ON FUZZY $\alpha$-CONTINUOUS MULTIFUNCTIONS

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Abstract. In this paper we use fuzzy $\alpha$-sets in order to obtain certain characterizations and properties of upper (or lower) fuzzy $\alpha$-continuous multifunctions.

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1. INTRODUCTION

In 1968 Chang [3] introduced fuzzy topological spaces by using fuzzy sets [12]. Since then several workers have contributed to this area: various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good many of them have been extended to the setting of multifunctions.

In 1988 Neubrunn [6] and others [9] introduced the concept of $\alpha$-continuous multifunctions. Njasted [7] and Mashhour [4] introduced $\alpha$-open ($\alpha$-closed) sets, respectively. Bin Shahna in [2] defined these concepts in the fuzzy setting. In this paper our purpose is to define upper (lower) fuzzy $\alpha$-continuous multifunctions and to obtain several characterizations of upper (lower) fuzzy $\alpha$-continuous multifunctions.

Fuzzy sets on a universe $X$ will be denoted by $\mu, \rho, \eta$, etc. Fuzzy points will be denoted by $x_\epsilon, y_\nu$, etc. For any fuzzy points $x_\epsilon$ and any fuzzy set $\mu$, we write $x_\epsilon \in \mu$ iff $\epsilon \leq \mu(x)$. A fuzzy point $x_\epsilon$ is called quasi-coincident with a fuzzy set $\rho$, denoted by $x_\epsilon \ q \ \rho$, iff $\epsilon + \rho(x) > 1$.

A fuzzy set $\mu$ is called quasi-coincident with a fuzzy set $\rho$, denoted by $\mu \ q \ \rho$, iff there exists a $x \in X$ such that $\mu(x) + \rho(x) > 1$. [10, 11]

In this paper we use the concept of fuzzy topological space as introduced in [3]. By int $(\mu)$ and cl $(\mu)$, we mean the interior of $\mu$ and the closure of $\mu$, respectively.

Let $(X, \tau)$ be a topological space in the classical sense and $(Y, \nu)$ be a fuzzy topological space. $F : X \rightarrow Y$ is called a fuzzy multifunction iff for each $x \in X, F(x)$ is a fuzzy set in $Y$. [8]
Let \( F : X \to Y \) be a fuzzy multifunction from a fuzzy topological space \( X \) to a fuzzy topological space \( Y \). For any fuzzy set \( \mu \leq X \), \( F^+(\mu) \) and \( F^-(\mu) \) are defined by 
\[
F^+(\mu) = \{ x \in X : F(x) \leq \mu \}, \quad F^-(\mu) = \{ x \in X : F(x) > \mu \}. \quad [5]
\]

2. Fuzzy \( \alpha \)-continuous multifunction

**Definition 1.** Let \((X, \tau)\) be a fuzzy topological space and let \( \mu \leq X \) be a fuzzy set. Then it is said that:

(i) \( \mu \) is fuzzy \( \alpha \)-open set [2] if \( \mu \leq \text{int \, cl \, int \, } \mu \).

(ii) \( \mu \) is fuzzy \( \alpha \)-closed set [2] if \( \mu \geq \text{cl \, int \, cl \, } \mu \).

(iii) \( \mu \) is fuzzy semiopen set [1] if \( \mu \leq \text{cl \, int \, } \mu \).

(iv) \( \mu \) is fuzzy preopen set [2] if \( \mu \leq \text{int \, cl \, } \mu \).

**Definition 2.** Let \( F : X \to Y \) be a fuzzy multifunction from a fuzzy topological space \((X, \tau)\) to a fuzzy topological space \((Y, \upsilon)\). Then it is said that \( F \) is:

(1) Upper fuzzy \( \alpha \)-continuous at \( x \in X \) iff for each fuzzy open set \( \mu \) of \( Y \) containing \( F(x) \), there exists a fuzzy \( \alpha \)-open set \( \rho \) containing \( x \) such that \( \rho \leq F^+(\mu) \).

(2) Lower fuzzy \( \alpha \)-continuous at \( x \in X \) iff for each fuzzy open set \( \mu \) of \( Y \) such that \( x \in F^-(\mu) \), there exists a fuzzy \( \alpha \)-open set \( \rho \) containing \( x \) such that \( \rho \leq F^-(\mu) \).

(3) Upper (lower) fuzzy \( \alpha \)-continuous iff it has this property at each point of \( X \).

We know that a net \((x_{\alpha_\mu})\) in a fuzzy topological space \((X, \tau)\) is said to be eventually in the fuzzy set \( \mu \leq X \) if there exists an index \( \alpha_0 \in J \) such that \( x_{\alpha_\mu} \in \rho \) for all \( \alpha \geq \alpha_0 \).

The following theorem states some characterizations of upper fuzzy \( \alpha \)-continuous multifunction.

**Definition 3.** A sequence \((x_{\alpha_\mu})\) is said to \( \alpha \)-converge to a point \( X \) if for every fuzzy \( \alpha \)-open set \( \mu \) containing \( x_{\alpha}, \) there exists an index \( n_0 \) such that for \( n \geq n_0, x_{\alpha_n} \in \mu \). This is denoted by \( x_{\alpha_n} \to_{\alpha} x_{\alpha} \).

**Theorem 1.** Let \( F : X \to Y \) be a fuzzy multifunction from a fuzzy topological \((X, \tau)\) to a fuzzy topological space \((Y, \upsilon)\). Then the following statements are equivalent:

(i) \( F \) is upper fuzzy \( \alpha \)-continuous.

(ii) For each \( x_{\alpha_\mu} \in \) and for each fuzzy open set \( \mu \) such that \( x_{\alpha_\mu} \in F^+(\mu) \), there exists a fuzzy \( \alpha \)-open set \( \rho \) containing \( x_{\alpha_\mu} \) such that \( \rho \leq F^+(\mu) \).

(iii) \( F^+(\mu) \) is a fuzzy \( \alpha \)-open set for any fuzzy open set \( \mu \in X \).

(iv) \( F^-(\mu) \) is a fuzzy \( \alpha \)-closed set for any fuzzy open set \( \mu \in Y \).

(v) For each \( x_{\alpha_\mu} \in \) and for each net \((x_{\alpha_\mu}^\alpha)\) which \( \alpha \)-converges to \( x_{\alpha_\mu} \) in \( X \) and for each fuzzy open set \( \mu \in Y \) such that \( x_{\alpha_\mu} \in F^+(\mu) \), the net \((x_{\alpha_\mu}^\alpha)\) is eventually in \( F^+(\mu) \).
Proof. (i)⇔(ii) this statement is obvious.
(i)⇔(iii). Let \( x_\varepsilon \in F^+(\mu) \) and let \( \mu \) be a fuzzy open set. It follows from (i) that there exists a fuzzy \( \alpha \)-open set \( \rho \) containing \( x_\varepsilon \) such that \( \rho x_\varepsilon \leq F^+(\mu) \). It follows that \( F^+(\mu) = \bigvee_{x_\varepsilon \in F^+(\mu)} \rho x_\varepsilon \) and hence \( F^+(\mu) \) is fuzzy \( \alpha \)-open.

The converse can be shown easily.

(iii)⇒(iv) Let \( \mu \leq Y \) be a fuzzy open set. We have that \( Y \setminus \mu \) is a fuzzy open set. From (iii), \( F^+(Y \setminus \mu) = X \setminus F^-(\mu) \) is a fuzzy \( \alpha \)-open set. Then it is obtained that \( F^-(\mu) \) is a fuzzy \( \alpha \)-closed set.

(i)⇒(v). Let \( (x^\alpha_{e_\alpha}) \) be a net which \( \alpha \)-converges to \( x_\varepsilon \) in \( X \) and let \( \mu \leq Y \) be any fuzzy open set such that \( x_\varepsilon \in F^+(\mu) \). Since \( F \) is an upper fuzzy \( \alpha \)-continuous multifunction, it follows that there exists a fuzzy \( \alpha \)-open set \( \rho \leq X \) containing \( x_\varepsilon \) such that \( \rho \leq F^+(\mu) \). Since \( (x^\alpha_{e_\alpha}) \) \( \alpha \)-converges to \( x_\varepsilon \), it follows that there exists an index \( \alpha_0 \in J \) such that \( (x^\alpha_{e_\alpha}) \in \rho \) for all \( \alpha \geq \alpha_0 \) from here, we obtain that \( x^\alpha_{e_\alpha} \leq \rho \leq F^+(\mu) \) for all \( \alpha \geq \alpha_0 \). Thus the net \( (x^\alpha_{e_\alpha}) \) is eventually in \( F^+(\mu) \).

(v)⇒(i). Suppose that is not true. There exists a point \( x_\varepsilon \) and a fuzzy open set \( \mu \) with \( x_\varepsilon \in F^+(\mu) \) such that \( \rho \leq F^+(\mu) \) for each fuzzy \( \alpha \)-open set \( \rho \leq X \) containing \( x_\varepsilon \). Let \( x_{e_\rho} \in \rho \) and \( x_\varepsilon \notin F^+(\mu) \) for each fuzzy \( \alpha \)-open set \( \rho \leq X \) containing \( x_\varepsilon \). Then for the \( \alpha \)-neighborhood net \( (x_{e_\rho}) \), \( x_{e_\rho} \rightarrow_\alpha x_\varepsilon \), but \( (x_{e_\rho}) \) is not eventually in \( F^+(\mu) \). This is a contradiction. Thus, \( F \) is an upper fuzzy \( \alpha \)-continuous multifunction.

Remark 1. For a fuzzy multifunction \( F : X \rightarrow Y \) from a fuzzy topological \((X, \tau)\) to a fuzzy topological space \((Y, \upsilon)\), the following implication holds:

Upper fuzzy continuous \( \implies \) Upper fuzzy \( \alpha \)-continuous.

The following example show that the reverse need not be true.

Example 1. Let \( X = \{x, y\} \) with topologies \( \tau = \{X, \phi, \mu\} \) and \( \upsilon = \{X, \phi, \rho\} \), where the fuzzy sets \( \mu, \rho \) are defined as:

\[
\mu(x) = 0.3, \quad \mu(y) = 0.6 \\
\rho(x) = 0.7, \quad \rho(y) = 0.4
\]

A fuzzy multifunction \( F : (X, \tau) \rightarrow (Y, \upsilon) \) given by \( x_\varepsilon \rightarrow F(x_\varepsilon) = \{x_\varepsilon\} \) is upper \( \alpha \)-continuous, but it is not upper continuous.

The following theorem states some characterizations of a lower fuzzy \( \alpha \)-continuous multifunction.

Theorem 2. Let \( F : X \rightarrow Y \) be a fuzzy multifunction from a fuzzy topological \((X, \tau)\) to a fuzzy topological space \((Y, \upsilon)\). Then the following statements are equivalent.

(i) \( F \) is lower fuzzy \( \alpha \)-continuous.

(ii) For each \( x_\varepsilon \in X \) and for each fuzzy open set \( \mu \) such that \( x_\varepsilon \in F^-(\mu) \) there exists a fuzzy \( \alpha \)-open set \( \rho \) containing \( x_\varepsilon \) such that \( \rho \leq F^-(\mu) \).

(iii) \( F^-(\mu) \) is a fuzzy \( \alpha \)-open set for any fuzzy open set \( \mu \leq Y \).
(iv) $F^+(\mu)$ is a fuzzy $\alpha$-closed set for any fuzzy open set $\mu \leq Y$.

(v) For each $x_\epsilon \in X$ and for each net $(x^\alpha_\eta)$ which $\alpha$-converges to $x_\epsilon$ in $X$ and for each fuzzy open set $\mu \leq Y$ such that $x_\epsilon \in F^- (\mu)$, the net $(x^\alpha_\eta)$ is eventually in $F^- (\mu)$.

Proof. It can be obtained similarly as Theorem 1.

**Theorem 3.** Let $F : X \to Y$ be a fuzzy multifunction from a fuzzy topological $(X, \tau)$ to a fuzzy topological space $(Y, \upsilon)$ and let $F(X)$ be endowed with subspace fuzzy topology. If $F$ is an upper fuzzy $\alpha$-continuous multifunction, then $F : X \to F(X)$ is an upper fuzzy $\alpha$-continuous multifunction.

Proof. Since $F$ is an upper fuzzy $\alpha$-continuous, $F(X \wedge F(X)) = F^+(\mu) \wedge F^+(F(X)) = F^+(\mu)$ is fuzzy $\alpha$-open for each fuzzy open subset $\mu$ of $Y$. Hence $F : X \to F(X)$ is an upper fuzzy $\alpha$-continuous multifunction.

**Definition 4.** Suppose that $(X, \tau), (Y, \upsilon)$ and $(Z, \omega)$ are fuzzy topological spaces. It is known that if $F_1 : X \to Y$ and $F_2 : Y \to Z$ are fuzzy multifunctions, then the fuzzy multifunction $F_1 \circ F_2 : X \to Z$ is defined by $(F_1 \circ F_2)(x_\epsilon) = F_2(F_1(x_\epsilon))$ for each $x_\epsilon \in X$.

**Theorem 4.** Let $(X, \tau), (Y, \upsilon)$ and $(Z, \omega)$ be fuzzy topological space and let $F : X \to Y$ and $G : Y \to Z$ be fuzzy multifunction. If $F : X \to Y$ is an upper (lower) fuzzy continuous multifunction and $G : Y \to Z$ is an upper (lower) fuzzy $\alpha$-continuous multifunction. Then $G \circ F : X \to Z$ is an upper (lower) fuzzy $\alpha$-continuous multifunction.

Proof. Let $\lambda \leq Z$ be any fuzzy open set. From the definition of $G \circ F$, we have $(G \circ F)^+ (\lambda) = F^+(G^+(\lambda))((G \circ F)^- (\lambda) = F^-(G^-(\lambda)))$, since $G$ is an upper (lower) fuzzy $\alpha$-continuous, it follows that $G^+(\lambda)(G^-(\lambda))$ is a fuzzy open set. Since $F$ is an upper (lower) fuzzy continuous, it follows that $F^+(G^+(\lambda))(F^-(G^-(\lambda)))$ is a fuzzy $\alpha$-open set, this shows that $G \circ F$ is an upper (lower) fuzzy $\alpha$-continuous.

**Theorem 5.** Let $F : X \to Y$ be a fuzzy multifunction from a fuzzy topological $(X, \tau)$ to a fuzzy topological space $(Y, \upsilon)$. If $F$ is a lower(upper) fuzzy $\alpha$-continuous multifunction and $\mu \leq X$ is a fuzzy set, then the restriction multifunction $F|_\mu : \mu \to Y$ is an lower (upper) fuzzy $\alpha$-continuous multifunction.

Proof. Suppose that $\beta \leq Y$ is a fuzzy open set. Let $x_\epsilon \in \mu$ and let $x_\epsilon \in F^-|_\mu (\beta)$. Since $F$ is a lower fuzzy $\alpha$-continuous multifunction, if follows that there exists a fuzzy open set $x_\epsilon \in \rho$ such that $\rho \leq F^- (\beta)$. From here we obtain that $x_\epsilon \in \rho \wedge \mu$ and $\rho \wedge \mu \leq F^-|_\mu (\beta)$. Thus, we show that the restriction multifunction $F|_\mu$ is lower fuzzy $\alpha$-continuous multifunction.

The proof for the case of the upper fuzzy $\alpha$-continuity of the multifunction $F|_\mu$ is similar to the above.
Theorem 6. Let \( F : X \rightarrow Y \) be a fuzzy multifunction from a fuzzy topological \((X, \tau)\) to a fuzzy topological space \((Y, \nu)\), let \( \{ \lambda_\gamma : \gamma \in \Phi \} \) be a fuzzy open cover of \( X \). If the restriction multifunction \( F_\gamma = F_{\lambda_\gamma} \) is lower (upper) fuzzy \( \alpha \)-continuous multifunction for each \( \gamma \in \Phi \), then \( F \) is lower (upper) fuzzy \( \alpha \)-continuous multifunction.

Proof. Let \( \mu \leq Y \) be any fuzzy open set. Since \( F_\gamma \) is lower fuzzy \( \alpha \)-continuous for each \( \gamma \), we know that \( F^-_\gamma (\mu) \leq \text{int} \lambda_\gamma (F^-_\gamma (\mu)) \) and from here \( F^- (\mu) \wedge \lambda_\gamma \leq \text{int} \lambda_\gamma (F^- (\mu)) \wedge \lambda_\gamma \) and \( F^- (\mu) \wedge \lambda_\gamma \leq \text{int} (F^- (\mu)) \wedge \lambda_\gamma \). Since \( \{ \lambda_\gamma : \gamma \in \Phi \} \) is a fuzzy open cover of \( X \). It follows that \( F^- (\mu) \leq \text{int} (F^- (\mu)) \). Thus, we obtain that \( F \) is lower(upper) fuzzy \( \alpha \)-continuous multifunction. \( \square \)

The proof of the upper fuzzy \( \alpha \)-continuity of \( F \) is similar to the above.

Definition 5. Suppose that \( F : X \rightarrow Y \) is a fuzzy multifunction from a fuzzy topological space \( X \) to a fuzzy topological space \( Y \). The fuzzy graph multifunction \( G_F : X \rightarrow X \times Y \) of \( F \) is defined as \( G_F(x_\epsilon) = \{ x_\epsilon \} \times F(x_\epsilon) \).

Theorem 7. Let \( F : X \rightarrow Y \) be a fuzzy multifunction from a fuzzy topological \((X, \tau)\) to a fuzzy topological space \((Y, \nu)\). If the graph function of \( F \) is lower(upper) fuzzy \( \alpha \)-continuous multifunction, then \( F \) is lower(upper) fuzzy \( \alpha \)-continuous multifunction.

Proof. For the fuzzy sets \( \beta \leq X, \eta \leq Y \), we take

\[
(\beta \times \eta)(z, y) = \begin{cases} 
0 & \text{if } z \notin \beta \\
\eta(y) & \text{if } z \in \beta
\end{cases}
\]

Let \( x_\epsilon \in X \) and let \( \mu \leq Y \) be a fuzzy open set such that \( x_\epsilon \in F^- (\mu) \). We obtain that \( x_\epsilon \in G^- (X \times \mu) \) and \( X \times \mu \) is a fuzzy open set. Since fuzzy graph multifunction \( G_F \) is lower fuzzy \( \alpha \)-continuous, it follows that there exists a fuzzy \( \alpha \)-open set \( \rho \leq X \) containing \( x_\epsilon \) such that \( \rho \leq G^-_\gamma (X \times \mu) \). From here, we obtain that \( \rho \leq F^- (\mu) \). Thus, \( F \) is lower fuzzy \( \alpha \)-continuous multifunction.

The proof of the upper fuzzy \( \alpha \)-continuity of \( F \) is similar to the above.

Theorem 8. Suppose that \((X, \tau)\) and \((X_\alpha, \tau_\alpha)\) are fuzzy topological space where \( \alpha \in J \). Let \( F : X \rightarrow \prod_{\alpha \in J} X_\alpha \) be a fuzzy multifunction from \( X \) to the product space \( \prod_{\alpha \in J} X_\alpha \) and let \( P_\alpha : \prod_{\alpha \in J} X_\alpha \rightarrow X_\alpha \) be the projection multifunction for each \( \alpha \in J \) which is defined by \( P_\alpha ((x_\alpha)) = \{ x_\alpha \} \). If \( F \) is an upper (lower) fuzzy \( \alpha \)-continuous multifunction, then \( P_\alpha \circ F \) is an upper (lower) fuzzy \( \alpha \)-continuous multifunction for each \( \alpha \in J \).

Proof. Take any \( \alpha_0 \in J \). Let \( \mu_{\alpha_0} \) be a fuzzy open set in \((X_{\alpha_0}, \tau_{\alpha_0})\). Then \((P_{\alpha_0} \circ F)^+(\mu_{\alpha_0}) = F^+(P_{\alpha_0}^+(\mu_{\alpha_0})) = F^+(\mu_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha) \) (resp., \((P_{\alpha_0} \circ F)^-(\mu_{\alpha_0}) = F^-((P_{\alpha_0}^-(\mu_{\alpha_0})) = F^-((\mu_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)) \).
Since $F$ is upper (lower) fuzzy $\alpha$-continuous multifunction and since $\mu_\alpha \times \prod_{\alpha \neq \alpha_0} X_\alpha$ is a fuzzy open set, it follows that $F^+(\mu_\alpha \times \prod_{\alpha \neq \alpha_0} X_\alpha)$ (resp., $F^-(\mu_\alpha \times \prod_{\alpha \neq \alpha_0} X_\alpha)$) is fuzzy $\alpha$- open in $(X, \tau)$. It shows that $P_\alpha \circ F$ is upper (lower) fuzzy $\alpha$-continuous multifunction.

Hence, we obtain that $P_\alpha \circ F$ is an upper (lower) fuzzy $\alpha$-continuous multifunction for each $\alpha \in J$. \hfill $\square$

**Theorem 9.** Suppose that for each $\alpha \in J$, $(X_\alpha, \tau_\alpha)$ and $(Y_\alpha, \nu_\alpha)$ are fuzzy topological spaces. Let $F_\alpha : X_\alpha \rightarrow Y_\alpha$ be a fuzzy multifunction for each $\alpha \in J$ and let $F : \prod_{\alpha \in J} X_\alpha \rightarrow \prod_{\alpha \in J} Y_\alpha$ be defined by $F((x_\alpha)) = \prod_{\alpha \in J} F_\alpha(x_\alpha)$ from the product space $\prod_{\alpha \in J} X_\alpha$ to product space $\prod_{\alpha \in J} Y_\alpha$. If $F$ is an upper (lower) fuzzy $\alpha$-continuous multifunction, then each $F_\alpha$ is an upper (lower) fuzzy $\alpha$-continuous multifunction for each $\alpha \in J$.

**Proof.** Let $\mu_\alpha \leq Y_\alpha$ be a fuzzy open set. Then $\mu_\alpha \times \prod_{\alpha \neq \beta} Y_\beta$ is a fuzzy open set. Since $F$ is an upper (lower) fuzzy $\alpha$-continuous multifunction, it follows that $F^+(\mu_\alpha \times \prod_{\alpha \neq \beta} Y_\beta) = F^+(\mu_\alpha) \times \prod_{\alpha \neq \beta} X_\beta$. $(F^- (\mu_\alpha \times \prod_{\alpha \neq \beta} Y_\beta) = F^-(\mu_\alpha) \times \prod_{\alpha \neq \beta} X_\beta)$ is a fuzzy $\alpha$-open set. Consequently, we obtain that $F^+(\mu_\alpha)$ (resp., $F^-(\mu_\alpha)$) is a fuzzy $\alpha$-open set. Thus, we show that $F_\alpha$ is an upper (lower) fuzzy $\alpha$-continuous multifunction. \hfill $\square$

**Theorem 10.** Suppose that $(X_1, \tau_1)$, $(X_2, \tau_2)$, $(Y_1, \nu_1)$ and $(Y_2, \nu_2)$ are fuzzy topological spaces and $F_1 : X_1 \rightarrow Y_1$, $F_2 : X_2 \rightarrow Y_2$ are fuzzy multifunctions and suppose that if $\eta \times \beta$ is fuzzy $\alpha$-open set then $\eta$ and $\beta$ are fuzzy $\alpha$-open sets for any fuzzy $\eta \leq Y_1, \beta \leq Y_2$. Let $F_1 \times F_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ be a fuzzy multifunction which is defined by $(F_1 \times F_2)(x_\epsilon, y_\nu) = (F_1(x_\epsilon) \times F_2(y_\nu)$. If $F_1 \times F_2$ is an upper (lower) fuzzy $\alpha$-continuous multifunction, then $F_1$ and $F_2$ are upper (lower) fuzzy $\alpha$-continuous multifunctions.

**Proof.** We know that $(\mu^* \times \beta^*)(x_\epsilon, y_\nu) = \min \{\mu^*(x), \beta^*(y)\}$ for any fuzzy sets $\mu^*, \beta^*$ and for any fuzzy point $x_\epsilon, y_\nu$.

Let $\mu \times \beta \leq Y_1 \times Y_2$ be a fuzzy open set. It known that $(F_1 \times F_2)^+(\mu \times \beta) = F_1^+(\mu) \times F_2^+(\beta)$. Since $F_1 \times F_2$ is an upper fuzzy $\alpha$-continuous multifunction, it follows that $F_1^+(\mu) \times F_2^+(\beta)$ is a fuzzy $\alpha$-open set. From here, $F_1^+(\mu)$ and $F_2^+(\beta)$ are fuzzy $\alpha$-open sets. Hence, it is obtain that $F_1$ and $F_2$ are upper fuzzy $\alpha$-continuous multifunctions. \hfill $\square$

The proof of the lower fuzzy $\alpha$-continuity of the multifunctions $F_1$ and $F_2$ is similar to the above.

**Theorem 11.** Suppose that $(X, \tau)$, $(Y, \nu)$ and $(Z, \omega)$ are fuzzy topological spaces and $F_1 : X \rightarrow Y$, $F_2 : X \rightarrow Z$ are fuzzy multifunction and suppose that if $\eta \times \beta$ is a fuzzy $\alpha$-open set, then $\eta$ and $\beta$ are fuzzy $\alpha$-open sets for any fuzzy sets $\eta \leq Y, \beta \leq Z$. Let $F_1 \times F_2 : X \rightarrow Y \times Z$ be a fuzzy multifunction which is defined by
(\(F_1 \times F_2\))(x_\varepsilon) = F_1(x_\varepsilon) \times F_2(x_\varepsilon). If \(F_1 \times F_2\) is an upper (lower) fuzzy \(\alpha\)-continuous multifunction, then \(F_1\) and \(F_2\) are upper (lower) fuzzy \(\alpha\)-continuous multifunctions.

Proof. Let \(x_\varepsilon \in X\) and let \(\mu \leq \gamma, \beta \leq Z\) be fuzzy \(\alpha\)-open sets such that \(x_\varepsilon \in F_1^+(\mu)\) and \(x_\varepsilon \in F_2^+(\beta)\). Then we obtain that \(F_1(x_\varepsilon) \leq \mu\) and \(F_2(x_\varepsilon) \leq \beta\) and from here, \(F_1(x_\varepsilon) \times F_2(x_\varepsilon) = (F_1 \times F_2)(x_\varepsilon) \leq \mu \times \beta\). We have \(x_\varepsilon \in (F_1 \times F_2)^+(\mu \times \beta)\). Since \(F_1 \times F_2\) is an upper fuzzy \(\alpha\)-continuous multifunction, it follows that there exist a fuzzy \(\alpha\)-open set \(\rho\) containing \(x_\varepsilon\) such that \(\rho \leq (F_1 \times F_2)^+(\mu \times \beta)\). We obtain that \(\rho \leq F_1^+(\mu)\) and \(\rho \leq F_2^+(\beta)\). Thus we obtain that \(F_1\) and \(F_2\) are fuzzy \(\alpha\)-continuous multifunctions. \(\Box\)

The proof of the lower fuzzy \(\alpha\)-continuity of the multifunctions \(F_1\) and \(F_2\) is similar to the above.

Lemma 1 ([2]). A fuzzy set in fuzzy topological space \(X\) is a fuzzy \(\alpha\)-open set if and only if it is fuzzy semiopen and fuzzy preopen.

Theorem 12. Let \(F : X \to Y\) be a fuzzy multifunction from a fuzzy topological \((X, \tau)\) to a fuzzy topological space \((Y, \upsilon)\). Then \(F\) is an upper fuzzy \(\alpha\)-continuous if and only if it is an upper fuzzy semicontinuous and upper fuzzy precontinuous.

Proof. Let \(F\) be upper fuzzy semicontinuous and upper fuzzy precontinuous, and let \(\mu\) be a fuzzy open set in \(Y\). Then \(F^+(\mu)\) is fuzzy semiopen and fuzzy preopen, it follows from lemma 1 that \(F^+(\mu)\) is a fuzzy \(\alpha\)-open set, and hence \(F\) is an upper fuzzy \(\alpha\)-continuous multifunction. The converse is immediate. \(\Box\)

Theorem 13. Let \(F : X \to Y\) be a fuzzy multifunction from a fuzzy topological space \((X, \tau)\) to a fuzzy topological space \((Y, \upsilon)\). Then \(F\) is a lower fuzzy \(\alpha\)-continuous if and only if it is lower fuzzy semicontinuous and lower fuzzy precontinuous.

Proof. Similar to that of Theorem 12 and is omitted. \(\Box\)

References


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