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2-ABSORBING AND n-WEAKLY PRIME SUBMODULES

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Abstract. Let R be a commutative ring with identity, and let $n > 1$ be an integer. A proper submodule N of an R-module M will be called 2-absorbing [resp. n-weakly prime], if $r, s \in R$ and $x \in M$ with $rsx \in N$ [resp. $rsx \in N \setminus (N : M)^{n-1}N$] implies that $rs \in (N : M)$ or $rx \in N$, or $sx \in N$. These concepts are generalizations of the notions of 2-absorbing ideals and weakly prime submodules, which have been studied in [3, 4, 6, 7]. We will study 2-absorbing and n-weakly prime submodules in this paper. Among other results, it is proved that if $(N : M)^{n-1}N \neq (N : M)^2N$, then N is 2-absorbing if and only if it is n-weakly prime.

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1. INTRODUCTION

Throughout this paper all rings are commutative with identity and all modules are unitary. Also we take R as a commutative ring with identity, M as an R-module, and $n > 1$ is a positive integer.

Let N be a submodule of M. The ideal $\{r \in R | rM \subseteq N\}$ is denoted by $(N : M)$.

It is said that a proper submodule N of M is *prime* if for $r \in R$ and $a \in M$ with $ra \in N$, either $a \in N$ or $r \in (N : M)$. If N is a prime submodule of M, then one can easily see that $P = (N : M)$ is a prime ideal of R, and we say N is a P-prime submodule. Prime submodules have been studied extensively in many papers (see, for example, [2], [4], [3]), so studying its generalization can be helpful in the amplification of this theory.

As a generalization of prime submodules, a proper submodule N of M is called *weakly prime*, if $r, s \in R$ and $x \in M$ with $rsx \in N$ implies that $rx \in N$ or $sx \in N$ (see [3, 4, 7]).

In this paper, we will introduce and study two generalizations of weakly prime submodules.

2. 2-ABSORBING SUBMODULES

According to [6] an ideal I of a ring R is called *2-absorbing*, if $abc \in I$ for $a, b, c \in I$ implies that $ab \in I$ or $bc \in I$ or $ac \in I$.

A generalization of weakly prime submodules, which is also a module version of 2-absorbing ideals, is introduced as follows:

Definition 1. A proper submodule N of M will be called 2-absorbing if for $r, s \in R$ and $x \in M$, $rsx \in N$ implies that $rs \in (N : M)$ or $rx \in N$ or $sx \in N$.

Lemma 1 (Theorem 2.1, Theorem 2.4, and Theorem 2.5 in [6])**.** *Let I be a 2-absorbing ideal of R with $\sqrt{I} = J$. Then*

(1) *J is a 2-absorbing ideal of R with $J^2 \subseteq I \subseteq J = \{r \in R \mid r^2 \in I\}$.*
(2) *$\{(I : r)\}_{r \in J \setminus I}$ is a chain of prime ideals.*
(3) *Either J is a prime ideal of R, or $J = P_1 \cap P_2$ with $P_1 P_2 \subseteq I$, where P_1, P_2 are the only distinct prime ideals of R, which are minimal over I.*

For each $r \in R$ and every submodule N of M, we consider $N_r = (N :_M r) = \{x \in M \mid rx \in N\}$.

Part (ii) of the following lemma proves that 2-absorbing submodules are not too far from prime submodules.

Proposition 1. *Let N be a 2-absorbing submodule of M with $\sqrt{(N : M)} = J$. Then*

(i) *$(N : M)$ and J are 2-absorbing ideals of R. Furthermore $J^2 \subseteq (N : M) \subseteq J = \{r \in R \mid r^2 \in (N : M)\}$.*
(ii) *If $(N : M) \neq J$, then for every $r \in J \setminus (N : M)$, N_r is a prime submodule containing N with $J \subseteq (N_r : M)$. Moreover $\{(N_r : M)\}_{r \in J \setminus (N:M)}$ is a chain of prime ideals.*
(iii) *Either J is a prime ideal of R, or $J = P_1 \cap P_2$, where P_1, P_2 are the only distinct minimal prime ideals over $(N : M)$ and $P_1 P_2 \subseteq (N : M)$.*

Proof. (i) Let $s, t, r \in R$ with $str \in (N : M)$. If $sr, tr \notin (N : M)$, then there exist $x, y \in M \setminus N$ such that $srx, try \notin N$.

Since $st(r(x + y)) \in N$ and N is 2-absorbing, $st \in (N : M)$ or $sr(x + y) \in N$ or $tr(x + y) \in N$. If $sr(x + y) \in N$, then since $srx \notin N$, we have $sry \notin N$. So as $st(ry) \in N$ and $try \notin N$, $st \in (N : M)$.

Similarly in case $tr(x + y) \in N$, we get $st \in (N : M)$.

Now since $(N : M)$ is a 2-absorbing ideal, by Lemma 1(1), J is also a 2-absorbing ideal with $J^2 \subseteq (N : M) \subseteq J = \{r \in R \mid r^2 \in (N : M)\}$.

(ii) To prove that N_r is a prime submodule, let $sx \in N_r$, where $s \in R \setminus (N_r : M)$ and $x \in M$. Then by the definition of N_r, $rsx \in N$ and as N is 2-absorbing, $rs \in (N : M)$ or $rx \in N$ or $sx \in N$.

If $rs \in (N : M)$, then $srM \subseteq N$, that is $s \in (N_r : M)$, which is a contradiction. If $rx \in N$, then $x \in N_r$ by the definition of N_r, which completes the proof.

Now suppose $sx \in N$. By part (i), $r^2 \in J^2 \subseteq (N : M)$, so $rM \subseteq N_r$, particularly $rx \in N_r$. Then $(r+s)x \in N_r$, that is $r(r+s)x \in N$, and since N is 2-absorbing, $rx \in N$ or $(r+s)x \in N$ or $r(r+s) \in N$.

If $rx \in N$, then $x \in N_r$, which completes the proof. Also if $(r+s)x \in N$, then from $sx \in N$, again we get $rx \in N$ and so $x \in N_r$.

Now assume $r(r+s) \in (N : M)$. According to part (i), $r^2 \in J^2 \subseteq (N : M)$, hence $rs \in (N : M)$, and so $s \in (N_r : M)$. Whence N_r is a prime submodule of M.

One can easily see that $((N : M) : r) = (N_r : M)$. By part (i), $rJ \subseteq J^2 \subseteq (N : M)$, so $J \subseteq ((N : M) : r) = (N_r : M)$.

For the proof of the rest of this part note that by part (i), $(N : M)$ is a 2-absorbing ideal. Hence by Lemma 1(2), $\{((N : M) : r)\}_{r \in J \setminus (N:M)}$ is a chain of prime ideals and $(N_r : M) = ((N : M) : r)$.

(iii) By part (i), $(N : M)$ is a 2-absorbing ideal, so the proof is clear by Lemma 1(3). □

Let S be a multiplicatively closed subset of R, and W a submodule of $S^{-1}M$ as $S^{-1}R$-module. We consider $W^c = \{x \in M \mid \frac{x}{1} \in W\}$.

The proof of the following lemma is easy and we leave it to the reader.

Lemma 2. *Let N be an 2-absorbing submodule of M, and S a multiplicatively closed subset of R.*

(i) *If $S^{-1}N \neq S^{-1}M$, then $S^{-1}N$ is a 2-absorbing submodule of $S^{-1}M$.*

(ii) *If W is a 2-absorbing submodule of a $S^{-1}R$-module $S^{-1}M$, then W^c is a 2-absorbing submodule of M.*

Lemma 3 (Proposition 1 in [9])**.** *Let S be a multiplicatively closed subset of R. If N is a P-prime submodule of M such that $(N : M) \cap S = \varnothing$, then $S^{-1}N$ is a prime submodule of $S^{-1}M$ as an $S^{-1}R$-module.*

Let N be a 2-absorbing submodule of M with $(N : M) \neq \sqrt{(N : M)}$. Then evidently $(N_r : M) = ((N : M) : r)$, so

according to Proposition 1(ii), $\mathfrak{P} = \cap\{((N : M) : r) \mid r \in \sqrt{(N : M)} \setminus (N : M)\}$ is a prime ideal. In this case we say $\mathfrak{P}$ is the prime ideal related to N.

Corollary 1. *Let N be a 2-absorbing submodule of M with $(N : M) \neq \sqrt{(N : M)}$ and $dim\ R < \infty$. Suppose S is a multiplicatively closed subset of R, and $\mathfrak{P}$ is the prime ideal related to N.*

(i) *If $S \cap \mathfrak{P} = \varnothing$, then $S^{-1}N$ is a 2-absorbing submodule of $S^{-1}M$.*

(ii) *$N_{\mathfrak{P}}$ is a 2-absorbing submodule of the $R_{\mathfrak{P}}$-module $M_{\mathfrak{P}}$.*

Proof. (i) By Lemma 2(i), it is enough to prove that $S^{-1}M \neq S^{-1}N$. According to Proposition 1(ii), $\{(N_r : M)\}_{r \in \sqrt{(N:M)} \setminus (N:M)}$ is a chain of prime ideals, and since $dim\ R < \infty$, this chain has a minimal element, say $(N_{r_0} : M)$. Now since $(N_r : M) = ((N : M) : r)$ for each $r \in \sqrt{(N : M)} \setminus (N : M)$, by our assumption we get

$S\cap(N_{r_0}:M)=S\cap\mathfrak{P}=\varnothing$. Now according to Proposition 1(ii) and Lemma 3, $S^{-1}N_{r_0}$ is a prime submodule of $S^{-1}M$ containing $S^{-1}N$. Hence $S^{-1}N\neq S^{-1}M$.

(ii) The proof is clear by part (i). □

Lemma 4. *Let N be an P-primary submodule of M. Then N is 2-absorbing if and only if $P^2\subseteq(N:M)$. In particular for every maximal submodule K of M, $(K:M)^2$ is a 2-absorbing ideal of R.*

Proof. If N is 2-absorbing, then by Proposition 1(i), $P^2\subseteq(N:M)$.

For the converse suppose that $rsx\in N$ for some $r,s\in R$ and $x\in M$. If $rx,sx\notin N$, then since N is P-primary, $r,s\in P$ and so $rs\in P^2\subseteq(N:M)$. Therefore N is 2-absorbing. □

Example 1. Let $\mathfrak{M}$ be a maximal ideal of R.

(a) Evidently, every weakly prime submodule is 2-absorbing. In particular if $\{P_i\}_{i\in\mathbb{N}}$ is a chain of prime ideals, then it is easy to see that for the free R-module $\oplus_{i\in\mathbb{N}}R$, the submodule $\oplus_{i\in\mathbb{N}}P_i$ is 2-absorbing.

(b) Let F be a faithfully flat R-module. Then $\mathfrak{M}F$ and $\mathfrak{M}^2F$ are 2-absorbing submodules, particularly if F is a free module, or a projective module over an integral domain.

(c) Let R be a Noetherian domain which is not a field. If F is a free R-module, then $\mathfrak{M}^kF$ is a primary submodule for $2<k\in\mathbb{N}$, but it is not 2-absorbing.

(d) Let R be a Dedekind domain domain which is not a field. If F is a free R-module, then $\mathfrak{M}^2F$ is a 2-absorbing submodule but it is not weakly prime.

(e) If R is a unique factorization domain and p is an irreducible element of R, then for the free R-module $R\oplus R$, the submodule $N=Rp\oplus Rp^2$ is 2-absorbing, but it is not weakly prime.

Proof. (a) The proof is easy , so it is omitted.

(b) Since F is faithfully flat, $\mathfrak{M}F$ and $\mathfrak{M}^2F$ are proper submodules of F. Clearly $\sqrt{(N:F)}=\mathfrak{M}$, where $N=\mathfrak{M}^kF$ for $k\in\mathbb{N}$. Then N is a primary submodule, since $\sqrt{(N:F)}$ is a maximal ideal. Evidently $\mathfrak{M}^2\subseteq(\mathfrak{M}^2F:F)$ and $\mathfrak{M}^2\subseteq(\mathfrak{M}F:F)$, so by Lemma 4, the submodules $\mathfrak{M}F$ and $\mathfrak{M}^2F$ are 2-absorbing.

(c) It is easy to see that in case F is a free module, $(IF:F)=I$ for each ideal I of R. As it was proved in part (b), $\mathfrak{M}^kF$ is a primary submodule. However, if $\mathfrak{M}^kF$ is 2-absorbing, then $\mathfrak{M}^2\subseteq(\mathfrak{M}^kF:F)=\mathfrak{M}^k\subseteq\mathfrak{M}^2$ according to Lemma 4. Thus $\mathfrak{M}^2=\mathfrak{M}^k$. Now by Nakayama's lemma, there exists $r\in R$ such that $r\mathfrak{M}^2=0$ and $r-1\in\mathfrak{M}^{k-2}$. Then either $r=0$, or $\mathfrak{M}=0$, and both are impossible.

(d) Note that for every weakly prime submodule N of a module M, the ideal $(N:M)$ is prime. Although $(\mathfrak{M}^2F:F)=\mathfrak{M}^2$ is not a prime ideal, consequently $\mathfrak{M}^2F$ is not weakly prime.

(e) A straightforward calculation shows that N is 2-absorbing. But N is not weakly prime, because $p.p(1,1)\in N$, however $p(1,1)\notin N$. □

Lemma 5 (Lemma 4 in [5]). *Let M be a finitely generated R-module and B a submodule of M. If $(B:M)\subseteq P$, where P is a prime ideal of R, then there exists a P-prime submodule N of M containing B.*

Let P be a prime ideal of R. For simplification, we denote the submodule $((P^2)_P M_P)^c$ of M by $P^{(2)}M$.

The following corollary supplies abundant examples of 2-absorbing submodules.

Corollary 2. *Let P be a prime ideal of R. If one of the following holds, then $P^{(2)}M$ is 2-absorbing.*

(i) $(P^2)_P M_P \neq M_P$.

(ii) *M is finitely generated and $ann(M)\subseteq P$.*

Proof. (i) Evidently $(P^2)_P \subseteq ((P^2)_P M_P : M_P)$, so $P_P \subseteq \sqrt{((P^2)_P M_P : M_P)}$, and since P_P is a maximal ideal, $\sqrt{((P^2)_P M_P : M_P)} = P_P$. Therefore $(P^2)_P M_P$ is a P_P-primary submodule of M_P. Then clearly $P^{(2)}M$ is a P-primary submodule of M. Now the proof is given by Lemma 4, as $P^2 \subseteq (P^{(2)}M : M)$.

(ii) By part (i), it is enough to prove that $(P^2)_P M_P \neq M_P$.

According to Lemma 5, there exists a P-prime submodule N of M. Then by Lemma 3, N_P is a P_P-prime submodule of M_P. Now from $P_P M_P \subseteq N_P$, we get $(P^2)_P M_P \subseteq N_P$. Consequently $(P^2)_P M_P \neq M_P$. □

In the following, if

$$\mathcal{A} = \{N \mid N \text{ is a } P\text{-primary and 2-absorbing submodule of } M\} = \varnothing,$$

then we consider $\bigcap \mathcal{A} = M$.

Corollary 3. *If P is a prime ideal of R, then*

$$P^{(2)}M = \bigcap \{N \mid N \text{ is a } P\text{-primary and 2-absorbing submodule of } M\}.$$

Proof. Set $\mathcal{A} = \{N \mid N \text{ is a } P\text{-primary and 2-absorbing submodule of } M\}$.

If $P^{(2)}M = M$, then $\mathcal{A} = \varnothing$, because if N is a P-primary and 2-absorbing submodule of M, by Lemma 4, $P^2M \subseteq N$. Therefore $M = P^{(2)}M \subseteq (N_P)^c = N$, which is impossible. Hence $\mathcal{A} = \varnothing$, and so in this case $\bigcap \mathcal{A} = M = P^{(2)}M$.

Now let $P^{(2)}M \neq M$. By Corollary 2(i), $P^{(2)}M$ is 2-absorbing. Also in the proof of Corollary 2(i), we showed that $P^{(2)}M$ is P-primary, so $P^{(2)}M \in \mathcal{A}$. Consequently $\bigcap \mathcal{A} \subseteq P^{(2)}M$.

Now suppose that N' is a P-primary and 2-absorbing submodule of M. Then Lemma 4 implies that $P^{(2)}M \subseteq (N'_P)^c = N'$. Consequently $P^{(2)}M = \bigcap \mathcal{A}$. □

A prime ideal P of R is said to be a divided prime ideal if $P \subseteq Rr$ for every $r \in R \setminus P$.

We consider $T(M) = \{m \in M \mid \exists 0 \neq r \in R,\ rm = 0\}$. If M is a nonzero module with $T(M) = 0$, then it is easy to see that R is an integral domain, and in this case we say M is a torsion-free module.

Theorem 1. *Let M be a nonzero finitely generated module and P a divided prime ideal. If $T(M) \subseteq P^2M$, then P^2M is 2-absorbing and*

$$P^2M = \bigcap\{N \mid N \text{ is a } P\text{-primary and 2-absorbing submodule of } M\},$$

particularly if M is a torsion-free module.

Proof. First we show that P^2M is a proper submodule of M. If $P^2M = M$, then by Nakayama's lemma, there exists $a \in R$ such that $1-a \in P^2$ and $aM = 0$. Since $1-a \in P$, $a \notin P$ and as P is a divided prime ideal, $1-a \in P \subseteq Ra$. Thus there exists $t \in R$ with $1-a = ta$. Therefore $M = (1-a)M = taM = 0$, which is impossible.

Now by Corollary 3 and Lemma 4, it suffices to show that P^2M is P-primary. Suppose that $rx = s_1t_1y_1 + \cdots + s_nt_ny_n \in P^2M$, where $s_i, t_i \in P$, $y_i, x \in M$, and $r \in R$. If $r \notin P$, then since P is a divided prime, $P \subseteq Rr$, and hence there exist $r_1, \ldots, r_n \in R$ such that $s_i = rr_i \in P$, for $i = 1, \ldots, n$. Thus for each i, $r_i \in P$ and $r(r_1t_1y_1 + \cdots + r_nt_ny_n) = rx \in P^2M$. Hence as $x - (r_1t_1y_1 + \cdots + r_nt_ny_n) \in T(M) \subseteq P^2M$, and $r_1t_1y_1 + \cdots + r_nt_ny_n \in P^2M$, we have $x \in P^2M$, which completes the proof. □

According to [1] an ideal I of R is called an *n-almost prime* ideal if for $a, b \in R$ with $ab \in I \setminus I^n$, either $a \in I$ or $b \in I$. The case $n = 2$ is called an *almost prime* ideal and it is due to [8].

Theorem 2. *Let R be a Noetherian domain, which is not a field. Then the following are equivalent.*

(i) *R is Dedekind domain.*
(ii) *If I is a 2-absorbing ideal of R, then I is almost prime or $I = P_1 \cap P_2$ or $I = P^2$, where P, P_1, P_2 are prime ideals of R.*

Proof. (i) $\Rightarrow$ (ii) The proof is given by [6, Theorem 3.14].

(ii) $\Rightarrow$ (i) We prove that every localization of R at any nonzero prime ideal has the property introduced in (ii).

Let J be a 2-absorbing ideal of $R_{\mathfrak{P}}$, where $\mathfrak{P}$ is a nonzero prime ideal of R. By Lemma 2, J^c is a 2-absorbing ideal of R, and hence by our assumption, J^c is almost prime or $J^c = P_1 \cap P_2$ or $J^c = P^2$, for some prime ideals P, P_1, P_2 of R.

By [10, Proposition 2.10(ii)], the localization of an almost prime ideal is almost prime if it is a proper ideal. Hence if J^c is an almost prime ideal, then $(J^c)_{\mathfrak{P}} = J \neq R$, and so J is an almost prime ideal of $R_{\mathfrak{P}}$.

If $J^c = P_1 \cap P_2$, then $J = (J^c)_{\mathfrak{P}} = (P_1)_{\mathfrak{P}} \cap (P_2)_{\mathfrak{P}}$, and since J is a proper ideal, at least one of $(P_1)_{\mathfrak{P}}$ or $(P_2)_{\mathfrak{P}}$ is a prime ideal. So in this case either J is a prime ideal or the intersection of two prime ideals.

In case $J^c = P^2$, then $J = (J^c)_{\mathfrak{P}} = (P_{\mathfrak{P}})^2$, and as J is proper, the ideal $(P)_{\mathfrak{P}}$ is prime.

Therefore by considering the localization of R, we may suppose that $\mathfrak{M}$ is the only maximal ideal of R. If $\mathfrak{M} = \mathfrak{M}^2$, then by Nakayama's lemma, $\mathfrak{M} = 0$, that is R is a field. Now let $s \in \mathfrak{M} \setminus \mathfrak{M}^2$, and set $I = \mathfrak{M}^2 + Rs$.

First we prove that every ideal K with $\mathfrak{M}^2 \subset K$ is almost prime. $(*)$

Evidently $\sqrt{K} = \mathfrak{M}$, and so K is a primary ideal with $\mathfrak{M}^2 \subseteq K$. So by Lemma 4, K is 2-absorbing and the hypothesis in (ii) implies that K is almost prime, or $K = P_1 \cap P_2$ or $K = P^2$, where P, P_1, P_2 are prime ideals of R. If $K = P^2$, then $\mathfrak{M}^2 \subseteq K = P^2$, and so $\mathfrak{M} = P$. Thus $K = \mathfrak{M}^2$, which is impossible. If $K = P_1 \cap P_2$, then $\mathfrak{M}^2 \subseteq P_1$ and $\mathfrak{M}^2 \subseteq P_2$ and so $P_1 = P_2 = \mathfrak{M}$, that is in this case $K = \mathfrak{M}$, so evidently K is (almost) prime.

By $(*)$ in above, I is an almost prime ideal. We will prove that $I^2 = \mathfrak{M}^2$. On the contrary let $a, b \in \mathfrak{M}$ such that $ab \notin I^2$. Thus $ab \in I \setminus I^2$, and since I is almost prime, we have $a \in I$ or $b \in I$ and not both, as $ab \notin I^2$, then suppose $a \in I$ and $b \notin I$. Note that $b^2 \in \mathfrak{M}^2 \subseteq I$. Hence $b(a+b) \in I$. If $b(a+b) \notin I^2$, then $b \in I$ or $a+b \in I$, which is impossible. Hence $b(a+b) \in I^2$, and $ab \notin I^2$, therefore $b^2 \notin I^2$. Then $b^2 \in I \setminus I^2$, and so $b \in I$, which is a contradiction.

Consequently $\mathfrak{M}^2 = I^2 = \mathfrak{M}^4 + \mathfrak{M}^2 s + Rs^2 = \mathfrak{M}^2(\mathfrak{M}^2 + Rs) + Rs^2$. Hence by Nakayama's lemma $\mathfrak{M}^2 = Rs^2 \subseteq Rs$, and as $s \notin \mathfrak{M}^2$, we have $\mathfrak{M}^2 \subset Rs$. Thus again by $(*)$, Rs is almost prime. By [8, Lemma 2.6], every principal and almost prime ideal is a prime ideal, hence Rs is a prime ideal. Now since $\mathfrak{M}^2 \subseteq Rs$, $\mathfrak{M} = Rs$, that is $\mathfrak{M}$ is a principal ideal. Therefore R is a discrete valuation domain, in case R is local.

Now for the general case, note that every localization of R is a discrete valuation domain, hence R is a Dedekind domain. □

3. n-WEAKLY PRIME SUBMODULES

Another generalization of weakly prime submodules is introduced in the following. The following definition is also a generalization and a module version of n-almost prime ideals which was introduced and studied in [1].

Definition 2. Let $n > 1$ be an integer. A proper submodule N of M will be called n-weakly prime, if for $r, s \in R$ and $x \in M$, $rsx \in N \setminus (N : M)^{n-1} N$ implies that $rs \in (N : M)$ or $rx \in N$ or $sx \in N$.

If we consider R as an R-module, then evidently a proper ideal I of R is n-weakly prime if for $a, b, c \in R$, $abc \in I \setminus I^n$ implies that $ab \in I$ or $bc \in I$ or $ac \in I$.

Remark 1. For any submodule, we have the following implications:

(1) $Prime \Longrightarrow weakly\ prime \Longrightarrow 2-absorbing \Longrightarrow$n$-weakly\ prime.$

(2) n-weakly prime $\Longrightarrow$ $(n-1)$-weakly prime, for each $n > 2$.

Evidently the zero submodule is n-weakly prime, but it is not necessarily 2-absorbing. The following example introduces non trivial n-weakly prime submodules, which are not 2-absorbing.

Example 2. Let $R = \frac{K[X_1,X_2,X_3,X_4]}{\langle X_1{}^2,X_2{}^2,X_3{}^2,X_4{}^2,X_1X_2X_3,X_1X_2X_4,X_1X_3X_4,X_2X_3X_4\rangle}$, where K is a field of characteristic 2 and X_1, X_2, X_3, X_4 are independent indeterminates. Consider $M = R \oplus R$ and $I = \langle \bar{X_1}\bar{X_2} + \bar{X_3}\bar{X_4}\rangle$. Then the two submodules $N = \{(x,x) \mid x \in I\}$ and $N' = I \oplus I$ are n-weakly prime, but they are not 2-absorbing.

Proof. Evidently $(R,\mathfrak{M})$ is a local ring with $\mathfrak{M}^3 = 0$, where $\mathfrak{M} = \langle \bar{X_1}, \bar{X_2}, \bar{X_3}, \bar{X_4}\rangle$. First we prove that $\bar{X_1}\bar{X_2} + \bar{X_3}\bar{X_4}$ is irreducible.

Suppose $fg = \bar{X_1}\bar{X_2} + \bar{X_3}\bar{X_4} \quad (*)$, with f, g non unit. Note that $\mathfrak{M}^3 = 0$, then we can consider $f = a_1\bar{X_1} + a_2\bar{X_2} + a_3\bar{X_3} + a_4\bar{X_4} \in \mathfrak{M}$, and $g = b_1\bar{X_1} + b_2\bar{X_2} + b_3\bar{X_3} + b_4\bar{X_4} \in \mathfrak{M}$, where $a_i, b_i \in K$. From $(*)$ we get:

$$\begin{array}{lll} (1)\ a_1b_2 + a_2b_1 = 1 & (2)\ a_1b_3 + a_3b_1 = \circ & (3)\ a_1b_4 + a_4b_1 = \circ \\ (4)\ a_2b_3 + a_3b_2 = \circ & (5)\ a_2b_4 + a_4b_2 = \circ & (6)\ a_3b_4 + a_4b_3 = 1 \end{array}$$

By (2), $\circ = a_1b_4(a_1b_3 + a_3b_1)$ and by (3), $\circ = a_1b_3(a_1b_4 + a_4b_1)$ and so $a_1b_1(a_3b_4 - a_4b_3) = \circ$. Since the characteristic of K is 2, $-a_4b_3 = a_4b_3$ and so $a_1b_1(a_3b_4 + a_4b_3) = \circ$. Hence by (6), $a_1b_1 = \circ$. Then $a_1 = \circ$ or $b_1 = \circ$. The case $a_1 = b_1 = \circ$ is impossible, by (1). If $\circ = a_1$ and $\circ \neq b_1$, then (2) and (3) imply that $a_3 = \circ = a_4$ and this is a contradiction by (6).

In case $\circ \neq a_1$ and $\circ = b_1$, then by (2), (3) we get $b_3 = \circ = b_4$, which is a again impossible, according to (6). Consequently $\bar{X_1}\bar{X_2} + \bar{X_3}\bar{X_4}$ is irreducible.

One can easily see that $(N : M) = 0$, and so $(N : M)^{n-1}N = 0$. Also it is easy to see that $I \subseteq \mathfrak{M}^2$ and $(N' : M) = I$. Then $I^2 \subseteq \mathfrak{M}^4 = 0$, and thus $(N' : M)^{n-1}N' = 0$.

To show that N is n-weakly prime, let $(\circ,\circ) \neq rs(a,b) \in N$, where $r, s \in R$ and $(a,b) \in M$. We can assume $\circ \neq rsa \in I$. Then for some $h \in R$, $\circ \neq rsa = h(\bar{X_1}\bar{X_2} + \bar{X_3}\bar{X_4})$. But since $I\mathfrak{M} \subseteq \mathfrak{M}^3 = 0$, $h \in R \setminus \mathfrak{M}$. Thus h is unit and so $rsah^{-1} = \bar{X_1}\bar{X_2} + \bar{X_3}\bar{X_4}$ and it is irreducible, therefore r or sa is unit. Hence r or s is unit and so $s(a,b) \in r^{-1}N = N$ or $r(a,b) \in s^{-1}N = N$. This show that N is n-weakly prime. The same argument proves that N' is n-weakly prime.

Now if on the contrary N is a 2−absorbing submodule, then again by Proposition 1(i), $(N : M) = 0$ must be a 2−absorbing ideal and as $0 = \mathfrak{M}^3 \subseteq (N : M)$, we will have $\mathfrak{M}^2 \subseteq (N : M) = 0$, which is impossible. Thus N is not a 2-absorbing submodule.

If N' is a 2−absorbing submodule, then by Proposition 1(i), $(N' : M) = I$ is a 2−absorbing ideal of R and since $0 = \mathfrak{M}^3 \subseteq I$, then $\mathfrak{M}^2 \subseteq I$. Consequently $\bar{X_1}\bar{X_2} \in \mathfrak{M}^2 \subseteq I$. Then for some $h' \in R$, $\circ \neq \bar{X_1}\bar{X_2} = h'(\bar{X_1}\bar{X_2} + \bar{X_3}\bar{X_4})$. As $\mathfrak{M}^3 = 0$, h' is unit and since $\bar{X_1}\bar{X_2} + \bar{X_3}\bar{X_4}$ is irreducible, $\bar{X_1}$ or $\bar{X_2}$ is unit, which is impossible. □

Evidently $(N : M)^{n-1}N \subseteq (N : M)^2N$, for each submodule N of M for each $n > 2$. We now introduce a simple criteria for an n-weakly prime submodule to be 2-absorbing.

Theorem 3. *Let N be a submodule of M with $(N:M)^2N \not\subseteq (N:M)^{n-1}N$. Then N is 2-absorbing if and only if it is n-weakly prime.*

Proof. Let N be an n-weakly prime submodule. Suppose $rsx \in N$, where $r,s \in R$ and $x \in M$. If $rx, sx \notin N$ and $rs \notin (N:M)$, then we prove that $(N:M)^2N \subseteq (N:M)^{n-1}N$, which is impossible and so N is 2-absorbing.

First we show that the following facts hold:

(i) $rsx \in (N:M)^{n-1}N$.

(ii) $rsN \subseteq (N:M)^{n-1}N$.

(iii) $r(N:M)x, s(N:M)x \subseteq (N:M)^{n-1}N$.

(iv) $(N:M)^2x \subseteq (N:M)^{n-1}N$.

(v) $r(N:M)N, s(N:M)N \subseteq (N:M)^{n-1}N$.

(i) Since N is n-weakly prime and $rx, sx \notin N$ and $rs \notin (N:M)$, then $rsx \in (N:M)^{n-1}N$.

(ii) If $rsN \not\subseteq (N:M)^{n-1}N$, then for some $y \in N$ we have $rsy \notin (N:M)^{n-1}N$. So since $rsx \in (N:M)^{n-1}N$, $rs(x+y) \notin (N:M)^{n-1}N$. Hence $rs(x+y) \in N \setminus (N:M)^{n-1}N$ and then $r(x+y) \in N$ or $s(x+y) \in N$ or $rs \in (N:M)$. Thus $rx \in N$ or $sx \in N$ or $rs \in (N:M)$, which is impossible. Consequently $rsN \subseteq (N:M)^{n-1}N$.

(iii) Let $r(N:M)x \not\subseteq (N:M)^{n-1}N$. Then there exists $t \in (N:M)$ such that $rtx \in N \setminus (N:M)^{n-1}N$. Clearly $r(s+t)x \in N$. We have $r(s+t)x \notin (N:M)^{n-1}N$, otherwise since $rsx \in (N:M)^{n-1}N$, $rtx \in (N:M)^{n-1}N$, which is a contradiction. Then $r(s+t)x \in N \setminus (N:M)^{n-1}N$ and hence $rx \in N$ or $(s+t)x \in N$ or $r(s+t) \in (N:M)$, which implies $rx \in N$ or $sx \in N$ or $rs \in (N:M)$, a contradiction to our assumption. Therefore $r(N:M)x \subseteq (N:M)^{n-1}N$. Similarly $s(N:M)x \subseteq (N:M)^{n-1}N$.

(iv) Let $a,b \in (N:M)$. If $abx \notin (N:M)^{n-1}N$, then since $rsx \in N$, $(a+r)(b+s)x \in N$. we show that $(a+r)(b+s)x \notin (N:M)^{n-1}N$.

If $(a+r)(b+s)x \in (N:M)^{n-1}N$, then $rsx+rbx+asx+abx \in (N:M)^{n-1}N$, and so by parts (i), (iii), $rsx+rbx+asx \in (N:M)^{n-1}N$. Hence $abx \in (N:M)^{n-1}N$, which is impossible. Thus $(a+r)(b+s)x \notin (N:M)^{n-1}N$. Therefore $(a+r)(b+s)x \in N \setminus (N:M)^{n-1}N$ and so $(a+r)x \in N$ or $(b+s)x \in N$ or $(a+r)(b+s) \in (N:M)$, which implies $rx \in N$ or $sx \in N$ or $rs \in (N:M)$, and this is a contradiction. Then $abx \in (N:M)^{n-1}N$ and so $(N:M)^2x \subseteq (N:M)^{n-1}N$.

(v) If for some $b \in (N:M)$ and $y \in N$, $rby \notin (N:M)^{n-1}N$, then $r(s+b)(x+y) \in N$. By parts (i),(ii),(iii), $rsx+rsy+rbx \in (N:M)^{n-1}N$ and since $rby \notin (N:M)^{n-1}N$, then $r(s+b)(x+y) \notin (N:M)^{n-1}N$. Hence $r(x+y) \in N$ or $(s+b)(x+y) \in N$ or $r(s+b) \in (N:M)$. Then $rx \in N$ or $sx \in N$ or $rs \in (N:M)$, which is a contradiction. Consequently $r(N:M)N \subseteq (N:M)^{n-1}N$ and similarity $s(N:M)N \subseteq (N:M)^{n-1}N$.

Now we prove the theorem. Let $a,b \in (N:M)$ and $y \in N$. If $aby \notin (N:M)^{n-1}N$, then obviously $(a+r)(b+s)(x+y) \in N$. If $(a+r)(b+s)(x+y) \in (N:M)^{n-1}N$, then by previous parts $aby = (a+r)(b+s)(x+y) - (abx+asx+asy+rbx+$

$rby + rsx + rsy) \in (N : M)^{n-1}N$, which is impossible. Thus $(a+r)(b+s)(x+y) \notin (N : M)^{n-1}N$ and so $(a+r)(b+s)(x+y) \in N \setminus (N : M)^{n-1}N$. Hence $(a+r)(x+y) \in N$ or $(b+s)(x+y) \in N$ or $(a+r)(b+s) \in (N : M)$. Therefore $rx \in N$ or $sx \in N$ or $rs \in (N : M)$, which is impossible. Consequently $(N : M)^2N \subseteq (N : M)^{n-1}N$. □

Corollary 4. *Let $n > 3$ and M be a nonzero torsion-free Noetherian R-module. Then a submodule is 2-absorbing if and only if it is n-weakly prime.*

Proof. Let N be an n-weakly prime submodule. By Theorem 3, it is enough to prove that $(N : M)^{n-1}N \neq (N : M)^2N$. On the contrary suppose that $(N : M)^{n-1}N = (N : M)^2N$. Then by Nakayama's lemma there exists $a \in (N : M)^{n-3}$ such that $(a-1)(N : M)^2N = 0$. As M is torsion-free, we have $a = 1$, or $(N : M) = 0$ or $N = 0$.

If $a = 1$, then $N = M$, which is impossible. Evidently $N = 0$ is 2-absorbing. Now suppose $(N : M) = 0$. Assume $rsx \in N$, where $r, s \in R$ and $x \in M$. If $rsx \neq 0$, then $rsx \in N \setminus (N : M)^{n-1}N$, and since N is n-weakly prime, the proof is clear in this case.

In case $rsx = 0$, then $rs = 0 \in (N : M)$, or $x = 0 \in N$. □

Proposition 2. *Let $x \in M$ and $a \in R$.*

(i) *If $ann_M(a) \subseteq aM$, then the submodule aM is 2-absorbing if and only if it is n-weakly prime.*
(ii) *If $ann_R(x) \subseteq (Rx : M)$, then the submodule Rx is 2-absorbing if and only if Rx is n-weakly prime.*

Proof. (i) Let M be an n-weakly prime submodule and $r, s \in R$ and $x \in M$ with $rsx \in aM$. If $rsx \notin (aM : M)^{n-1}aM$, then $rs \in (aM : M)$ or $rx \in aM$ or $sx \in aM$. Therefore assume $rsx \in (aM : M)^{n-1}aM$. Clearly $r(s+a)x = rsx + rax \in aM$. If $r(s+a)x \notin (aM : M)^{n-1}aM$, then $r(s+a) \in (aM : M)$ or $rx \in aM$ or $(s+a)x \in aM$. So as $a \in (aM : M)$, $rs \in (aM : M)$ or $rx \in aM$ or $sx \in aM$.

Now suppose that $r(s+a)x \in (aM : M)^{n-1}aM$. Then since $rsx \in (aM : M)^{n-1}aM$, for some $y \in (aM : M)^{n-1}M$, we have $arx = ay$ and so $a(rx - y) = 0$. Hence $rx - y \in ann_M(a) \subseteq aM$ and $y \in (aM : M)^{n-1}M = (aM : M)^{n-2}(aM : M)M \subseteq aM$. Thus $rx \in aM$.

(ii) Let Rx be an n-weakly prime submodule and $r, s \in R, y \in M$ with $rsy \in Rx$. Since Rx is n-weakly prime, we may assume $rsy \in (Rx : M)^{n-1}Rx$. Evidently $rs(x+y) \in Rx$. If $rs(x+y) \notin (Rx : M)^{n-1}Rx$, then $rs \in (Rx : M)$ or $r(x+y) \in Rx$ or $s(x+y) \in Rx$. Hence $rs \in (Rx : M)$ or $ry \in Rx$ or $sy \in Rx$.

Now let $rs(x+y) \in (Rx : M)^{n-1}Rx$. Then as $rsy \in (Rx : M)^{n-1}Rx$, $rsx \in (Rx : M)^{n-1}Rx$ and so $rsx = tx$, for some $t \in (Rx : M)^{n-1} \subseteq (Rx : M)$. Hence $rs - t \in ann(x) \subseteq (Rx : M)$ and thus $rs \in (Rx : M)$. □

Example 3. Let R be a unique factorization domain, p an irreducible element of R, and $M = R \oplus R$.

(a) The submodule $N = p^2 M$ is 2-absorbing.
(b) The submodule $N = p^3 M$ is neither 2-absorbing, nor 2-weakly prime.

Proof. (a) Consider $ab(c,d) \in N$, where $a,b,c,d \in R$. Then a straightforward calculation shows that $a(c,d) \in N$ or $b(c,d) \in N$ or $p^2 \mid ab$.

(b) If N is 2-absorbing, then by Proposition 1(i), $(N : M)$ is 2-absorbing and evidently $p^3 \in (N : M)$, therefore $p^2 \in (N : M)$. Then $p^2(1,0) \in N = p^3 M$. Hence there exists $t \in R$ with $p^2 = p^3 t$. Then $pt = 1$, which is impossible. Therefore N is not 2-absorbing and by Proposition 2(i), N is not 2-weakly prime. □

Recall that the set of zero divisors of M, denoted by $Z(M)$ is defined by $Z(M) = \{r \in R | \exists 0 \neq x \in M,\ rx = 0\}$.

The following result studies the behavior of n-weakly prime submodules under localization. Its proof is not difficult and we leave it to the reader.

Proposition 3. *Let S be a multiplicatively closed subset of R.*

(i) *If N is an n-weakly prime submodule of M with $S^{-1}N \neq S^{-1}M$, then $S^{-1}N$ is an n-weakly prime submodule of $S^{-1}M$.*
(ii) *Let N be an n-weakly prime submodule of M with $Z(\frac{M}{N}) \cap S = \varnothing$. Then $S^{-1}N$ is an n-weakly prime submodule of $S^{-1}M$ and $(S^{-1}N)^c = N$. Moreover $S^{-1}(N : M) = (S^{-1}N : S^{-1}M)$.*

We can introduce the concept of n-weak prime as follows:

A proper submodule N of M will be called *n-weakly prime*, if for $r,s \in R$ and $x \in M$, $rsx \in N \setminus (N : M)^{n-1}N$ implies that $rx \in N$ or $sx \in N$.

Then similar to the proof of Theorem 3, Corollary 4 and Proposition 2 we can prove the following results:

(1) Let N be a submodule of M with $(N : M)^2 N \nsubseteq (N : M)^{n-1}N$. Then N is weakly prime if and only if it is n-weak prime.
(2) Let $n > 3$ and M be a nonzero torsion-free Noetherian R-module. Then a submodule is weakly prime if and only if it is n-weak prime.
(3) Let $a \in R$ with $ann_M(a) \subseteq aM$. Then aM is a weakly prime if and only if it is n-weak prime.

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