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GEODESIC MAPPINGS OF (PSEUDO-) RIEMANNIAN MANIFOLDS PRESERVE CLASS OF DIFFERENTIABILITY

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Abstract. In this paper, we prove that geodesic mappings of (pseudo-) Riemannian manifolds preserve the class of differentiability ($C^r, r \geq 1$). Also, if the Einstein space V_n admits a non-trivial geodesic mapping onto a (pseudo-) Riemannian manifold $\bar{V}_n \in C^1$, then $\bar{V}_n$ is an Einstein space. If a four-dimensional Einstein space with non-constant curvature globally admits a geodesic mapping onto a (pseudo-) Riemannian manifold $\bar{V}_4 \in C^1$, then the mapping is affine and, moreover, if the scalar curvature is non-vanishing, then the mapping is homothetic, i. e. $\bar{g} = \text{const} \cdot g$.

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1. Introduction

The paper is devoted to the geodesic mapping theory of (pseudo-) Riemannian manifolds with respect to differentiability of their metrics. Most of the results in this area are formulated for "sufficiently" smooth, or analytic, geometric objects, as usual in differential geometry. It can be observed in most of the monographs and researches dedicated to the study of the theory of geodesic mappings and transformations, see [1, 3, 5–11, 13–19, 23–36].

Let $V_n = (M, g)$ and $\bar{V}_n = (\bar{M}, \bar{g})$ be (pseudo-) Riemannian manifolds, where M and $\bar{M}$ are n-dimensional manifolds with dimension $n \geq 2$, g and $\bar{g}$ are metrics. All the manifolds are assumed to be connected.

Definition 1. A diffeomorphism $f\colon V_n \to \bar{V}_n$ is called a *geodesic mapping* of V_n onto $\bar{V}_n$ if f maps any geodesic in V_n onto a geodesic in $\bar{V}_n$.

Hinterleitner and Mikeš [11] have proved the following theorem:

Theorem 1. *If the (pseudo-) Riemannian manifold V_n ($V_n \in C^r$, $r \geq 2$, $n \geq 2$) admits a geodesic mapping onto $\bar{V}_n \in C^2$, then $\bar{V}_n$ belongs to C^r.*

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Here and later, $V_n = (M, g) \in C^r$ means that $g \in C^r$, i. e., in a coordinate neighborhood (U, x) for the components of the metric g, $g_{ij}(x) \in C^r$ holds. If $V_n \in C^r$, then $M \in C^{r+1}$. This means that the atlas on the manifold M has the differentiability class C^{r+1}, i. e., for non-disjoint charts (U, x) and (U', x') on $U \cap U'$, it is true that the transformation $x' = x'(x) \in C^{r+1}$.

We suppose that the differentiability class r is equal to $0, 1, 2, \dots, \infty, \omega$, where $0, \infty$ and ω denote continuous, infinitely differentiable, and real analytic functions, respectively.

In the paper, we prove more general results. The following theorem holds:

Theorem 2. *If the (pseudo-) Riemannian manifold V_n ($V_n \in C^r$, $r \geq 1$, $n \geq 2$) admits a geodesic mapping onto $\bar{V}_n \in C^1$, then $\bar{V}_n$ belongs to C^r .*

Briefly, this means that *the geodesic mapping preserves the class of smoothness of the metric*.

Remark 1. It's easy to prove that the Theorems 1 and 2 are valid also for $r = \infty$ and for $r = \omega$. This follows from the theory of solvability of differential equations. Of course, we can apply this theorem only locally, because differentiability is a local property.

Remark 2. To require V_n, $\bar{V}_n \in C^1$ is a minimal requirement for geodesic mappings.

T. Levi-Civita [13] found metrics (*Levi-Civita metrics*) which admit geodesic mappings, see [1, 5], [25, p. 173], [27, p. 325]. From these metrics, we can easily see examples of non-trivial geodesic mappings $V_n \to \bar{V}_n$, where

- $V_n, \bar{V}_n \in C^r$ and $\notin C^{r+1}$ for $r \in \mathbb{N}$;
- $V_n, \bar{V}_n \in C^\infty$ and $\notin C^\omega$;
- $V_n, \bar{V}_n \in C^\omega$.

2. Geodesic mappings of Einstein manifolds

These results may be applied to geodesic mappings of Einstein manifolds V_n onto pseudo-Riemannian manifolds $\bar{V}_n \in C^1$.

Geodesic mappings of Einstein spaces have been studied by many authors starting by A. Z. Petrov (see [27]). Einstein spaces V_n are characterized by the condition $\mathrm{Ric} = \mathrm{const} \cdot g$.

An Einstein space V_3 is a space of constant curvature. It is known that Riemannian spaces of constant curvature form a closed class with respect to geodesic mappings (Beltrami theorem [5, 23, 25, 27, 29, 31]). In 1978 (see [15] and PhD. thesis [14], and see [16, 20, 22], [23, p. 125], [25, p. 188]), Mikeš proved that under the conditions $V_n, \bar{V}_n \in C^3$, the following theorem holds (locally):

Theorem 3. *If the Einstein space V_n admits a non-trivial geodesic mapping onto a (pseudo-) Riemannian manifold $\bar{V}_n$, then $\bar{V}_n$ is an Einstein space.*

Many properties of Einstein spaces appear when $V_n \in C^3$ and $n > 3$. Moreover, it is known (D. M. DeTurck and J. L. Kazdan [4], see [2, p. 145]), that Einstein space V_n belongs to C^ω, i. e., for all points of V_n a local coordinate system x exists, for which $g_{ij}(x) \in C^\omega$ (*analytic coordinate system*).

It implies global validity of Theorem 3 and, on the basis of Theorem 2, the following more general theorem holds:

Theorem 4. *If the Einstein space V_n admits a nontrivial geodesic mapping onto a (pseudo-) Riemannian manifold $\bar{V}_n \in C^1$, then $\bar{V}_n$ is an Einstein space.*

The present Theorem is true globally, because the function Ψ which determines the geodesic mapping is real analytic on an analytic coordinate system and so ψ $(= \nabla\Psi)$ is vanishing only on a point set of zero measure. This simplifies the proof given in [11].

Finally, based on the results (see [16,20–22], [23, p. 128], [25, p. 194]) for geodesic mappings of four-dimensional Einstein manifolds, the following theorem holds:

Theorem 5. *If a four-dimensional Einstein space V_4 with non-constant curvature globally admits a geodesic mapping onto a (pseudo-) Riemannian manifold $\bar{V}_4 \in C^1$, then the mapping is affine and, moreover, if the scalar curvature is non-vanishing, then the mapping is homothetic, i. e. $\bar{g} = \text{const} \cdot g$.*

3. Geodesic mapping theory for $V_n \to \bar{V}_n$ of class C^1

Let us briefly recall some main facts of geodesic mapping theory of (pseudo-) Riemannian manifolds which were found by T. Levi-Civita [13], L. P. Eisenhart [5,6] and N. S. Sinyukov [31], see [1, 9–11, 14, 16, 18, 19, 23, 25–32, 34–36]. In these results, no details about the smoothness class of the metric were stressed. They were formulated "for sufficiently smooth" geometric objects.

Since a geodesic mapping $f\colon V_n \to \bar{V}_n$ is a diffeomorphism, we can suppose $\bar{M} = M$. A (pseudo-) Riemannian manifold $V_n = (M, g)$ admits a geodesic mapping onto $\bar{V}_n = (M, \bar{g})$ if and only if the *Levi-Civita equations*

$$\bar{\nabla}_X Y = \nabla_X Y + \psi(X)Y + \psi(Y)X \tag{3.1}$$

hold for any tangent fields X, Y and where ψ is a differential form on M. Here, ∇ and $\bar{\nabla}$ are Levi-Civita connections of g and $\bar{g}$, respectively. If $\psi \equiv 0$, then f is *affine* or *trivially geodesic*.

Let (U, x) be a chart from the atlas on M. Then, equation (3.1) on U has the following local form: $\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_i \delta^h_j + \psi_j \delta^h_i$, where Γ^h_{ij} and $\bar{\Gamma}^h_{ij}$ are the Christoffel symbols of V_n and $\bar{V}_n$, ψ_i are components of ψ and δ^h_i is the Kronecker delta. Equations (3.1) are equivalent to the following Levi-Civita equations

$$\nabla_k \bar{g}_{ij} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi \bar{g}_{ik} \tag{3.2}$$

where $\bar{g}_{ij}$ are components of $\bar{g}$.

It is known that

$$\psi_i = \partial_i \Psi, \quad \Psi = \frac{1}{2(n+1)} \ln \left| \frac{\det \bar{g}}{\det g} \right|, \quad \partial_i = \frac{\partial}{\partial x^i}.$$

N.S. Sinyukov proved that the Levi-Civita equations (3.1) and (3.2) are equivalent to ([31, p. 121], [16], [23, p. 108], [25, p. 167], [29, p. 63]):

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik}, \tag{3.3}$$

where

$$\text{(a)}\ a_{ij} = \mathrm{e}^{2\Psi} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}; \quad \text{(b)}\ \lambda_i = -\mathrm{e}^{2\Psi} \bar{g}^{\alpha\beta} g_{\beta i} \psi_\alpha. \tag{3.4}$$

From (3.3) follows $\lambda_i = \partial_i(\frac{1}{2} a_{\alpha\beta} g^{\alpha\beta})$, $(g^{ij}) = (g_{ij})^{-1}$ and $(\bar{g}^{ij}) = (\bar{g}_{ij})^{-1}$.

On the other hand [29, p. 63]:

$$\bar{g}_{ij} = \mathrm{e}^{2\Psi} \hat{g}_{ij}, \quad \Psi = \frac{1}{2} \ln \left| \frac{\det \hat{g}}{\det g} \right|, \quad (\hat{g}_{ij}) = (g^{i\alpha} g^{j\beta} a_{\alpha\beta})^{-1}. \tag{3.5}$$

We can rewrite equations (3.3) and (3.4) in the following equivalent form (see [18], [25, p. 150]):

$$\nabla_k a^{ij} = \lambda^i \delta^j_k + \lambda^j \delta^i_k, \tag{3.6}$$

where

$$\text{(a)}\ a^{ij} = \mathrm{e}^{2\Psi} \bar{g}^{ij} \quad \text{and} \quad \text{(b)}\ \lambda^i = -\psi_\alpha a^{\alpha i}. \tag{3.7}$$

Evidently, it follows

$$\lambda^i = \frac{1}{2} g^{ik} \partial_k (a^{\alpha\beta} g_{\alpha\beta}). \tag{3.8}$$

The above formulas (3.1), (3.2), (3.3), (3.6), are the criterion for geodesic mappings $V_n \to \bar{V}_n$ globally as well as locally. These formulas are true only under the condition $V_n, \bar{V}_n \in C^1$.

4. Geodesic mapping theory for $V_n \in C^2 \to \bar{V}_n \in C^1$

In this section, we prove the main Theorem 2 from above. It is easy to see that Theorem 2 follows from Theorem 1 and the following theorem.

Theorem 6. *If $V_n \in C^2$ admits a geodesic mapping onto $\bar{V}_n \in C^1$, then $\bar{V}_n \in C^2$.*

Proof. Below, we prove Theorem 6.

4.1. We will suppose that the (pseudo-) Riemannian manifold $V_n \in C^2$ admits the geodesic mapping onto the (pseudo-) Riemannian manifold $\bar{V}_n \in C^1$. Furthermore, we can assume that $\bar{M} = M$.

We study the coordinate neighborhood (U, x) of any point $p = (0, 0, \dots, 0)$ at M. Evidently, components $g_{ij}(x) \in C^2$ and $\bar{g}_{ij}(x) \in C^1$ on $U \subset M$. On (U, x), formulas (3.1)–(3.8) hold. From that fact, it follows that the functions $g^{ij}(x) \in C^2$, $\bar{g}^{ij}(x) \in C^1$, $\Psi(x) \in C^1$, $\psi_i(x) \in C^0$, $a^{ij}(x) \in C^1$, $\lambda^i(x) \in C^0$, and $\Gamma^h_{ij}(x) \in C^1$, where $\Gamma^h_{ij} = \frac{1}{2} g^{hk}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij})$ are Christoffel symbols.

4.2. It is easy to see that in a neighborhood of the point p in $V_n \in C^r$ there exist a semigeodesic coordinate system (U, x) for which the metric $g \in C^r$ has the following form (see [5], [25, p. 64])

$$\mathrm{d}s^2 = e(\mathrm{d}x^1)^2 + g_{ab}(x^1, \dots, x^n)\mathrm{d}x^a\mathrm{d}x^b, \quad e = \pm 1, \quad a, b > 1. \tag{4.1}$$

Evidently, for $a > 1$:

$$g_{11} = g^{11} = e = \pm 1, \quad g_{1a} = g^{1a} = 0 \text{ and } \Gamma^1_{11} = \Gamma^1_{1a} = \Gamma^a_{11} = 0. \tag{4.2}$$

We can construct such a coordinate system using a coordinate transformation of class C^{r+1} for a basis of non-isotropic hypersurfaces $\Sigma \in C^{r+1}$ in a neighborhood of $p \in \Sigma$. Moreover, we can assume at p that

$$g_{ij}(0) = e_i\,\delta_{ij}; \quad e_i = \pm 1. \tag{4.3}$$

4.3. We write equations (3.6) in the following form

$$\partial_k a^{ij} = \lambda^i \delta^j_k + \lambda^j \delta^i_k - a^{i\alpha}\Gamma^j_{\alpha k} - a^{j\alpha}\Gamma^i_{\alpha k}. \tag{4.4}$$

Because $a^{ij} \in C^1$ and $\Gamma^j_{\alpha k} \in C^1$ from equation (4.4), we have the existence of the derivative immediately

$$\partial_{kl}a^{ii},\ \partial_{kk}a^{ii},\ \partial_{ki}a^{ii}(\equiv \partial_{ik}a^{ii}),\ \partial_{kl}a^{ij},\ \partial_{kk}a^{ij},\ \partial_{ki}a^{ij}(\equiv \partial_{ik}a^{ij}),$$

for each set of different indices i, j, k, l. Derivatives do not depend on the order because they are continuous functions.

We compute formula (4.4) for $i = j = k$ and for $i \neq j = k$:

$$\partial_i a^{ii} = 2\lambda^i - 2a^{i\alpha}\Gamma^i_{\alpha i} \text{ and } \partial_k a^{ik} = \lambda^i - a^{k\alpha}\Gamma^i_{\alpha k} - a^{i\alpha}\Gamma^k_{\alpha k}$$

where, for an index k, we do not carry out the Einstein summation and after eliminating λ^i, we obtain

$$\tfrac{1}{2}\,\partial_i a^{ii} - \partial_k a^{ik} = a^{k\alpha}\Gamma^i_{\alpha k} + a^{i\alpha}\Gamma^k_{\alpha k} - a^{i\alpha}\Gamma^i_{\alpha i} \tag{4.5}$$

Because there exists the partial derivative $\partial_{ik}a^{ii}$, formula (4.5) implies the existence of the partial derivatives $\partial_{kk}a^{ik}$.

4.4. In the semigeodesic coordinate system (4.1), we compute (4.4) for $i = j = k = 1$: $\lambda^1 = \frac{1}{2}\,\partial_1 a^{11}$, and from (3.8): $\lambda^1 = \frac{1}{2}\,\partial_1(a^{11} + ea^{\alpha\beta}g_{\alpha\beta})$, we obtain $\partial_1(a^{\alpha\beta}g_{\alpha\beta}) = 0$. Here and later $\alpha, \beta > 1$.

Further (4.4) for $i = j = 1$ and $k = 2$, we have the following expression $\partial_1 a^{12} + a^{1\gamma}\Gamma^2_{\gamma 1} + a^{2\gamma}\Gamma^1_{\gamma 1} = \lambda^2$. Using (3.8), we have

$$\partial_1 a^{12} = \tfrac{1}{2}\,g^{2\gamma}\cdot\partial_\gamma(a^{11} + a^{\alpha\beta}g_{\alpha\beta}) - a^{1\gamma}\Gamma^2_{\gamma 1}, \quad \gamma > 1,$$

and after integration, we obtain

$$a^{12} = \frac{1}{2}\left(\int_0^{x^1} g^{2\gamma}(\tau^1, x^2, \dots, x^n)d\tau^1\right)\cdot \partial_\gamma(a^{\alpha\beta}\cdot g_{\alpha\beta}) + \frac{1}{2}\int_0^{x^1} g^{2\gamma}(\tau^1, x^2, \dots, x^n)\cdot \partial_\gamma a^{11} d\tau^1 - \int_0^{x^1} a^{1\gamma}\Gamma^2_{\gamma 1} d\tau^1 + A(x^2, \dots, x^n). \quad (4.6)$$

As $a^{12}(0, x^2, \dots, x^n) \equiv A(x^2, \dots, x^n)$, the function $A \in C^1$.

After differentiating the formula (4.6) by x^2 and using the law of commutation of derivatives and integrals, see [12, p. 300], we can see that

$$\frac{\partial}{\partial x^2}\left\{\left(\int_0^{x^1} g^{2\gamma}(\tau^1, x^2, \dots, x^n)d\tau^1\right)\cdot \partial_\gamma(a^{\alpha\beta}\cdot g_{\alpha\beta})\right\} \quad (4.7)$$

exists. From (4.5) for $i = 2$ and $k = c \neq 2$, we obtain $\partial_c a^{c2} = \frac{1}{2}\partial_2 a^{22} + a^{c\delta}\Gamma^2_{\delta c} + a^{2\delta}\Gamma^c_{\delta c} - a^{2\delta}\Gamma^2_{\delta 2}$. Using this formula, we can rewrite the bracket (4.7) in the following form

$$\left\{\left(\int_0^{x^1} g^{2\gamma}(\tau^1, x^2, \dots, x^n)d\tau^1\right)\cdot g_{2\gamma}\ \cdot\ \partial_2 a^{22} + f\right\},$$

where f is the rest of this parenthesis, which is evidently differentiable by x^2.

Since the parenthesis and also the coefficients by $\partial_2 a^{22}$ are differentiable with respect to x^2, there exists $\partial_{22}a^{22}$ if

$$\left(\int_0^{x^1} g^{2\gamma}(\tau^1, x^2, \dots, x^n)d\tau^1\right)\cdot g_{2\gamma} \neq 0.$$

Using (3.3), this inequality is true for all x in a neighborhood of the point p excluding the point for which $x^1 = 0$.

For these reasons, in this domain, there exists the derivative $\partial_{22}a^{22}$ as well as all second derivatives a^{ij}. This follows from the derivative of the formula (4.5).

So, $a^{ij} \in C^2$ and $\lambda^i \in C^1$, from the formula (3.7b), it follows $\psi_i \in C^1$ and it means that $\Psi \in C^2$. From (3.7a) follows $\bar{g}^{ij} \in C^2$ and also $\bar{g}_{ij} \in C^2$. This is a proof of Theorem 6. □

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