NEW INTEGRAL INEQUALITIES INVOLVING BETA FUNCTION VIA $P$-CONVEXITY

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Abstract. In this note we establish some estimates, involving the Euler Beta function, of the integral

$$\int_a^b (x-a)^p (b-x)^q f(x) \, dx$$

for functions when a power of the absolute value is $P$-convex. An extension to functions of several variables is also obtained.

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1. INTRODUCTION

Let $I$ be an interval in $\mathbb{R}$. Then $f : I \rightarrow \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

The notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f : [a,b] \rightarrow \mathbb{R}$ is said to be quasi-convex on $[a,b]$ if

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}$$

holds for any $x, y \in [a,b]$ and $t \in [0,1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [11]).

The generalized quadrature formula of Gauss-Jacobi type has the form

$$\int_a^b (x-a)^p (b-x)^q f(x) \, dx = \sum_{k=0}^m B_{m,k} f(\gamma_k) + R_m[f]$$

(1.1)

for certain $B_{m,k}, \gamma_k$ and rest term $R_m[f]$ (see [22]).

In [17], Özdemir et al. established several integral inequalities concerning the left-hand side of (1.1) via some kinds of convexity. Especially, they discussed the following result connecting with quasi-convex function:
Theorem 1. Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) such that \( f \in L([a, b]) \), 0 < \( a < b < \infty \). If \( f \) is quasi-convex on \([a, b]\), then for some fixed \( p, q > 0 \), we have
\[
\int_a^b (x-a)^p (b-x)^q f(x) \, dx \leq (b-a)^{p+q+1} \beta(p+1, q+1) \max\{f(a), f(b)\},
\]
where \( \beta(x, y) \) is the Euler Beta function.

Recently, Liu [12] established some new integral inequalities for quasi-convex functions as follows:

Theorem 2. Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) such that \( f \in L([a, b]) \), 0 < \( a < b < \infty \) and let \( k > 1 \). If \( |f|^{\frac{k}{k-1}} \) is quasi-convex on \([a, b]\), for some fixed \( p, q > 0 \), then
\[
\int_a^b (x-a)^p (b-x)^q f(x) \, dx 
\leq (b-a)^{p+q+1} \left[ \beta(kp + 1, kq + 1) \right]^{\frac{1}{k}} \left( \max\{|f(a)|^{\frac{k}{k-1}}, |f(b)|^{\frac{k}{k-1}}\} \right)^{\frac{k-1}{k}}.
\]

Theorem 3. Let \( f : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) such that \( f \in L([a, b]) \), 0 < \( a < b < \infty \) and let \( l \geq 1 \). If \( |f|^l \) is quasi-convex on \([a, b]\), for some fixed \( p, q > 0 \), then
\[
\int_a^b (x-a)^p (b-x)^q f(x) \, dx 
\leq (b-a)^{p+q+1} \beta(p+1, q+1) \left( \max\{|f(a)|^l, |f(b)|^l\} \right)^{\frac{1}{l}}.
\]

On the other hand, Dragomir et al. in [6] defined the following class of functions of \( P \)-convex.

Definition 1. Let \( I \subseteq \mathbb{R} \) be an interval. The function \( f : I \to \mathbb{R} \) is said to belong to the class \( P(I) \) (or to be \( P \)-convex) if it is nonnegative and, for all \( x, y \in I \) and \( t \in [0, 1] \), satisfies the inequality
\[
f(tx + (1-t)y) \leq f(x) + f(y).
\]

Note that \( P(I) \) contain all nonnegative convex and quasiconvex functions. Since then numerous articles have appeared in the literature reflecting further applications in this category; see [1, 2, 4, 5, 7–10, 13–16, 18–21, 23–26] and references therein.

The main purpose of this note is to establish some new estimates of the integral \( \int_a^b (x-a)^p (b-x)^q f(x) \, dx \) for functions when a power of the absolute value is \( P \)-convex. An extension to functions of several variables is also obtained. That is, this study is a continuation and further generalization of [12, 17] via \( P \)-convexity.
2. NEW INTEGRAL INEQUALITIES VIA $P$-CONVEXITY

In this section we generalize Theorems 1-3 with a $P$-convex function setting. For this purpose, we need the following lemma (see [17, Lemma 2.1]):

**Lemma 1.** Let $f : [a, b] \subset [0, \infty) \to \mathbb{R}$ be continuous on $[a, b]$ such that $f \in L([a, b])$, $a < b$. Then the equality
\[
\int_a^b (x-a)^p (b-x)^q f(x) \, dx \leq (b-a)^{p+q+1} \int_0^1 (1-t)^p \beta(ta+(1-t)b) \, dt
\]
holds for some fixed $p, q > 0$.

The next theorem gives a new result for $P$-convex functions.

**Theorem 4.** Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ such that $f \in L([a, b])$, $0 \leq a < b < \infty$. If $|f|$ is $P$-convex on $[a, b]$, for some fixed $p, q > 0$, then
\[
\int_a^b (x-a)^p (b-x)^q f(x) \, dx \leq (b-a)^{p+q+1} \beta(p + 1, q + 1)(|f(a)| + |f(b)|),
\]
where $\beta(x, y)$ is the Euler Beta function.

**Proof.** By Lemma 1, the Beta function which is defined for $x, y > 0$ as
\[
\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt
\]
and the fact that $f$ is $P$-convex on $[a, b]$, we have
\[
\int_a^b (x-a)^p (b-x)^q f(x) \, dx \leq (b-a)^{p+q+1} \int_0^1 (1-t)^p \beta(ta+(1-t)b) \, dt
\]
\[
\leq (b-a)^{p+q+1} \int_0^1 (1-t)^p \beta(|f(a)| + |f(b)|) \, dt
\]
\[
= (b-a)^{p+q+1} \beta(q + 1, p + 1)(|f(a)| + |f(b)|),
\]
which completes the proof. \(\square\)

The corresponding version for powers of the absolute value is incorporated in the following result.

**Theorem 5.** Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ such that $f \in L([a, b])$, $0 \leq a < b < \infty$ and let $k > 1$. If $|f|^k$ is $P$-convex on $[a, b]$, for some fixed $p, q > 0$, then
\[
\int_a^b (x-a)^p (b-x)^q f(x) \, dx
\]
\[(b-a)^{p+q+1} \left[ \beta(kp + 1, kq + 1) \right]^{\frac{1}{p+1}} \left( |f(a)|^{\frac{k}{p+1}} + |f(b)|^{\frac{k}{p+1}} \right)^{\frac{k-1}{k}}. \quad (2.3) \]

**Proof.** By Lemma 1, Hölder’s inequality, the definition of Beta function and the fact that \(|f|^{\frac{k}{p+1}}\) is \(P\)-convex on \([a, b]\), we have

\[
\int_a^b (x-a)^p (b-x)^q f(x) \, dx 
\leq (b-a)^{p+q+1} \left[ \int_0^1 (1-t)^{kp+1} t^{kq} \, dt \right]^{\frac{1}{p+1}} \left[ \int_0^1 \left( |f(ta + (1-t)b)|^{\frac{k}{p+1}} \right) \, dt \right]^{\frac{k-1}{k}} 
\leq (b-a)^{p+q+1} \left[ \beta(kp + 1, kq + 1) \right]^{\frac{1}{p+1}} \left[ \int_0^1 \left( |f(a)|^{\frac{k}{p+1}} + |f(b)|^{\frac{k}{p+1}} \right) \, dt \right]^{\frac{k-1}{k}} 
\leq (b-a)^{p+q+1} \beta(p+1, q+1) \left( |f(a)|^l + |f(b)|^l \right)^{\frac{1}{l}}. \quad (2.4)
\]

which completes the proof. □

A more general inequality using Lemma 1 is as follows:

**Theorem 6.** Let \(f : [a, b] \to \mathbb{R}\) be continuous on \([a, b]\) such that \(f \in L([a, b])\), \(0 \leq a < b < \infty\) and let \(l > 1\). If \(|f|^l\) is \(P\)-convex on \([a, b]\), for some fixed \(p, q > 0\), then

\[
\int_a^b (x-a)^p (b-x)^q f(x) \, dx 
\leq (b-a)^{p+q+1} \beta(p+1, q+1) \left( |f(a)|^l + |f(b)|^l \right)^{\frac{1}{l}}. \quad (2.4)
\]

**Proof.** By Lemma 1, Hölder’s inequality, the definition of Beta function and the fact that \(|f|^l\) is \(P\)-convex on \([a, b]\), we have

\[
\int_a^b (x-a)^p (b-x)^q f(x) \, dx 
= (b-a)^{p+q+1} \int_0^1 \left[ (1-t)^{p+1} t^q \right]^{\frac{1}{p+1}} \left[ (1-t)^{p+1} t^q \right]^{\frac{1}{p+1}} \left( f(ta + (1-t)b) \right) \, dt 
\leq (b-a)^{p+q+1} \left[ \int_0^1 (1-t)^{p+1} t^q \, dt \right]^{\frac{1}{p+1}} \left[ \int_0^1 (1-t)^{p+1} t^q \left( |f(a)|^l + |f(b)|^l \right) \, dt \right]^{\frac{1}{p+1}} 
\leq (b-a)^{p+q+1} \beta(q+1, p+1) \left( |f(a)|^l + |f(b)|^l \right)^{\frac{1}{l}} 
\leq (b-a)^{p+q+1} \beta(p+1, q+1) \left( |f(a)|^l + |f(b)|^l \right)^{\frac{1}{l}},
\]

which completes the proof. □
3. AN EXTENSION TO FUNCTIONS OF SEVERAL VARIABLES

In this section some new integral inequalities for functions of several variables on convex subsets of \( \mathbb{R}^n \) will be given. First we recall the notion of \( P \)-convexity for functions on a convex subset \( U \) of \( \mathbb{R}^n \).

**Definition 2 ([3, Definition 3.1]).** The function \( f : U \rightarrow \mathbb{R} \) is said to be \( P \)-convex on \( U \) if it is nonnegative and, for all \( x, y \in U \) and \( \lambda \in [0, 1] \), satisfies the inequality
\[
f(\lambda x + (1 - \lambda)y) \leq f(x) + f(y).
\]

The following proposition will be used throughout this section.

**Proposition 1 ([3, Proposition 3.2]).** Let \( U \subseteq \mathbb{R} \) be a convex subset of \( \mathbb{R} \) and \( f : U \rightarrow \mathbb{R} \) be a function. Then \( f \) is \( P \)-convex on \( U \) if and only if, for every \( x, y \in U \), the function \( \varphi : [0, 1] \rightarrow \mathbb{R} \), defined by
\[
\varphi(t) := f((1-t)x + ty),
\]
is \( P \)-convex on \( I \) with \( I = [0, 1] \).

We have the following inequalities for functions of several variables on convex subsets of \( \mathbb{R}^n \).

**Theorem 7.** Let \( U \subseteq \mathbb{R} \) be a convex subset of \( \mathbb{R} \). Assume that \( f : U \rightarrow \mathbb{R}^+ \) is a \( P \)-convex function on \( U \). Then, for every \( x, y \in U \) and every \( [a, b] \in [0, 1] \) with \( a < b \), the following inequality holds:
\[
\int_a^b (t-a)^p (b-t)^q f((1-t)x + ty) dt \\
\leq (b-a)^{p+q+1} \beta (p+1,q+1)[f((1-a)x + ay) + f((1-b)x + by)].
\] (3.1)

**Proof.** Let \( x, y \in U \) and every \( [a, b] \in [0, 1] \) with \( a < b \). Since \( f : U \rightarrow \mathbb{R}^+ \) is a \( P \)-convex function, by Proposition 1 the function \( \varphi : [0, 1] \rightarrow \mathbb{R}^+ \) defined by
\[
\varphi(t) := f((1-t)x + ty),
\]
is \( P \)-convex on \( I \) with \( I = [0, 1] \). Applying Theorem 4 to the function \( \varphi \) implies that
\[
\int_a^b (t-a)^p (b-t)^q \varphi(t) dt \\
\leq (b-a)^{p+q+1} \beta (p+1,q+1)(|\varphi(a)| + |\varphi(b)|),
\]
and we deduce that (3.1) holds. \( \square \)

Similarly, we have
Theorem 8. Let $U \subseteq \mathbb{R}$ be a convex subset of $\mathbb{R}$ and let $k > 1$. Assume that $f^{k+1} : U \rightarrow \mathbb{R}^+$ is a $P$-convex function on $U$. Then, for every $x, y \in U$ and every $[a, b] \in [0, 1]$ with $a < b$, the following inequality holds:

$$
\int_a^b (t-a)^p (b-t)^q f((1-t)x + ty) dt \\
\leq (b-a)^{p+q+1} \left[ \beta(kp+1,kq+1) \right]^\frac{1}{k} \\
\left[ f^{\frac{k}{k+1}}((1-a)x + ay) + f^{\frac{k}{k+1}}((1-b)x + by) \right]^{\frac{k-1}{k}}.
$$

Theorem 9. Let $U \subseteq \mathbb{R}$ be a convex subset of $\mathbb{R}$ and let $l > 1$. Assume that $f^l : U \rightarrow \mathbb{R}^+$ is a $P$-convex function on $U$. Then, for every $x, y \in U$ and every $[a, b] \in [0, 1]$ with $a < b$, the following inequality holds:

$$
\int_a^b (t-a)^p (b-t)^q f((1-t)x + ty) dt \\
\leq (b-a)^{p+q+1} \beta(p+1,q+1) \left[ f^l((1-a)x + ay) + f^l((1-b)x + by) \right]^\frac{1}{l}.
$$

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References

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