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GEOMETRY OF ALMOST CLIFFORDIAN MANIFOLDS: NIJENHUIS TENSOR

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Abstract. We generalize some classical results on Nijenhuis tensor for an almost Cliffordian manifold based on arbitrary Clifford algebra and suggest its relations with the integrability of the corresponding G-structure. We prove the set of properties for Nijenhus tensors with respect to arbitrary Clifford algebra.

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1. Almost Cliffordian manifolds

Let $\mathcal{O} = \mathcal{C}\ell(s,t)$ be a Clifford algebra. If M is an km-dimensional manifold, where $k = 2^{s+t}$ and $m \in \mathbb{N}$, then an almost Clifford manifold is given by a reduction of the structure group $\mathrm{GL}(km,\mathbb{R})$ of the principal frame bundle of M to

$$\mathrm{GL}(m,\mathcal{O}) = \{A \in \mathrm{GL}(km,\mathbb{R}) | AI = IA, I \in \mathcal{O}\},$$

where $\mathcal{O}$ is arbitrary Cliffford algebra. In other words, an almost Clifford manifold is a smooth manifold equipped by the set of anti commuting and commuting affinors $I_i, i = 1,\dots,t,\ I_i^2 = -E$ and $J_j,\ j = 1,\dots,s,\ J_j^2 = E$ such that the free associative unitary algebra generated by $\langle I_i, J_j, E\rangle$ is isomorphically equivalent to $\mathcal{O}$. In particular, on the elements of this reduced bundle, one can define affinors in the form $F_1,\dots,F_k$ globally.

Definition 1. Let M be a smooth manifold such that $\dim(M) = m$. Let A be a smooth ℓ-dimensional ($\ell < m$) vector subbundle in $T^*M \otimes TM$ such that the identity affinor $E = \mathrm{id}_{TM}$ restricted to T_xM belongs to $A_xM \subset T_x^*M \otimes T_xM$ at each point $x \in M$. We say that M is equipped with an ℓ-dimensional *A-structure.*

It is easy to see that an almost Clifford structure is not an A-structure because the affinors in the form $F_0,\dots,F_\ell \in A$ have to be defined only locally.

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Definition 2. The A-structure, where A is a Clifford algebra $\mathcal{O}$, is called an almost Cliffordian manifold.

In particular, the almost Clifford and almost Cliffordian structures are G-structures based on Clifford algebras. Two most important examples are an almost hypercomplex geometry and an almost quaternionic geometry, which are based on Clifford algebra $\mathcal{C}\ell(0,2)$. Note that the geometric property of an almost hypercomplex structure reads that there is no nontrivial G-invariant subspace $\mathcal{D}$ in $\mathbb{V} \otimes \wedge^2\mathbb{V}^*$ because the first prolongation $\mathfrak{g}^{(1)}$ of the Lie algebra $\mathfrak{g}$ vanishes. For an almost quaternionic structure, the situation is more complicated, because $\mathfrak{g}^{(1)} = \mathbb{V}^*$ and there is a class of these structures indexed by $\mathbb{V}$, see [1]. Note that for Cliffordian structures based on $\mathcal{C}\ell(0,3)$ is $\mathfrak{g}^{(1)} = \mathbb{V}^*$ and there is a class of these structures indexed by $\mathbb{V}$ too, see [2].

2. Nijenhuis tensor

The Nijenhuis tensor plays an important role in the theory of integrability. As a classical concept, Nijenhuis introduced $N_J \in \wedge^2 T^*M \otimes TM$ of an almost complex structure $J \in T^*M \otimes TM$. This tensor is an obstruction for an almost complex structure which distinguished it from the complex structure, i. e. their integrability. Recall that Nijenhuis tensor $N(P,Q) \in \wedge^2 T^*M \otimes TM$ for a pair of tensors $P, Q \in T^*M \otimes TM$ is given by the expression

$$\begin{aligned} N(P,Q)(X,Y) = [PX,QY] - P[QX,Y] - Q[X,PY] + [QX,PY] \\ - Q[PX,Y] - P[X,QY] + (PQ+QP)[X,Y]. \end{aligned}$$

An almost quaternionic manifold M is integrable if and only if the Nijenhuis tensors $N(I,I)$ and $N(J,J)$ vanish, where $I, J, IJ \in T^*M \otimes TM$ is a quaternionic structure. Let us finally note that in [3], the author proved a similar fact for Clifford algebra $\mathcal{C}\ell(0,3)$.

If $P = Q$, then, by straightforward computing,

$$N(P,P)(X,Y) = 2([PX,PY] - P[PX,Y] - P[X,PY] + P^2[X,Y])$$

and if $P = E$, where E is an identity, then

$$N(E,Q)(X,Y) = 0.$$

Following [6,7], we reformulate Nijenhuis tensor calculus with respect to arbitrary Clifford algebra. In those papers, a set of classical results for $\mathcal{C}\ell(0,2)$ and $\mathcal{C}\ell(2,0)$ was proved. In fact, most of the arguments work generally, but we have to formulate and prove them for any Clifford algebra. First of all, following papers [6,7], we recall the operation $\bar{\wedge}$

$$(S \bar{\wedge} P)(X,Y) := S(PX,Y) + S(X,PY) \tag{1}$$

and

$$(P \bar{\wedge} S)(X, Y) := PS(X, Y) \tag{2}$$

where $S \in \otimes^2 \mathbb{V} \otimes \mathbb{V}^*$ and $N \in \mathbb{V} \otimes \mathbb{V}^*$. Now, one can easily check the following identities

$$N(L, QP) + N(Q, LP) = N(L, QP)\bar{\wedge} P + L\bar{\wedge} N(Q, P) + Q\bar{\wedge} N(L, P) \tag{3}$$

$$(S\bar{\wedge} Q)\bar{\wedge} P - (S\bar{\wedge} P)\bar{\wedge} Q = S\bar{\wedge} QP - S\bar{\wedge} PQ \tag{4}$$

$$(L\bar{\wedge} S)\bar{\wedge} P = L\bar{\wedge}(S\bar{\wedge} P) \tag{5}$$

Lemma 1. *Let $\mathcal{O} = \mathcal{C}\ell(s, t)$ be a Clifford algebra. If $F, G \in \mathcal{O}$ such that $F \neq G$, then the following identities hold:*

$$N(FG, F) = -\frac{1}{2}N(F, G)\bar{\wedge} F - \frac{1}{2}G\bar{\wedge} N(F, F) + \frac{1}{4}N(F, F)\bar{\wedge} G \tag{6}$$

$$0 = N(F, G)\bar{\wedge} F + 2F\bar{\wedge} N(F, G) + G\bar{\wedge} N(F, F) + \frac{1}{2}N(F, F)\bar{\wedge} G \tag{7}$$

$$N(G, H) = N(F, G)\bar{\wedge} G + G\bar{\wedge} N(F, G) + F\bar{\wedge} N(G, G). \tag{8}$$

Proof. Putting $L = Q = F,\ P = G$ in (3), we find

$$N(F, FG) + N(F, FG) = N(F, F)\bar{\wedge} G + F\bar{\wedge} N(F, G) + F\bar{\wedge} N(F, G),$$

that is,

$$N(FG, F) = \frac{1}{2}N(F, F)\bar{\wedge} G + F\bar{\wedge} N(F, G). \tag{9}$$

Putting $L = G,\ P = Q = F$ in (3), we find

$$N(G, F^2) + N(F, GF) = N(G, F)\bar{\wedge} F + G\bar{\wedge} N(F, F) + F\bar{\wedge} N(G, F),$$

that is,

$$N(F, GF) = N(G, F)\bar{\wedge} F + G\bar{\wedge} N(F, F) + F\bar{\wedge} N(G, F). \tag{10}$$

Adding (9) and (10) and dividing the sum by 2, we find

$$N(FG, F) = -\frac{1}{2}N(F, G)\bar{\wedge} F - \frac{1}{2}G\bar{\wedge} N(F, F) + \frac{1}{4}N(F, F)\bar{\wedge} G$$

and subtracting (10) from (9), we find

$$N(F, G)\bar{\wedge} F + 2F\bar{\wedge} N(F, G) + G\bar{\wedge} N(F, F) + \frac{1}{2}N(F, F)\bar{\wedge} G = 0 \tag{11}$$

Finally, putting $L = P = G,\ Q = F$ in (3), we find

$$N(F, G^2) + N(G, FG) = N(F, G)\bar{\wedge} G + F\bar{\wedge} N(G, G) + G\bar{\wedge} N(F, G),$$

that is,

$$N(G, H) = N(F, G)\bar{\wedge} G + G\bar{\wedge} N(F, G) + F\bar{\wedge} N(G, G).$$

□

Lemma 2. *Let $\mathcal{O} = \mathcal{C}\ell(s,t)$ be a Clifford algebra. If $F, G \in \mathcal{O}$ such that $F \neq G$, then the following identity holds:*

$$\epsilon_1 N(F,F) - \epsilon_2 N(G,G) + N(H,F)\bar{\wedge}G + G\bar{\wedge}N(H,F) \\ -N(G,H)\bar{\wedge}F + F\bar{\wedge}N(G,H) + 2H\bar{\wedge}N(F,G) = 0, \tag{12}$$

where $H = FG$ and $\epsilon_i = 1$ for $K^2 = -1$ and $\epsilon_i = -1$ for $K^2 = 1$.

Proof. Putting $L = FG,\ Q = F, P = G$ in (3), we find

$$N(FG,FG) - N(F,FGG) = N(FG,F)\bar{\wedge}G + FG\bar{\wedge}N(F,G) \\ +F\bar{\wedge}N(FG,G),$$

that is,

$$N(H,H) = \epsilon N(F,F) + N(FG,F)\bar{\wedge}G + FG\bar{\wedge}N(F,G) \\ + F\bar{\wedge}N(FG,G), \tag{13}$$

where $\epsilon = 1$ for $G^2 = -1$ and $\epsilon = -1$ for $G^2 = 1$.

Putting $L = FG,\ Q = G, P = F$ in (3), we find

$$N(FG,GF) - N(G,FGF) = N(FG,G)\bar{\wedge}F + FG\bar{\wedge}N(G,F) \\ + G\bar{\wedge}N(FG,F),$$

that is,

$$N(H,H) = \epsilon N(G,G) - N(G,H)\bar{\wedge}F - H\bar{\wedge}N(F,G) - G\bar{\wedge}N(H,F), \tag{14}$$

where $\epsilon = 1$ for $G^2 = -1$ and $\epsilon = -1$ for $G^2 = 1$. Thus, from (13) and (14), we find

$$N(H,H) = \frac{1}{2}\{\epsilon N(F,F) + \epsilon N(G,G) + N(H,F)\bar{\wedge}G \\ + G\bar{\wedge}N(H,F) - N(G,H)\bar{\wedge}F + F\bar{\wedge}N(G,H) + 2H\bar{\wedge}N(F,G)\} = 0$$

and

$$\epsilon N(F,F) - \epsilon N(G,G) + N(H,F)\bar{\wedge}G \\ + G\bar{\wedge}N(H,F) - N(G,H)\bar{\wedge}F + F\bar{\wedge}N(G,H) + 2H\bar{\wedge}N(F,G) = 0,$$

which completes the proof. □

Lemma 3. *Let $\mathcal{O} = \mathcal{C}\ell(s,t)$ be a Cliffod algebra. If $F \in \mathcal{O}$, then the following identity holds:*

$$N(F,F)\bar{\wedge}F = -2F\bar{\wedge}N(F,F).$$

Proof. One can easily check that putting $L = P = Q = F$ gives $N(F,F^2) + N(F,F^2) = N(F,F)\bar{\wedge}F + F\bar{\wedge}N(F,F) + F\bar{\wedge}N(F,F)$ □

Theorem 1. *Let $\mathcal{O}$ be a Clifford algebra $\mathcal{C}\ell(s,t)$ and let $F, G \in \mathcal{O}$ such that $F \neq G$. If the Nijenhuis tensors $N(F,F)$ and $N(G,G)$ vanish, then $N(FG,FG)$ vanishes.*

Proof. Since $N(F,F) = 0$, we have from (9) $N(H,F) = F\bar{\wedge}N(F,G)$, and from (11),

$$N(F,G)\bar{\wedge}F = -2F\bar{\wedge}N(F,G). \tag{15}$$

Since $N(G,G) = 0$, we have from (9), where we changed F and G,

$$N(F,G)\bar{\wedge}G = -2G\bar{\wedge}N(F,G) \tag{16}$$

and from (11), where we changed F and G,

$$N(F,G)\bar{\wedge}G = -2G\bar{\wedge}N(F,G). \tag{17}$$

Now, if we substitute $N(F,F) = 0$, $N(G,G) = 0$, and (16) into (12), then the part containing ϵ_i vanishes and the proof is correct for any Clifford algebra, i. e., we find

$$(F\bar{\wedge}N(F,G))\bar{\wedge}G + G\bar{\wedge}(F\bar{\wedge}N(F,G)) - (G\bar{\wedge}N(F,G))\bar{\wedge}F - F\bar{\wedge}(G\bar{\wedge}N(F,G)) + 2H\bar{\wedge}N(F,G) = 0,$$

from which

$$(F\bar{\wedge}N(F,G))\bar{\wedge}G - (G\bar{\wedge}N(F,G))\bar{\wedge}F = 0, \tag{18}$$

since

$$G\bar{\wedge}(F\bar{\wedge}N(F,G)) = -F\bar{\wedge}(G\bar{\wedge}N(F,G)) = -H\bar{\wedge}N(F,G)$$

by virtue of $GF = -FG = -H$. Now, using (5), (15) and (17), we find, from (18)

$$\begin{aligned} F\bar{\wedge}(N(F,G)\bar{\wedge}G) - G\bar{\wedge}(N(F,G)\bar{\wedge}F) &= 0, \\ -2F\bar{\wedge}(G\bar{\wedge}N(F,G)) + 2G\bar{\wedge}(F\bar{\wedge}N(F,G)) &= 0, \\ -4FG\bar{\wedge}N(F,G) &= 0, \end{aligned}$$

that is,

$$H\bar{\wedge}N(F,G) = 0. \tag{19}$$

Since $H^2 = -1$, we have from (19) $N(F,G) = 0$. □

Corollary 1. *Let $\mathcal{O}$ be a Clifford algebra $\mathcal{C}\ell(s,t)$. If the Nijenhuis tensors $N(I_i, I_i)$ vanish, where I_i are the algebra generators of $\mathcal{O}$, then*

$$N(F_i, F_j) = 0,$$

where F_i are vector space generators.

3. Classes of subordinated connections

Recall the concept of A-planar curves on A-structures equipped with the linear connection ∇. For any tangent vector $X \in T_xM$, we shall write $A_x(X)$ for the vector subspace

$$A_x(X) = \{F_i(X) | F_i \in A_xM\} \subset T_xM$$

and call it the *A-hull of the vector X*. Similarly, *A-hull of a vector field* is a subbundle in TM obtained pointwise. Let M be a smooth manifold equipped with an A-structure and a linear connection ∇. A smooth curve $c : \mathbb{R} \to M$ is said to be *A-planar* if

$$\nabla_{\dot{c}}\dot{c} \in A(\dot{c}).$$

In [5], the authors proved a set of facts about the class of $\mathcal{D}$-connections. The theorems below, about Cliffordian structures, are proved in paper [5] and some examples of this concept can be found in papers [2,4]. The theorems about $\mathcal{D}$-connections can be found in [1].

Following [4,5], we have a set of results on Clifford and Cliffordian manifolds.

Corollary 2. *Let M be a smooth manifold equipped with a G-structure, where $G = \mathrm{GL}(n, \mathcal{O})$, $\mathcal{O} = \mathcal{C}\ell(s,t)$, $s + t > 1$, i. e. an almost Clifford manifold. Then the G-structure is of type 1 and there exists a unique $\mathcal{D}$-connection.*

One can see that an almost Cliffordian manifold M is given as a G-structure provided that there is a reduction of the structure group of the principal frame bundle of M to $G := \mathrm{GL}(m, \mathcal{O})\,\mathrm{GL}(1, \mathcal{O}) = \mathrm{GL}(m, \mathcal{O}) \times \mathrm{GL}(1, \mathcal{O})$, the action of G on T_xM looks like QXq, where $Q \in \mathrm{GL}(m, \mathcal{O}), q \in \mathrm{GL}(1, \mathcal{O})$, where the right action of $\mathrm{GL}(1, \mathcal{O})$ is blockwise. In this case the tensor fields in the form $F_1, \dots, F_k$ can be defined only locally. It is easy to see that the Lie algebra $\mathfrak{gl}(m, \mathcal{O})$ of a Lie group $\mathrm{GL}(m, \mathcal{O})$ is of the form

$$\mathfrak{gl}(m, \mathcal{O}) = \{A \in \mathfrak{gl}(km, \mathbb{R}) | AI_i = I_iA, AJ_j = J_jA\}$$

and the Lie algebra $\mathfrak{g}$ of a Lie group $\mathrm{GL}(m, \mathcal{O})\,\mathrm{GL}(1, \mathcal{O})$ is of the form $\mathfrak{g} = \mathfrak{gl}(m, \mathcal{O}) \oplus \mathfrak{gl}(1, \mathcal{O})$.

Let us note that the cases of $\mathcal{C}\ell(s,t)$, where $s + t = 2$, were studied in [6,7] and the case of $\mathcal{C}\ell(0,3)$ was studied in a detailed way in [2].

Corollary 3. *Let M be an almost Cliffordian manifold based on Clifford algebra $\mathcal{O} = \mathcal{C}\ell(s,t)$, where* $\dim(M) \geq 2(s + t)$*, i. e., a smooth manifold equipped with a G-structure, where $G = \mathrm{GL}(n, \mathcal{O})\,\mathrm{GL}(1, \mathcal{O})$ or equivalently an A-structure, where $A = \mathcal{O}$. Then the class of $\mathcal{D}$-connections preserves A and shares the same A-planar curves, which are isomorphic to $(\mathbb{R}^{km})^*$.*

4. CONCLUSION

From the classical theory, the Nijenhuis tensor is a part of a torsion of any almost complex connection ∇ and is J-antilinear in each argument ,

$$N_J(X,Y) = T_\nabla(X,Y) + JT_\nabla(JX,Y) + JT_\nabla(X,JY) - T_\nabla(JX,JY)$$

and the connection ∇ called minimal such that $N_J = 4T_\nabla$, i. e. the structure is integrable if and only if N_J vanishes. Let M be an almost quaternionic manifold or an almost quaternionic manifold of the second kind (paraquarernionic). In [6] and [7], the authors proved that the structure is integrable if and only if the structure tensor $T_Q = N(I,I) + N(J,J) + N(K,K)$ vanishes. In this case, there is the class of $\mathcal{D}$-connections without torsion.

A similar fact was proved for an almost Cliffordian manifold, where $\mathcal{C}\ell(0,3)$, see [3]. In this article, the autohors proved that the structure tensor which is locally generated by F_i is given by

$$T_Q = \sum_{i=1}^{6} N(F_i,F_i) + \sum_{i=1}^{6} N(F_i,F_i)\partial(\alpha_a \otimes F_a),$$

where ∂ denotes Spencer's operator of alternation. The following step is to find a description of the structure preserving connection based on the Nijenhuis tensor for any Cliffordian manifold.

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