{\nu j} = \begin{cases} \displaystyle\sum_{\substack{m=\nu\\(i,m)\in J_{\rho pl}}}^{j} q_{im}(x,\tau)\frac{B^i_{1\nu}(x)B^i_{\nu m}(x)}{B^i_{1m}(x)}\frac{B^i_{1m}(\tau)D^i_{mj}(\tau)}{B^i_{1j}(\tau)} & \text{if } 1\le\nu\le j\le n_i\\ 0 & \text{if } 1\le j<\nu\le n_i. \end{cases}$$

Using the notation

$$z_m(x) = \int_{x_3}^{x}\lambda_{1m}(\tau,x,z(\tau))\,d\tau + \int_{x}^{+\infty}\lambda_{2m}(\tau,x,z(\tau))\,d\tau \qquad (m=1,\dots,n), \tag{3.6}$$

we rewrite (3.5) as (3.6), where the functions $\lambda_{\rho m}$ $(\rho=1,2,\ \ m=1,\dots,n)$ are given in the following way: for all $\nu\in\{1,\dots,n_\eta\}$, we have

$$\lambda_{\rho N_\eta+2\nu-1}(\tau,x,z) = \sum_{i=1}^{s}\sum_{m=1}^{m_i}\sum_{\xi=1}^{s}\sum_{j=\nu}^{n_\eta}\left\{\left(Q_\eta^{(\rho)}(x,\tau)\right)_{\nu j}^{(1)}\left(G_{\eta\xi}(\tau)\right)_{2j-1\,m} - \right.$$

$$\left. -\left(Q_\eta^{(\rho)}(x,\tau)\right)_{\nu j}^{(2)}\left(G_{\eta\xi}(\tau)\right)_{2j\,m}\right\} z_{N_i+m} +$$

$$+\sum_{m=1}^{l}\sum_{j=\nu}^{n_\eta}\left\{\left(Q_\eta^{(\rho)}(x,\tau)\right)_{\nu j}^{(1)}\left(G_{\eta p}(\tau)\right)_{2j-1\,m} - \right.$$

$$\left. -\left(Q_\eta^{(\rho)}(x,\tau)\right)_{\nu j}^{(2)}\left(G_{\eta p}(\tau)\right)_{2j\,m}\right\}\frac{1}{U_{pl}(\tau)}\left(Y_{pl}(\tau)\right)_{N_p+m} +$$

$$+\sum_{j=\nu}^{n_\eta}\frac{1}{U_{pl}(\tau)}\left\{\left(Q_\eta^{(\rho)}(x,\tau)\right)_{\nu j}^{(1)} H_{\eta\,2j-1}(\tau)\widetilde{F}_{N_{\eta+2j-1}}(\tau,z) - \right.$$

$$\left. -\left(Q_\eta^{(\rho)}(x,\tau)\right)_{\nu j}^{(2)}\sum H_{\eta\,2j}(\tau)\widetilde{F}_{N_{\eta+2j}}(\tau,z)\right\}$$

and

$$\lambda_{\rho N_\eta+2\nu}(\tau,x,z) = \sum_{i=1}^{s}\sum_{m=1}^{m_i}\sum_{\xi=1}^{s}\sum_{j=\nu}^{n_\eta}\Big\{\Big(Q_\eta^{(\rho)}(x,\tau)\Big)_{\nu j}^{(2)}\Big(G_{\eta\xi}(\tau)\Big)_{2j-1\,m}+$$

$$+\Big(Q_\eta^{(\rho)}(x,\tau)\Big)_{\nu j}^{(1)}\Big(G_{\eta\xi}(\tau)\Big)_{2j\,m}\Big\}z_{N_i+m}+$$

$$+\sum_{m=1}^{l}\sum_{j=\nu}^{n_\eta}\Big\{\Big(Q_\eta^{(\rho)}(x,\tau)\Big)_{\nu j}^{(2)}\Big(G_{\eta p}(\tau)\Big)_{2j-1\,m}+$$

$$+\Big(Q_\eta^{(\rho)}(x,\tau)\Big)_{\nu j}^{(1)}\Big(G_{\eta p}(\tau)\Big)_{2j\,m}\Big\}\frac{1}{U_{pl}(\tau)}\Big(Y_{pl}(\tau)\Big)_{N_p+m}+$$

$$+\sum_{j=\nu}^{n_\eta}\frac{1}{U_{pl}(\tau)}\Big\{\Big(Q_\eta^{(\rho)}(x,\tau)\Big)_{\nu j}^{(2)}H_{\eta\,2j-1}(\tau)\widetilde{F}_{N_{\eta+2j-1}}(\tau,z)+$$

$$+\Big(Q_\eta^{(\rho)}(x,\tau)\Big)_{\nu j}^{(1)}H_{\eta\,2j}(\tau)\widetilde{F}_{N_{\eta+2j}}(\tau,z)\Big\},$$

when $\eta\in\{1,\dots,r\}$; and we have

$$\lambda_{\rho N_\eta+\nu}(\tau,x,z) = \sum_{i=1}^{s}\sum_{m=1}^{m_i}\sum_{\xi=1}^{s}\sum_{j=\nu}^{n_\eta}\Big(Q_\eta^{(\rho)}(x,\tau)\Big)_{\nu j}\Big(G_{\eta\xi}(\tau)\Big)_{j\,m}z_{N_i+m}+$$

$$+\sum_{m=1}^{l}\sum_{j=\nu}^{n_\eta}\Big(Q_\eta^{(\rho)}(x,\tau)\Big)_{\nu j}\Big(G_{\eta p}(\tau)\Big)_{j\,m}\frac{1}{U_{pl}(\tau)}\Big(Y_{pl}(\tau)\Big)_{N_p+m}+$$

$$+\sum_{j=\nu}^{n_\eta}\frac{1}{U_{pl}(\tau)}\Big(Q_\eta^{(\rho)}(x,\tau)\Big)_{\nu j}H_{\eta\,j}(\tau)\widetilde{F}_{N_{\eta+j}}(\tau,z)$$

when $\eta\in\{r+1,\dots,s\}$. Here,

$$\widetilde{F}_j(\tau,z)=F_j(\tau,\Delta^{-1}(\tau)U_{pl}(\tau)z+Y_{pl}(\tau))\qquad(j=1,\dots,n).$$

Let $c_1$ be chosen as in Lemma 1 and

$$\zeta(x,\tau)=\frac{c_1}{c_2}\left\{\sum_{i=1}^{r}\sum_{\nu=1}^{n_i}\sum_{j=\nu}^{n_i}\sum_{\substack{m=\nu\\(i,m)\in J_{1pl}}}^{j}|q_{im}(x,\tau)|\left(\left|\frac{B^i_{1m}(\tau)D^{i1}_{mj}(\tau)}{B^i_{1j}(\tau)}\right|+\right.\right.$$

$$\left.\left.+\left|\frac{B^i_{1m}(\tau)D^{i2}_{mj}(\tau)}{B^i_{1j}(\tau)}\right|\right)+\sum_{i=r+1}^{s}\sum_{\nu=1}^{n_i}\sum_{j=\nu}^{n_i}\sum_{\substack{m=\nu\\(i,m)\in J_{1pl}}}^{j}|q_{im}(x,\tau)|\left|\frac{B^i_{1m}(\tau)D^{i}_{mj}(\tau)}{B^i_{1j}(\tau)}\right|\right\}$$

if $J_{1pl}\neq\oslash$, and $\zeta(x,\tau)=0$ in the case where $J_{1pl}=\oslash$. We have put

$$c_2=\sum_{i=1}^{r}(2c_1)^{n_i}{n_i}^2\theta_i+\sum_{i=r+1}^{s}c_1(2c_1)^{n_i-1}{n_i}^2\theta,$$

where

$$\theta_i=\max\{\theta_{im}:1\leq m\leq n_i\}\qquad(i=1,\dots,s).$$

It is easy to see that

$$\zeta(x,\tau)\leq 1$$

for $x\geq\tau\geq x_3$.

For $p\in\{1,\dots,r\}$, we set

$$\begin{aligned}q(\tau)\;&=c_3\left\{\sum_{\eta=1}^{s}\sum_{\nu=1}^{n_\eta}\left(b\sum_{i=1}^{s}\sum_{m=1}^{m_i}\sum_{\xi=1}^{s}\sum_{j=\nu_\eta}^{m_\eta}\left|\left(G_{\eta\xi}(\tau)\right)_{jm}\right|+\right.\right.\\&+\frac{1}{U_{pl}(\tau)}\sum_{\theta=1}^{l}C_\theta\sum_{m=1}^{k}\sum_{j=2\nu-1}^{m_\eta}\sum_{\xi=m}^{k}\exp\int_a^\tau\omega_{pm}(s)ds\left(\left|B^{p1}_{m\xi}(\tau)\right|+\left|B^{p2}_{m\xi}(\tau)\right|\right)\times\\&\left.\left.\times\left(\left|\left(G_{\eta p}(\tau)\right)_{j\,2m-1}\right|+\left|\left(G_{\eta p}(\tau)\right)_{j\,2m}\right|\right)+\sum_{j=2\nu-1}^{m_\eta}\frac{\phi(\tau)}{U_{pl}(\tau)}\left|H_{\eta j}(\tau)\right|\right)\right\}\end{aligned}$$

and, for $p\in\{r+1,\dots,s\}$, we put

$$\begin{aligned}q(\tau)\;&=c_3\left\{\sum_{\eta=1}^{s}\sum_{\nu=1}^{n_\eta}\left[b\sum_{i=1}^{s}\sum_{m=1}^{m_i}\sum_{\xi=1}^{s}\sum_{j=\nu_\eta}^{m_\eta}\left|\left(G_{\eta\xi}(\tau)\right)_{jm}\right|+\right.\right.\\&+\frac{1}{U_{pl}(\tau)}\sum_{\theta=1}^{l}C_\theta\sum_{m=1}^{k}\sum_{j=\nu}^{m_\eta}\sum_{\xi=m}^{k}\exp\int_{x_0}^\tau\omega_{pm}(s)ds\left|B^{p}_{m\xi}(\tau)\right|\left|\left(G_{\eta p}(\tau)\right)_{jm}\right|+\\&\left.\left.+\sum_{j=2\nu-1}^{m_\eta}\frac{\phi(\tau)}{U_{pl}(\tau)}\left|H_{\eta j}(\tau)\right|\right]\right\}.\end{aligned}$$

Using now the conditions assumed in Theorem 2, we choose the number $x_3$ so large that the relation $\int_{x_3}^{+\infty} q(\tau)d\tau \le b$ for each $z \in \mathbb{R}_b^n$ be satisfied. Then

$$|\lambda_{1m}(\tau, x, z)| \le \zeta(x, \tau)q(\tau) \qquad \text{for } \; x_3 \le \tau \le x \quad (m = 1, \ldots, n)$$

and

$$|\lambda_{2m}(\tau, x, z)| \le q(\tau) \qquad \text{for } \; x_3 \le x \le \tau \quad (m = 1, \ldots, n).$$

Hence, system (3.6) has at least one solution $z_{pl}(x)$ such that $\lim\limits_{x\to+\infty} z_{pl}(x) = 0$. To complete the proof, we define the function $y_{pl}(x)$ as follows:

$$z_{pl}(x) = \frac{\Delta(x)y_{pl}(x) - \Delta(x)Y_{pl}(x)}{U_{pl}(x)} \quad .$$

The theorem is proved. □

*Remark* 1. If the function $U_{pl}(x)$ from Theorem 2 satisfies the condition

$$\sum_{i=N_p+1}^{N_p+j} \left|(u_{N_p+j}(x))_i\right| = \underline{O}(U_{pl}(x)) \qquad \text{for each } \; j \in \{1, \ldots, l\}$$

when $x \to +\infty$, then condition (ii) of Theorem 2 can be dropped.

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