ON ALGEBRAIC EXTENSIONS MODULO $I$

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Abstract. Let $I$ be a nonzero ideal of a ring $T$, let $\varphi : T \to E := T/I$ denote the canonical projection, let $D$ be a ring contained in $E$, and let $R = \varphi^{-1}(D)$. The main purpose of this paper is to characterize when the ring extension $R \subset T$ is $n$- (resp., universally) algebraic modulo $I$ in case $I$ is an intersection of finitely many maximal ideals of $T$.

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1. INTRODUCTION

All rings considered below are commutative with identity but not necessarily integral domains. All subrings and inclusions of rings are (unital) ring extensions; all ring/algebra homomorphisms are unital. Let $A$ be a ring and $n \geq 1$ be an integer. We denote by $A[n]$ the ring of polynomials in $n$ indeterminates over $A$ (for $n = 1$, $A[1] = A[X]$ is the ring of polynomials in one indeterminate). For convenience, we write $A = A[0]$.

Let $I$ be a nonzero ideal of a ring $T$, $\varphi : T \to E := T/I$ the natural projection, and $D$ a ring contained in $E$. Then $R = \varphi^{-1}(D)$ is the ring arising from the following pullback of canonical homomorphisms:

$$
\begin{array}{ccc}
R & \longrightarrow & D \\
\downarrow & & \downarrow \\
T & \longrightarrow & T/I = E
\end{array}
$$

Following [4], we say that $R$ is the ring of the $(T, I, D)$ construction and we set $R := (T, I, D)$. We shall assume that $D$ is properly contained in $E$ (and hence, that $R$ is properly contained in $T$), and we shall refer to this as a pullback diagram of type (□). If $I$ is an intersection of finitely many maximal ideals of $T$, we shall refer to this as a diagram (□_□). A very good account of pullback constructions has been given in [4, 5] and [6]. It has fashionable in recent years to study rings via pullback diagrams. It is well worth noting that pullback constructions provide a rich source of examples and counterexamples in commutative algebra (see [1–5, 11, 12]). Unless
otherwise specified, the symbols \(T, D, I, R\) have the above meaning throughout the paper.

In [8] the authors introduced the concept of \(n\)-algebraic extension modulo \(I\) for a diagram \((\square)\) when \(T\) and \(D\) are integral domains and \(n \geq 0\) is an integer. More precisely, the ring extension \(R \subset T\) (of integral domains) is said to be \(n\)-algebraic modulo \(I\) if for every two prime ideals \(Q' \subset Q\) of \(T[n]\) such that \(I[n] \not\subseteq Q'\), \(I[n] \subseteq Q\) and \(ht(Q \cap R[n]/Q' \cap R[n]) = 1\), then \(R[n]/(Q \cap R[n]) \subseteq T[n]/Q\) is algebraic. This concept was first used to characterize when an integral domain \(R\) of the form \(D + I\), (where \(I\) is a nonzero ideal of an integral domain \(T\) and \(D\) is a subring of \(T\) satisfying \(D \cap I = (0)\)) is a (stably) strong S-domain (cf. [8, Théorème 1.7]). In [2], the authors dealt with a more general situation and used this concept to characterize when a ring \(R\) arising from a diagram \((\square)\) is a (stably) strong S-domain. The main purpose of this paper is to study \(n\)-algebraic extensions modulo \(I\) for a diagram \((\square)\) in order to deepen our knowledge about such extensions. We first extend this notion to arbitrary commutative rings. Our motivation is an example constructed by Fontana et al (see [8, Exemple 1.8]) of a diagram \((\square)\) in order to produce a ring extension \(R \subset T\) which is 0-algebraic modulo \(I\) but not 1-algebraic modulo \(I\). For this reason, M. Fontana et al (see [8]) have introduced the following definition: The ring extension \(R \subset T\) is said to be universally algebraic modulo \(I\), if \(R \subset T\) is \(n\)-algebraic modulo \(I\) for each positive integer \(n\). Our contribution (see Theorem 1) is to prove that for a diagram \((\square)\), \(R \subset T\) is \(n\)-algebraic modulo \(I\) if and only if \(R \subset T\) is \((n - 1)\)-algebraic modulo \(I\) if and only if \(R \subset T\) is a residually algebraic extension. The key step (Lemma 1) is to show, for any diagram \((\square)\), that if \(R \subset T\) is \(n\)-algebraic modulo \(I\) (where \(n \geq 1\)), then \(R \subset T\) is \((n - 1)\)-algebraic modulo \(I\).

Throughout the paper, we use “\(\subset\)” to denote proper containment and “\(\subseteq\)” to denote containment. Transcendence degrees play an important role in our study; if \(A \subseteq B\) are two domains, we denote by \(tr.\text{deg}[B : A]\) the transcendence degree of the quotient field of \(B\) over that of \(A\). Any unexplained terminology is standard as in [9, 10]. Relevant terminology and results will be recalled as needed through the paper.

2. MAIN RESULTS

We extend Fontana-Izelgue-Kabbaj’s definition, mentioned in the introduction, to arbitrary commutative rings in the following way:

**Definition 1.** Let \(n \geq 0\) be an integer. For a diagram \((\square)\), the extension \(R \subset T\) is said to be \(n\)-algebraic modulo \(I\) if for every two prime ideals \(Q' \subset Q\) of \(T[n]\) such that \(I[n] \not\subseteq Q'\), \(I[n] \subseteq Q\) and \(ht(Q \cap R[n]/Q' \cap R[n]) = 1\), then \(R[n]/(Q \cap R[n]) \subseteq T[n]/Q\) is algebraic.

**Definition 2.** For a diagram \((\square)\), the extension \(R \subset T\) is said to be universally algebraic modulo \(I\) if \(R \subset T\) is \(n\)-algebraic modulo \(I\) for each integer \(n \geq 0\).
Recall that an extension of rings $A \subseteq B$ is said to be residually algebraic if for each prime ideal $Q$ of $B$, the extension $A/(Q \cap A) \subseteq B/Q$ is algebraic. It is clear that if $R \subseteq T$ is a residually algebraic extension, then so is $R[n] \subseteq T[n]$ for any positive integer $n$ (cf. [7, Lemme 1.4]). Hence $R \subseteq T$ is universally algebraic modulo $I$.

Recall from [10, Section 1-5] that if $p$ is a prime ideal of a ring $A$, and $Q$ is a prime ideal of $A[X]$ with $Q \cap A = p$, but with $Q \neq p[X]$, then we call $Q$ an upper to $p$ in $A[X]$ (or more simply, an upper to $p$, or just an upper).

The main result of this paper is the following theorem which identifies $n$-algebraic extensions modulo $I$ for a diagram $(\square_\gamma)$. We assume that all rings are finite-dimensional.

**Theorem 1.** Let $n \geq 1$ be an integer. For a diagram $(\square_\gamma)$, consider the following statements:

1. $R \subseteq T$ is 1-algebraic modulo $I$.
2. $tr.deg[T/M : R/(M \cap R)] = 0$ for each maximal ideal $M$ of $T$ containing $I$.
3. $R \subseteq T$ is a residually algebraic extension.
4. $R \subseteq T$ is universally algebraic modulo $I$.
5. $R \subseteq T$ is $n$-algebraic modulo $I$.
6. $R \subseteq T$ is 0-algebraic modulo $I$.

Then:

(a) In general, $(1) \iff (2) \iff (3) \iff (4) \iff (5) \implies (6)$.

(b) If, in addition, $I \in \text{Max}(T)$, then the above statements $(1) - (6)$ are equivalent.

To prove the implications $(5) \implies (1)$ and $(5) \implies (6)$ in Theorem 1, we need the following lemma.

**Lemma 1.** Let $n \geq 1$ be an integer. For a diagram $(\square_\gamma)$, if $R \subseteq T$ is $n$-algebraic modulo $I$, then $R \subseteq T$ is $(n-1)$-algebraic modulo $I$.

**Proof.** Let $Q' \subseteq Q$ be two prime ideals of $T[n-1]$ such that $I[n-1] \not\subseteq Q'$ and $I[n-1] \subseteq Q$. Set $P' = Q' \cap R[n-1]$, $P = Q \cap R[n-1]$ and suppose that $P' \subset P$ are consecutive. Our task is to show that $R[n-1]/P' \subseteq T[n-1]/Q$ is an algebraic extension. Let $Q' = Q' + X_n T[n-1][X_n]$ and $Q = Q + X_n T[n-1][X_n]$. It is obvious that $Q'$ respectively $Q$ are uppers to $Q'$ respectively $Q$. Set $P' = Q' \cap R[n]$ and $P = Q \cap R[n]$. One can check easily that $P' = P' + X_n R[n]$ and $P = P + X_n R[n]$. As $X_n R[n] \subseteq P' \subset P$, then $P' \subset P$ are consecutive. On the other hand, since $R \subseteq T$ is $n$-algebraic modulo $I$, then $tr.deg[T[n]/Q : R[n]/P] = 0$. As $T[n]/Q \cong T[n-1]/Q$ and $R[n]/P \cong R[n-1]/P$, it follows that $tr.deg[T[n-1]/Q : R[n-1]/P] = 0$, as desired. \qed

Before proceeding to the proof of Theorem 1 it is convenient to recall the following Cahen’s lemma [4, Proposition 4]. We shall make use of this result in the proof of
Theorem 1. Note that this lemma holds even for polynomial rings since if \( R := (T, I, D) \), then \( R[n] := (T[n], I[n], D[n]) \).

**Lemma 2.** For a diagram (\( \Box \)), if \( P_0 \subseteq \ldots \subseteq P_n \) is a chain of primes in \( R \) such that \( P_n \) is minimal among primes of \( R \) containing \( I \) and \( P_{n-1} \), then this chain lifts in \( T \).

We now prove Theorem 1.

**Proof of Theorem 1.** (a) (1)\( \Rightarrow \)(2) Let \( \Omega \) be the finite subset of \( \text{Max}(T) \) such that \( I = \bigcap_{M \in \Omega} M \). We discuss the following two cases.

Case 1. \( | \Omega | \geq 2 \). Since \( M + \bigcap_{M \in \Omega} M = T \), then there exist \( u \in \bigcap_{M \in \Omega} M \) and \( v \in M \) such that \( u + v = 1 \). Let \( P'_1 = ((X - u)T[X]) \cap R[X] \) and \( P_1 = (M[X] + (X - u)T[X]) \cap R[X] \). The prime ideals \( P'_1 \subseteq P_1 \) are not necessarily consecutive. Since \( T[X] \) is finite-dimensional, there exist two prime ideals \( P' \) and \( P \) of \( T[X] \) such that \( P' \) is maximal among the primes such that \( P'_1 \subseteq P' \subseteq P_1 \) and not containing \( I \), and \( P \) is minimal such that \( P'_1 \subseteq P' \subseteq P \). Therefore \( P' \) does not contain \( I \), \( P \) contains \( I \) and \( P' \subseteq P \) are consecutive. The chain \( P'_1 \subseteq P' \subseteq P \) lifts in \( T[X] \) as \( Q'_1 \subseteq Q' \subseteq Q \). Notice that \( Q'_1 = (X - u)T[X] \) because \( P'_1 \) does not contain \( I \) and so it lifts uniquely in \( T[X] \). Hence \( Q \) contains \( X - u \) and \( I \). The prime ideal \( Q \) cannot contain any prime containing \( u \) (if so, it would contain \( X \), thus \( X \in P_1 \) and hence \( u \in M \), which is absurd). Consequently \( Q \) is above \( M \). Furthermore \( Q \) is an upper to \( M \) because \( X - u \in Q \setminus M[X] \). The prime ideal \( P \) is above \( p = M \cap R \). Next, we demonstrate that \( P \) is an upper to \( p \). Consider the polynomial \( f = (X - u)(X - v) = X^2 - X + uv \). Since \( uv \in I \), then clearly \( f \) belongs to \( P'_1 = ((X - u)T[X]) \cap R[X] \). Thus \( f \in P \). As \( f \not\in p[X] \), we deduce that \( P \) is an upper to \( p \). As \( R \subseteq T \) is \( I \)-algebraic modulo \( I \), it follows that \( T[X]/Q \) is algebraic over \( R[X]/P \). Since \( Q \) and \( P \) are upperly related to \( M \) and \( p \), we deduce that \( T/M \) is algebraic over \( R/p \).

Case 2. \( | \Omega | = 1 \). In this case \( I = M \), where \( M \) is a maximal ideal of \( T \). The proof in this case proceeds along the same lines as in the proof of Case 1 with some modifications. Set \( P'_1 = ((X - 1)T[X]) \cap R[X] \) and \( P_1 = (M[X] + (X - 1)T[X]) \cap R[X] \). These prime ideals are not necessarily consecutive, so let \( P' \) be maximal among the primes such that \( P'_1 \subseteq P' \subseteq P \) and not containing \( I \), and \( P \) be minimal such that \( P'_1 \subseteq P' \subseteq P \). Therefore \( P' \) does not contain \( I \), \( P \) contains \( I \), \( P' \subseteq P \) are consecutive and the chain \( P'_1 \subseteq P' \subseteq P \) lifts in \( T[X] \) as \( Q'_1 = (X - 1)T[X] \subseteq Q' \subseteq Q \). It is clear that \( Q \cap T \) contains \( I \), and as \( I \) is a maximal ideal of \( T \), then \( Q \cap T = M \). Moreover, since \( Q \) contains \( X - 1 \), then \( Q \) is an upper to \( M \). The prime ideal \( P \) is above \( p = M \cap R \). We claim that \( P \) is an upper to \( p \). Consider the polynomial \( f = (X - 1)^2 = X^2 - 2X + 1 \). It is obvious that \( f \in P'_1 = ((X - 1)T[X]) \cap R[X] \) and \( f \not\in p[X] \). Hence \( f \in P \setminus p[X] \). Therefore \( P \) is an upper to \( p \) as claimed. Since \( R \subseteq T \) is \( I \)-algebraic modulo \( I \), it results that \( T[X]/Q \) is algebraic over \( R[X]/P \). As \( Q \) and \( P \) are upperly related to \( M \) and \( p \), it follows that \( T/M \) is algebraic.
(2)⇒(3) Let $q \in \text{Spec}(T)$. Our purpose is to show that $R/(q \cap R) \subseteq T/q$ is an algebraic extension. If $I \nsubseteq q$, then $T_q \cong R_q \cap R$ (see [4, Proposition 0]). So $\text{tr.deg}[T/q : R/(q \cap R)] = 0$. If $I \subseteq q$, then $q \in \mathcal{O}$. Hence $\text{tr.deg}[T/q : R/(q \cap R)] = 0$.

(3)⇒(4)⇒(5) are trivial.

(5)⇒(1) The conclusion is clear if $n = 1$. So assume that $n \geq 2$. The conclusion follows readily from Lemma 1.

(5)⇒(6) Follows readily from Lemma 1.

(b) We now assume that $I \in \text{Max}(T)$. We will prove that (6)⇒(2). To this end, we have only to show that $\text{tr.deg}[T/I : R/I] = 0$. Let $q'$ be a prime ideal of $T$ such that $q' \subseteq I$ are consecutive in $T$ (such ideal exists since $T$ is finite-dimensional). Let $p' = q' \cap R$, then $p' \subseteq I$ are also consecutive in $R$. Indeed, assume that there exists a prime ideal $p$ of $R$ such that $p' \subseteq p \subseteq I$. This chain lifts in $T$ to $q' \subseteq q \subseteq I$ (notice that the unique prime ideal of $T$ lying over $I$ is $I$ itself since $I \in \text{Max}(T)$). The desired contradiction since $q' \subseteq I$ are consecutive. As $R \subseteq T$ is 0-algebraic modulo $I$, then $\text{tr.deg}[T/I : R/I] = 0$, as asserted. 

Remark 1. If we leave out the assumption “$I \in \text{Max}(T)$” in the statement of Theorem 1 (b), the conclusion does not hold. More precisely, Fontana et al (see [8, Exemple 1.8]) have already constructed a diagram $(\square \cap)$, where $I$ is an intersection of two maximal ideals of $T$, such that $R \subseteq T$ is 0-algebraic modulo $I$, whereas $R \subseteq T$ is not 1-algebraic modulo $I$.

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