GLOBAL OPTIMAL SOLUTIONS FOR NONCYCLIC MAPPINGS IN G-METRIC SPACES

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1. INTRODUCTION

In 2011, Abkar et al. [2] studied the existence of solutions of some specific minimization problems for noncyclic mappings in metric spaces. In 2006, Mustafa et al. [11] introduced the G–metric spaces as a generalization of the notion of metric spaces. Fixed point results and other results in G–metric spaces have been proved by a number of authors, see, e.g., [1,3–5,12,14,15]. In this paper we investigate some minimization problems for noncyclic mappings in G–metric spaces. This work extends results of Abkar et al. [2] to the case of G–metric spaces.

2. PRELIMINARIES

Throughout this paper, \( \mathbb{N} \) is the set of all natural numbers and \( \mathbb{R} \) is the set of all real numbers. Generalizations of the notion of a metric space have been proposed by Gabler [8, 9] and by Dhage [6, 7]. Mustafa et al. [11] introduced a more appropriate notion of a generalized metric space as following.

Definition 1. Let \( X \) be a nonempty set, and \( G : X \times X \times X \rightarrow \mathbb{R}^+ \) be a function satisfying the following conditions:

1. \( G(x, y, z) = 0 \) if \( x = y = z \),
2. \( 0 < G(x, x, y) \) for all \( x, y \in X \) with \( x \neq y \),

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(3) \( G(x, x, y) \leq G(x, y, z) \) for all \( x, y, z \in X \) with \( y \neq z \),
(4) \( G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots \),
(5) \( G(x, y, z) \leq G(x, w, w) + G(w, y, z) \) for all \( x, y, z, w \in X \).

The function \( G \) is called a generalized metric, or, a \( G \)-metric on \( X \), and the pair \((X, G)\) is called a \( G \)-metric space.

Example 1. ([11, Example 6.3]) Let \((X, d)\) be a metric space and define the functions \( G_s \) and \( G_m \) with

\[
G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z), \quad \forall x, y, z \in X
\]
\[
G_m(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\}, \quad \forall x, y, z \in X
\]

Then \((X, G_s)\) and \((X, G_m)\) are \( G \)-metric space.

Now, we recall some of the basic concepts for \( G \)-metric spaces from ([11]).

Definition 2. Let \((X, G)\) be a \( G \)-metric space, and \( \{x_n\} \) be a sequence of points of \( X \), we say that \( \{x_n\} \) is \( G \)-convergent to \( x \) and write \( x_n \xrightarrow{G} x \) if \( \lim_{n,m \to \infty} G(x, x_n, x_m) = 0 \), that is, for any \( \epsilon > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( G(x, x_n, x_m) < \epsilon \), for all \( n, m \geq n_0 \).

Proposition 1. Let \((X, G)\) be a \( G \)-metric space, then the following are equivalent.

1. \( \{x_n\} \) is \( G \)-convergent to \( x \).
2. \( \lim_{n \to \infty} G(x, x_n, x_n) = 0 \).
3. \( \lim_{n \to \infty} G(x, x, x_n) = 0 \).

Definition 3. Let \((X, G)\) be a \( G \)-metric space, a \( \{x_n\} \) is called \( G \)-Cauchy for any \( \epsilon > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( G(x_n, x_m, x_i) < \epsilon \), for all \( n, m, i \geq n_0 \) that is \( \lim_{n,m,i \to \infty} G(x_n, x_m, x_i) = 0 \).

Proposition 2. Let \((X, G)\) be a \( G \)-metric space, then the following are equivalent.

1. \( \{x_n\} \) is \( G \)-Cauchy.
2. For any \( \epsilon > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( G(x_n, x_m, x_m) < \epsilon \), for all \( n, m \geq n_0 \).

Definition 4. Let \((X_1, G_1)\) and \((X_2, G_2)\) be \( G \)-metric spaces. A function \( f : (X_1, G_1) \to (X_2, G_2)\) is \( G \)-continuous at a point \( a \in X \) if for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( x, y \in X_1, \ G_1(a, x, y) < \delta \) implies \( G_2(f(a), f(x), f(y)) < \epsilon \). A function \( f \) is \( G \)-continuous on \( X \) if and only if it is \( G \)-continuous at all \( a \in X \).

Proposition 3. Let \((X_1, G_1)\) and \((X_2, G_2)\) be \( G \)-metric spaces. A function \( f : (X_1, G_1) \to (X_2, G_2)\) is \( G \)-continuous at a point \( x \in X \) if and only if whenever \( \{x_n\} \) is \( G \)-convergent to \( x \), \( \{f(x_n)\} \) is \( G \)-convergent to \( f(x) \).

Definition 5. A \( G \)-metric space \((X, G)\) is said to be \( G \)-complete if every \( G \)-Cauchy sequence in \((X, G)\) is \( G \)-convergent in \((X, G)\).
Definition 6. Let \((X, G)\) be a \(G\)-metric space. A \(G\)-Ball with center \(x_0\) and radius \(r\) is
\[
B_G(x_0, r) = \{ x \in X : G(x_0, y, y) < r \}.
\]

Definition 7. Let \((X, G)\) be a \(G\)-metric space and \(\epsilon > 0\) be given, then a set \(A \subset X\) is called \(\epsilon\)-net of \((X, G)\) if given any \(x\) there is at least one point \(a \in A\) such that \(x \in B_G(a, \epsilon)\). If the \(A\) is finite then \(A\) is called a finite \(\epsilon\)-net of \((X, G)\). Note that if \(A\) is an \(\epsilon\)-net then \(X = \bigcup_{a \in A} B_G(a, \epsilon)\).

Definition 8. A \(G\)-metric space \((X, G)\) is called \(G\)-totally bounded if for every \(\epsilon > 0\) there exists a finite \(\epsilon\)-net.

Definition 9. A \(G\)-metric space \((X, G)\) is called \(G\)-compact if it is \(G\)-totally bounded.

Proposition 4. Let \((X, G)\) be a \(G\)-metric space, then the following are equivalent.
1. \((X, G)\) is a \(G\)-compact space.
2. \((X, G)\) is \(G\)-sequentially compact, that is, if the sequence \(\{x_n\} \subset X\) is such that \(\sup \{G(x_n, x_m, x_l) : n, m, l \in \mathbb{N}\} < \infty\), then \(\{x_n\}\) has a \(G\)-convergent subsequence.

Theorem 1 ([12], Theorem 2.1). Let \((X, G)\) be a \(G\)-metric space and \(T : X \to X\) be a mapping which satisfies the following condition, for all \(x, y, z \in X\),
\[
G(T(x), T(y), T(z)) \leq k \max\{G(x, y, z), G(x, T(x), T(x)), G(y, T(y), T(y)),
G(z, T(z), T(z)), G(x, T(y), T(y)),
G(y, T(z), T(z)), G(z, T(x), T(x))\},
\]
where \(k \in [0, 1/2]\). Then \(T\) has a unique fixed point (say \(u\)) and \(T\) is \(G\)-continuous at \(u\).

Definition 10. Let \(A, B, C\) be subsets of a \(G\)-metric space \((X, G)\). A mapping \(T : A \cup B \cup C \to A \cup B \cup C\) is called relatively \(G\)-nonexpansive if
\[
G(T(x), T(y), T(z)) \leq G(x, y, z), \quad \forall (x, y, z) \in A \times B \times C.
\]

Definition 11. Let \((X, G)\) be a \(G\)-metric space and \(A, B, C \subset X\), then
\[
dist(A, B, C) = \inf\{G(a, b, c) : a \in A, b \in B, c \in C\}.
\]

Example 2. Let \(R\) be equipped with the usual metric, and \(A = [-1, 0]\) and \(B = N_o\) and \(C = N_e\) where \(N_o\) and \(N_e\) are the set of odd natural numbers and even numbers, respectively. Let \(G_m(x, y, z) = \max\{|x - y|, |x - z|, |y - z|\}\), then \(dist(A, B, C) = 2\).

Definition 12. Let \((X, G)\) be a \(G\)-metric space and \(A, B, C \subset X\), \(T : A \cup B \cup C \to A \cup B \cup C\) is said noncyclic mapping, if
\[
T(A) \subset A, \quad T(B) \subset B, \quad T(C) \subset C.
\]
We consider the following minimization problem: Find
\[
\min_{a \in A} \{G(a, T(a), T(a))\}, \quad \min_{b \in B} \{G(b, T(b), T(b))\}, \\
\min_{b \in B} \{G(c, T(c), T(c))\}, \quad \min_{(a, b, c) \in A \times B \times C} \{G(a, b, c)\}
\] (2.2)
We say that \((x^*, y^*, z^*) \in A \times B \times C\) is a solution of above problem, if
\[
Tx^* = x^*, \quad Ty^* = y^*, \quad Tz^* = z^*.
\]
and
\[
G(x^*, y^*, z^*) = \text{dist}(A, B, C).
\]

**Definition 13.** Let \((X, G)\) be a \(G\)-metric space and \(A, B, C \subseteq X\), we set
\[
A_0 = \{a \in A : G(a, b, c) = \text{dist}(A, B, C), \text{ for some } b \in B, c \in C\}
\]
\[
B_0 = \{b \in B : G(a, b, c) = \text{dist}(A, B, C), \text{ for some } a \in A, c \in C\}
\]
\[
C_0 = \{c \in C : G(a, b, c) = \text{dist}(A, B, C), \text{ for some } a \in A, b \in B\}
\]

**Definition 14.** Let \((X, G)\) be a \(G\)-metric space and \(A, B, C\) be nonempty subsets of \(X\), with \(A_0 \neq \emptyset\). We say that \(A, B, C\) have \(P\)-property iff
\[
\left\{ \begin{array}{l}
G(x_1, y_1, z_1) = \text{dist}(A, B, C) \\
G(x_2, y_2, z_2) = \text{dist}(A, B, C) \\
G(x_3, y_3, z_3) = \text{dist}(A, B, C)
\end{array} \right.
\]
then
\[
G(x_1, x_2, x_3) = G(y_1, y_2, y_3) = G(z_1, z_2, z_3),
\]
where \(x_1, x_2, x_3 \in A_0\) and \(y_1, y_2, y_3 \in B_0\) and \(z_1, z_2, z_3 \in C_0\).

The above definition were found in the case of metric space in ([13]).

**Example 3.** Let \(A, B, C\) be nonempty subsets of a \(G\)-metric space \((X, G)\) such that \(A_0 \neq \emptyset\) and \(\text{dist}(A, B, C) = 0\), then \(A, B, C\) have \(P\)-property.

**Definition 15.** Let \((X, G)\) be a \(G\)-metric space and \(T : X \to X\) be a mapping. \(T\) is called expansive if for all \(x, y, z \in X\),
\[
G(T(x), T(y), T(z)) \geq G(x, y, z).
\]

**Definition 16.** Let \((X, G)\) be a \(G\)-metric space and \(T : X \to X\) be a mapping. \(T\) is said to be asymptotically regular iff \(\lim_{n \to \infty} G(T^n x, T^{n+1} x, T^{n+1} x) = 0\), for all \(x \in X\).
3. MAIN RESULTS

We start this section with the following theorem.

**Theorem 2.** Let $A, B, C$ be nonempty and closed subsets of a $G-$complete space $(X, G)$ such that $A_0 \neq \emptyset$ and $A, B, C$ satisfies the $P-$property. Let $T : A \cup B \cup C \rightarrow A \cup B \cup C$ be a noncyclic mapping. Suppose that

1. $T|_A$ be a mapping which satisfies in (2.1).
2. $T$ is relatively $G-$nonexpansive.

Then the minimization problem (2.2) has a solution.

**Proof.** If $x \in A_0$, then there exist $y \in B$ and $z \in C$ such that $G(x, y, z) = dist(A, B, C)$. Since $T$ is relatively $G-$nonexpansive then

$$G(T(x), T(y), T(z)) \leq G(x, y, z) = dist(A, B, C)$$

Hence $Tx \in A_0$.

Let $x_0 \in A_0$ by Theorem 1 if $x_n = T^n(x_0)$ then $x_n \xrightarrow{G} x^*$ where $x^*$ is unique fixed point of $T$ in $A$. Since $x_0 \in A_0$ there exist $y_0 \in B$ and $z_0 \in C$ such that $G(x_0, y_0, z_0) = dist(A, B, C)$. Since $x_1 = Tx_0 \in A_0$, there exist $y_1 \in B$ and $z_1 \in C$ such that $G(x_1, y_1, z_1) = dist(A, B, C)$. Using this process, we have a sequence $\{y_n\}$ in $B$ and $\{z_n\}$ in $C$ such that

$$G(x_n, y_n, z_n) = dist(A, B, C) \quad \forall n \in N \cup \{0\}.$$ 

Since $A, B, C$ have the $P-$property, we have for all $m, n, l \in N \cup \{0\}$

$$G(x_n, x_m, x_l) = G(y_n, y_m, y_l) = G(z_n, z_m, z_l).$$

This implies that $\{y_n\}$ and $\{z_n\}$ are $G-$Cauchy sequences, and there exist $y^* \in B$ and $z^* \in C$ such that $y_n \xrightarrow{G} y^*$ and $z_n \xrightarrow{G} z^*$. Thus

$$G(x^*, y^*, z^*) = \lim_{n \to \infty} G(x_n, y_n, z_n) = dist(A, B, C)$$

Since

$$G(T(x^*), T(y^*), T(z^*)) \leq G(x^*, y^*, z^*) = dist(A, B, C)$$

Therefore by the $P-$property, we have

$$G(x^*, T(x^*), T(x^*)) = G(y^*, T(y^*), T(y^*)) = G(z^*, T(z^*), T(z^*))$$

Thus $(x^*, y^*, z^*) \in A \cup B \cup C$ is a solution of the minimization problem (2.2). □

**Example 4.** Let $R$ be equipped with the usual metric, and $G_m(x, y, z) = \max\{|x-y|, |x-z|, |y-z|\}$. Let $A = [-2, 0]$ and $B = \{1\}$ and $C = [2, 3]$. It is obvious that
\[ A_0 = \{0\}, B_0 = \{1\}, C_0 = \{2\}. \]
Define \( T : A \cup B \cup C \rightarrow A \cup B \cup C \) with
\[
T(x) = \begin{cases} 
\frac{x}{4} & x \in A \\
1 & x \in B \\
\frac{x + 2}{2} & x \in C 
\end{cases}
\]
It is easy to check that all the conditions of Theorem 2 hold. Therefore, the minimization problem (2.2) has a solution \((x^*, y^*, z^*) = (0, 1, 2)\).

**Theorem 3.** Let \( A, B, C \) be nonempty subsets of a \( G \)-complete space \((X, G)\) such that \( A \) is \( G \)-compact and \( B \) and \( C \) are \( G \)-closed. Let \( A_0 \neq \emptyset \) and \( A, B, C \) satisfy the \( P \)-property. Let \( T : A \cup B \cup C \rightarrow A \cup B \cup C \) be a noncyclic mapping. Then the minimization problem (2.2) has a solution provided that the following conditions are satisfied:

1. \( T \) is relatively \( G \)-nonexpansive.
2. \( T|_A \) is a \( G \)-expansive.
3. \( T|_B \) and \( T|_C \) be mappings which satisfy in (2.1).

**Proof.** If \( x \in A_0 \), and \( x_{n+1} = Tx_n, (n \in N \cup \{0\}) \). By argument similar in the proof of Theorem 2 we obtain that \( T(A_0) \subset A_0 \) and there exist \( y_n \) in \( B \) and \( z_n \) in \( C \) such that
\[
G(x_n, y_n, z_n) = dist(A, B, C) \quad \forall n \in N \cup \{0\}.
\]
Since \( A \) is \( G \)-compact, by Proposition 4 there exist a subsequence \( \{x_{n_k}\} \) of the \( \{x_n\} \) such that \( x_{n_k} \xrightarrow{G} x^* \in A \). Since \( A, B, C \) satisfy the \( P \)-property,
\[
G(x_{n_k}, x_{n_k}, x_{n_k}) = G(y_{n_k}, y_{n_k}, y_{n_k}) = G(z_{n_k}, z_{n_k}, z_{n_k}), \quad (k, s, l \in N).
\]
This implies that \( \{y_n\} \) and \( \{z_n\} \) are \( G \)-Cauchy sequences and there exist \( y^* \in B \) and \( z^* \in C \) such that \( y_{n_k} \xrightarrow{G} y^* \) and \( z_{n_k} \xrightarrow{G} z^* \). Thus
\[
G(x^*, y^*, z^*) = \lim_{n \rightarrow \infty} G(x_{n_k}, y_{n_k}, z_{n_k}) = dist(A, B, C)
\]
Now we prove that \( x^*, y^*, z^* \in F(T) \). Since \( T \) is relatively \( G \)-nonexpansive,
\[
G(T^2(x^*), T^2(y^*), T^2(z^*)) = G(T(x^*), T(y^*), T(z^*)) = dist(A, B, C).
\]
Since \( A, B, C \) satisfy the \( P \)-property, we have
\[
G(x^*, T(x^*), T(x^*)) = G(y^*, T(y^*), T(y^*)) = G(z^*, T(z^*), T(z^*))
\]
and
\[
G(T(x^*), T^2(x^*), T^2(x^*)) = G(T(y^*), T^2(y^*), T^2(y^*)) = G(T(z^*), T^2(z^*), T^2(z^*)).
\]
Now let \( Ty^* \neq T^2y^* \), since \( T|_B \) satisfies in (2.1),
\[
G(T(y^*), T(T(y^*)), T(T(y^*))) \leq kG(y^*, T(y^*), T(y^*))
\]
Thus since \( T|_A \) is a \( G \)-expansive, we have
\[
G(T(y^*), T^2(y^*), T^2(y^*)) = G(T(y^*), T(T(y^*)), T(T(y^*))) \\
\leq kG(y^*, T(y^*), T(y^*)) \\
= kG(x^*, T(x^*), T(x^*)) \\
\leq kG(T(x^*), T^2(x^*), T^2(x^*)) \\
= kG(T(y^*), T^2(y^*), T^2(y^*)�,
\]
which is a contraction. Therefore \( Ty^* = T^2y^* \). A similar argument implies that \( Tz^* = T^2z^* \). Thus \( x^* = T(x^*) \) and \( y^* = T(y^*) \) and \( z^* = T(z^*) \).

Example 5. Let \( X = \mathbb{R}^3 \) and
\[
G((x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)) = \max\{G_m(x_1, x_2, x_3), \\
G_m(y_1, y_2, y_3), G_m(z_1, z_2, z_3)\},
\]
where \( G_m(x, y, z) = \max\{|x - y|, |x - z|, |y - z|\} \). Let \( A = \{(x, 0, 0) : -1 \leq x \leq 0\} \) and \( B = \{(0, y, 0) : 0 \leq y \leq 1\} \) and \( C = \{(0, 0, z) : -1 \leq z \leq 1\} \). It is obvious that \( A_0 = B_0 = C_0 = \{(0, 0, 0)\} \) and \( dist(A, B, C) = 0 \), therefore \( A, B, C \) have the \( P \)-property. Define \( T : A \cup B \cup C \to A \cup B \cup C \) with
\[
T(x, 0, 0) = (-x, 0, 0), \ T(0, y, 0) = (0, \frac{y}{4}, 0) \ and \ T(0, 0, z) = (0, 0, \frac{z}{4}).
\]
It is easy to check that all the conditions of Theorem 3 hold. Therefore the minimization problem (2.2) has a solution \( x^* = y^* = z^* = (0, 0, 0) \).

Theorem 4. Let \( A, B, C \) be nonempty subsets of a \( G \)-complete space \( (X, G) \) such that \( A \) is \( G \)-compact and \( B \) and \( C \) are \( G \)-closed. Let \( A_0 \neq \emptyset \) and \( A, B, C \) satisfy the \( P \)-property. Let \( T : A \cup B \cup C \to A \cup B \cup C \) be a noncyclic mapping. Then the minimization problem (2.2) has a solution provided that the following conditions are satisfied:

1. \( T \) is relatively \( G \)-nonexpansive.
2. \( T|_A \) is \( G \)-continuous and asymptotically regular.

Proof. Let \( \{x_n\}, \{y_n\}, \{z_n\}, \{x_{n_k}\}, \{y_{n_k}\}, \{z_{n_k}\}, x^*, y^* \) and \( z^* \) be as in Theorem 3. We have \( x_n \not\rightarrow x^* \in A, y_n \not\rightarrow y^* \in B, z_n \not\rightarrow z^* \in C \) and \( G(x^*, y^*, z^*) = dist(A, B, C) \). From Proposition 3, since \( T|_A \) is \( G \)-continuous, we have
\[
x_{n_k+1} = T(x_{n_k}) \not\rightarrow T(x^*).
\]
Also by the asymptotic regularity of \( T|_A \), we obtain
\[
G(x^*, T(x^*), T(x^*)) = \lim_{k \to \infty} G(x_{n_k}, T(x_{n_k}), T(x_{n_k})) \\
= \lim_{k \to \infty} G(T^{n_k}(x_0), T^{n_k+1}(x_0), T^{n_k+1}(x_0)) \\
= 0.
\]
This implies that \( T(x^*) = x^* \). Since \( T \) is relatively \( G \)--nonexpansive, we have

\[
G(T(x^*), T(y^*), T(z^*)) \leq dist(A, B, C)
\]

Therefore by the \( P \)--property, we have

\[
G(x^*, T(x^*), T(x^*)) = G(y^*, T(y^*), T(y^*)) = G(z^*, T(z^*), T(z^*))
\]

Hence \( T(y^*) = y^* \) and \( T(z^*) = z^* \). □

**QUESTION:** In 2011, Karapinar [10] obtain some common fixed point results in partial metric spaces. Can one study the minimization problem (2.2) for two mappings in partial metric spaces?

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