Miskolc Mathematical Notes
Vol. 16 (2015), No. 2, pp. 769–779
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2015.1343

LIE IDEALS AND ACTION OF GENERALIZED DERIVATIONS IN RINGS

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Received 19 September, 2014

Abstract. Let R be a prime ring of characteristic not 2, U a nonzero square closed Lie ideal of R and F, G, H be the generalized derivations with associated derivations d, δ, h of R respectively. In the present paper, we study the situations if one the follwoing holds (1) $F(u)G(v) \pm H(uv) \in Z(R)$, (2) $F(u)F(v) \pm H(vu) \in Z(R)$, for all $u, v \in U$, then $U \subseteq Z(R)$.

2010 *Mathematics Subject Classification:* 16W25; 16N60.

Keywords: prime ring, derivation, generalized derivation, extended centroid, Utumi quotient ring

1. INTRODUCTION

Let R be a prime ring with center $Z(R)$. For any pair of elements $x, y \in R$, we shall write $[x, y]$ for the commutator $xy - yx$. An additive subgroup U of R is said to be a Lie ideal of R, if $[U, R] \subseteq U$. The centralizer of U is denoted by $C_R(U)$ and defined by $C_R(U) = \{x \in R \mid [x, U] = 0\}$. An additive mapping $d : R \to R$ is called a derivation, if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. By a generalized inner derivation on R, one usually means an additive mapping $F : R \to R$ if $F(x) = ax + xb$ for fixed $a, b \in R$. For such a mapping F, it is easy to see that $F(xy) = F(x)y + x[y, b] = F(x)y + xI_b(y)$, where I_b is an inner derivation determined by b. This observation leads to the definition given in [8] : an additive mapping $F : R \to R$ is called generalized derivation associated with a derivation d if $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Obviously any derivation is a generalized derivation. Other basic examples of generalized derivations are the following: (i) $F(x) = ax + xb$ for $a, b \in R$; (ii) $F(x) = ax$ for some $a \in R$. Clearly, if $d = 0$, then F is a left multiplier map of R. An additive subgroup U of R is said to be a Lie ideal if $[u, r] \in U$ for all $u \in U$ and $r \in R$. A Lie ideal U of R is said to be square closed if $u^2 \in U$ for all $u \in U$.

In [5], Ashraf and Rehman established that a prime ring R with a nonzero ideal I must be commutative, if R admits a nonzero derivation d satisfying $d(xy) + xy \in Z(R)$ for all $x, y \in I$ or $d(xy) - xy \in Z(R)$ for all $x, y \in I$. Recently in [4] Ashraf et al. studied the case by replacing derivation d with a generalized derivation F in a prime ring R. More precisely, they proved that the prime ring R with a nonzero ideal I must be commutative, if R admits a generalized derivation F associated with a nonzero derivation d satisfying any one of the following situations: (i) $F(xy)-xy \in Z(R)$, (ii) $F(xy)+xy \in Z(R)$, (iii) $F(xy)-yx \in Z(R)$, (iv) $F(xy)+yx \in Z(R)$, (v) $F(x)F(y)-xy \in Z(R)$, (vi) $F(x)F(y)+xy \in Z(R)$; for all $x,y \in I$. In several papers, all these identities are also investigated in some appropriate subsets of prime and semiprime rings. For further details, we refer to [1, 3, 13, 14, 17, 18, 20]. Golbasi and Koc [13] studied all the cases (i) - (vi) in a square closed Lie ideal U in a 2-torsion free prime ring R and obtained that if $d \neq 0$, then $U \subseteq Z(R)$. It is natural to consider the situation $F(x)F(y) \pm yx \in Z(R)$ for all x, y in some suitable subset of R. Recently, in [11], Dhara et al. considered this situation in a square closed Lie ideal U in a 2-torsion free prime ring R and obtained that if $d \neq 0$, then $U \subseteq Z(R)$.

The present paper is motivated by the previous results and our aim is to generalizes all the above results by considering three generalized derivations.

2. Preliminaries

Let U be a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. Therefore, for any $u,v \in U$, we get $uv+vu = (u+v)^2-u^2-v^2 \in U$. Again in the same way, we have $uv-vu \in U$. Combining these two we get $2uv \in U$ for all $u,v \in U$.

Following results are needed for the proof of our main results.

Lemma 1 ([19, Lemma 2.6]). *Let R be a prime ring with char $(R) \neq 2$. If U is a commutative Lie ideal of R, then $U \subseteq Z(R)$.*

Lemma 2 ([7, Lemma 4]). *Let R be a prime ring with char $(R) \neq 2$. If $U \not\subseteq Z(R)$ is a Lie ideal of R and $aUb = 0$, then either $a = 0$ or $b = 0$.*

Lemma 3 ([15, Theorem 5]). *Let R be a prime ring with char $(R) \neq 2$. If d be a nonzero derivation of R and U be a nonzero Lie ideal of R such that $[u, d(u)] \in Z(R)$ for all $u \in U$, then $U \subseteq Z(R)$.*

Lemma 4 ([12, Theorem 1])). *Let R be a prime ring with char $(R) \neq 2$. If d be a nonzero derivation of R and U be a nonzero Lie ideal of R such that $u[[d(u),u],u] = 0$ for all $u \in U$, then $U \subseteq Z(R)$.*

Lemma 5 ([9, Lemma 2]). *If R is prime with a nonzero central ideal, then R is commutative.*

Lemma 6 ([6, Theorem 4]). *Let R be a prime ring and I be a nonzero left ideal of R. If R admits a nonzero derivation d which is centralizing on I, then R is commutative.*

Lemma 7 ([16, Theorem 2]). *Let R be a prime ring with a nonzero derivation d of R and I a nonzero ideal of R. If $x^p[d(x^q), x^r]_k = 0$ for all $x \in I$, where p,q,r,k are fixed positive integers, then R must be commutative.*

3. Results on Lie ideals in prime rings

Theorem 1. *Let R be a prime ring of characteristic not 2, U a nonzero square closed Lie ideal of R, and F, G and H generalized derivations associated to the derivations d, δ and h of R respectively. Suppose that $F(u)G(v) - H(uv) \in Z(R)$ for all $u, v \in U$. If $d \neq 0$ and $\delta \neq 0$, then $U \subseteq Z(R)$.*

Proof. We assume that $U \nsubseteq Z(R)$ and prove that a contradiction. Now by the given hypothesis we have

$$F(u)G(v) - H(uv) \in Z(R) \text{ for all } u, v \in U. \tag{3.1}$$

Replacing v by $2vw$ in (3.1) we get

$$2(F(u)(G(v)w + v\delta(w)) - H(uv)w - uvh(w)) \in Z(R) \text{ for all } u, v, w \in U.$$

Since char $(R) \neq 2$, this gives $(F(u)(G(v)w + v\delta(w)) - H(uv)w - uvh(w)) \in Z(R)$ that is,

$$(F(u)G(v) - H(uv))w + F(u)v\delta(w) - uvh(w) \in Z(R) \text{ for all } u, v, w \in U. \tag{3.2}$$

Commuting with w, we get

$$[(F(u)G(v) - H(uv))w, w] + [F(u)v\delta(w) - uvh(w), w] = 0 \text{ for all } u, v, w \in U. \tag{3.3}$$

Since $F(u)G(v) - H(uv) \in Z(R)$ for all $u, v \in U$, above relation reduces to

$$[F(u)v\delta(w) - uvh(w), w] = 0 \text{ for all } u, v, w \in U. \tag{3.4}$$

Now, replacing u by $2ux$ in (3.4) and then using the restriction on characteristic, we obtain

$$[(F(u)x + ud(x))v\delta(w) - uxvh(w), w] = 0 \text{ for all } u, v, w, x \in U. \tag{3.5}$$

Again, putting $v = 2xv$ in (3.4) we get

$$[F(u)xv\delta(w) - uxvh(w), w] = 0 \text{ for all } u, v, w, x \in U. \tag{3.6}$$

Subtracting (3.6) from (3.5), we have

$$[ud(x)v\delta(w), w] = 0 \text{ for all } u, v, w, x \in U. \tag{3.7}$$

Replacing u by $2tu$ and using (3.7) and char $(R) \neq 2$, we get

$$\begin{aligned} 0 &= [tud(x)v\delta(w), w] \\ &= t[ud(x)v\delta(w), w] + [t, w]ud(x)v\delta(w) \\ &= [t, w]ud(x)v\delta(w) \text{ for all } u, v, w, x, t \in U. \end{aligned} \tag{3.8}$$

By Lemma 2, for each $w \in U$, either $[t, w] = 0$ for all $t \in U$ or $d(x)v\delta(w) = 0$ for all $x, v \in U$. Let $T_1 = \{w \in U \mid [U, w] = (0)\}$ and $T_2 = \{w \in U \mid d(U)U\delta(w) = (0)\}$. Then T_1 and T_2 are two additive subgroups of U such that $T_1 \cup T_2 = U$. Since a group cannot be union of its two proper subgroups, therefore either $T_1 = U$ or $T_2 = U$.

Let $T_1 = U$. Then $[U,U] = 0$ implying by Lemma 1 that $U \subseteq Z(R)$, a contradiction. Now let $T_2 = U$. Then $d(U)U\delta(U) = 0$. Again by Lemma 2, either $d(U) = 0$ or $\delta(U) = 0$. By Lemma 3, both of these imply $U \subseteq Z(R)$, a contradiction. □

Theorem 2. *Let R be a prime ring of characteristic not 2, U a nonzero square closed Lie ideal of R and F, G and H generalized derivations associated to the derivations d, δ and h of R respectively. Suppose that $F(u)G(v) + H(uv) \in Z(R)$ for all $u,v \in U$. If $d \neq 0$ and $\delta \neq 0$, then $U \subseteq Z(R)$.*

Proof. We note that $-H$ is a generalized derivations of R with associated derivations $-h$. Hence replacing H by $-H$ in Theorem 1, we have $F(u)G(v) - (-H)uv \in Z(R)$ for all $u,v \in U$, that is $F(u)G(v) + H(uv) \in Z(R)$ for all $u,v \in U$ implies $U \subseteq Z(R)$. □

In particular, when $F = d$ and $G = \delta$ are two nonzero derivations of R, then we have the following corollary:

Corollary 1. *Let R be a prime ring of characteristic not 2, U a nonzero square closed Lie ideal of R, d,δ two nonzero derivation of R and H a generalized derivation associated to the derivation h of R. If $d(u)\delta(v) \pm H(uv) \in Z(R)$ for all $u,v \in U$, then $U \subseteq Z(R)$.*

In particular, when H is an identity map, then we have the following:

Corollary 2. *Let R be a prime ring of characteristic not 2, U a nonzero square closed Lie ideal of R and F,G generalized derivations associated with the derivations d and δ of R respectively. Suppose that $F(u)G(v) \pm uv \in Z(R)$ for all $u,v \in U$. If $d \neq 0$ and $\delta \neq 0$, then $U \subseteq Z(R)$.*

Theorem 3. *Let R be a prime ring of characteristic not 2, U a nonzero square closed Lie ideal of R and F,G generalized derivations associated with the derivations d and δ of R respectively. Suppose that $F(u)F(v) - H(vu) \in Z(R)$ for all $u,v \in U$. If $d \neq 0$, then $U \subseteq Z(R)$.*

Proof. On contrary assume that $U \not\subseteq Z(R)$. To prove our theorem we have to prove that this assumption leads to a contradiction. By the hypothesis, we have

$$F(u)F(v) - H(vu) \in Z(R) \text{ for all } u,v \in U. \tag{3.9}$$

Putting $v = 2vw$ in (3.9)and using char$(R) \neq 2$, we have

$$F(u)(F(v)w + vd(w)) - H(v)wu - v\delta(wu) \in Z(R) \tag{3.10}$$

which gives

$$F(u)F(v)w + F(u)vd(w) - H(v)wu - v\delta(wu) \in Z(R). \tag{3.11}$$

Commuting with w, we have

$$[F(u)F(v)w + F(u)vd(w) - H(v)wu - v\delta(wu), w] = 0 \tag{3.12}$$

i.e.,

$$[F(u)F(v),w]w+[F(u)vd(w),w]-[H(v)wu,w]-[v\delta(wu),w]=0. \tag{3.13}$$

From (3.9), we can write that $[F(u)F(v)-H(vu),w]=0$ for all $u,v,w\in U$, that is, $[F(u)F(v),w]=[H(vu),w]$ for all $u,v,w\in U$. Thus (3.13) reduces to

$$[H(vu),w]w+[F(u)vd(w),w]-[H(v)wu,w]-[v\delta(wu),w]=0. \tag{3.14}$$

Putting $u=w^2$ in (3.14), we have

$$\begin{aligned}&[H(v)w^2+v\delta(w^2),w]w+[(F(w)w+wd(w))vd(w),w]\\&\qquad-[H(v)w^3,w]-[v\delta(w^3),w]=0,\end{aligned} \tag{3.15}$$

i.e.,

$$[(F(w)w+wd(w))vd(w),w]-[vw^2\delta(w),w]=0 \text{ for all } v,w\in U. \tag{3.16}$$

Putting $v=2wv$ and $u=w$ in (3.14), then using $\operatorname{char}(R)\neq 2$, we have

$$[H(wvw),w]w+[F(w)wvd(w),w]-[H(wv)w^2,w]-[wv\delta(w^2),w]=0 \tag{3.17}$$

i.e.,

$$[F(w)wvd(w),w]-[wvw\delta(w),w]=0 \text{ for all } u,v,w\in U. \tag{3.18}$$

Subtracting (3.18) from (3.16), we get

$$[wd(w)vd(w),w]-[vw^2\delta(w),w]+[wvw\delta(w),w]=0 \text{ for all } v,w\in U. \tag{3.19}$$

Now putting $v=2wv$ in (3.19) and using $\operatorname{char}(R)\neq 2$ we get

$$[wd(w)wvd(w),w]-w[vw^2\delta(w),w]+w[wvw\delta(w),w]=0 \text{ for all } v,w\in U. \tag{3.20}$$

Left multiplying (3.19) by w and then subtracting from (3.20), we get

$$[w[d(w),w]vd(w),w]=0 \text{ for all } v,w\in U. \tag{3.21}$$

Replacing v with $2vw$ in (3.21) and using $\operatorname{char}(R)\neq 2$, we have

$$[w[d(w),w]vwd(w),w]=0 \text{ for all } v,w\in U. \tag{3.22}$$

Now right multiplying (3.21) by w and then subtracting from (3.22), we have

$$[w[d(w),w]v[d(w),w],w]=0 \tag{3.23}$$

and again replacing v with $2vw$, we get

$$[w[d(w),w]vw[d(w),w],w]=0 \text{ for all } v,w\in U, \tag{3.24}$$

i.e.,

$$w[d(w),w]vw[d(w),w]w-w^2[d(w),w]vw[d(w),w]=0 \text{ for all } v,w\in U. \tag{3.25}$$

Now we put $v=8vw[d(w),w]u$ in (3.25) and using $\operatorname{char}(R)\neq 2$, obtain

$$\begin{aligned}&w[d(w),w]vw[d(w),w]uw[d(w),w]w\\&\qquad-w^2[d(w),w]vw[d(w),w]uw[d(w),w]=0 \text{ for all } u,v,w\in U.\end{aligned}$$

By (3.25), this can be written as

$$w[d(w),w]vw^2[d(w),w]uw[d(w),w]-w[d(w),w]vw[d(w),w]wuw[d(w),w]=0$$

i.e.,

$$w[d(w),w]vw[[d(w),w],w]uw[d(w),w]=0 \text{ for all } u,v,w\in U.$$

By Lemma 2, this implies that $w[[d(w),w],w]=0$ for all $w\in U$. Then, by Lemma 4, we have $U\subseteq Z(R)$, a contradiction. □

Theorem 4. *Let R be a prime ring of characteristic not 2, U a nonzero square closed Lie ideal of R, and F, G are two generalized derivations associated to the derivations d and δ of R respectively. Suppose that $F(u)F(v)+H(vu)\in Z(R)$ for all $u,v\in U$. If $d\neq 0$, then $U\subseteq Z(R)$.*

Proof. Replacing H by $-H$ and h by $-h$ in Theorem 3, we get our conclusion. □

In particular, when $F=d$ is a nonzero derivation of R, then we have the following corollary:

Corollary 3. *Let R be a prime ring of characteristic not 2, U a nonzero square closed Lie ideal of R, d a nonzero derivation of R and H a generalized derivation associated to the derivation h of R. If $d(u)d(v)\pm H(vu)\in Z(R)$ for all $u,v\in U$, then $U\subseteq Z(R)$.*

In particular, when H is identity map of R, then we have the following corollary.

Corollary 4. *Let R be a prime ring of characteristic not 2, U a nonzero square closed Lie ideal of R and F a generalized derivation of R associated to the nonzero derivation d of R. If $F(u)F(v)\pm vu\in Z(R)$ for all $u,v\in U$, then $U\subseteq Z(R)$.*

We know that any both sided ideal is also a Lie ideal of R. If R is a prime ring and I is a nonzero ideal of R, then $aIb=0$ implies either $a=0$ or $b=0$. Moreover, similar Lemmas are holds for both sided ideals (see Lemma 5, Lemma 6 and Lemma 7) in prime rings without assumption of char $(R)\neq 2$, therefore, we see that if we replace Lie ideal with a both sided ideal of R in the above Theorems, then the conclusion remain valid even without assumption of characteristic on R. Thus the following corollaries are straightforward.

Corollary 5. *Let R be a prime ring, I a nonzero ideal of R and F,G and H generalized derivations associated to the derivations d, δ and h of R respectively. Suppose that $F(x)G(y)\pm H(xy)\in Z(R)$ for all $x,y\in I$. If $d\neq 0$ and $\delta\neq 0$, then R must be commutative.*

Corollary 6. *Let R be a prime ring, I a nonzero ideal of R and F and H generalized derivations associated to the derivations d and δ of R respectively. Suppose that $F(x)F(y)\pm H(yx)\in Z(R)$ for all $x,y\in I$. If $d\neq 0$, then R must be commutative.*

4. Results on semiprime rings with identity element

In this section we discussed the identity $F(x^n y^m) = F(x^n)F(y^m)$ for all $x, y \in R$. Let us introduce some well known and elementary definitions for the sake of completeness. For any nonempty subset S of R. If $F(xy) = F(x)F(y)$ or $F(xy) = F(y)F(x)$ for all $x, y \in S$, then F is called a generalized derivation which acts as a homomorphism or an anti-homomorphism on S, respectively.

Before the beginning our proofs, we would like to recall Ali et al. results, more precisely we refer to Theorem 4.1 and Theorem 4.3 in [2]. All that we need here is to remind the conclusions contained in [2] in the case F is a generalized derivation associated with derivation d in semiprime ring, because for $x, y, z \in R$, $F((xy)z) = F(x(yz))$ implies $F(xy)z + xyd(z) = F(x)yz + xd(yz)$, that is, $F(x)yz + xd(y)z + xyd(z) = F(x)yz + xd(yz)$, implying $R(d(yz) - d(y)z - yd(z)) = (0)$. Since R is semiprime ring, this implies that d is a derivation of R and hence F is a generalized derivation of R.

We summarize these reduced results in the following lemmas:

Lemma 8 ([2, Theorem 4.1]). *Let R be an $n!$-torsion free semiprime ring with identity 1, where $n \geq 2$ is a fixed integer and let $F, d : R \to R$ be additive mappings such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. If $F((xy)^n) = F(x^n y^n)$ holds for all $x, y \in R$, then $[d(x), x] = 0$ for all $x \in R$.*

Moreover, if R is prime and d is a nonzero derivation of R, then R is commutative.

Lemma 9 ([2, Theorem 4.3]). *Let R be a $(m \vee n)!$-torsion free semiprime ring with identity 1, where m and n are positive integers and $F, d : R \to R$ be additive mappings such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. If $F(x^m y^n) = F(y^n x^m)$ for all $x, y \in R$, then $[d(x), x] = 0$ for all $x \in R$.*

Moreover, if R is prime and d is a nonzero derivation of R, then R is commutative.

Lemma 10 ([10, Theorem 2.2]). *Let R be a semiprime ring, I a nonzero ideal of R and F a nonzero generalized derivation of R associated with a derivation d. If $F(xy) = F(x)F(y)$ for all $x, y \in I$, then $d(I) = 0$ and F is a commuting left multiplier mapping on I.*

In particular, if R is a prime ring, then $d = 0$ and F is identity mapping of R.

Lemma 11 ([10, Theorem 2.4]). *Let R be a semiprime ring, I a nonzero ideal of R and F a nonzero generalized derivation of R associated with a derivation d. If $F(xy) = F(y)F(x)$ for all $x, y \in I$, then $d(I) = 0$ or R contains a nonzero central ideal.*

In particular, if R is a prime ring, then R is commutative and and F is left multiplier mapping of R.

We are now ready to prove our theorems.

Theorem 5. *Let R be a $(m \vee n)!$-torsion free semiprime ring with identity 1, where m and n are two fixed positive integers and F a nonzero generalized derivation associated with a derivation d of R . If $F(x^n y^m) = F(x^n)F(y^m)$ for all $x, y \in R$, then $d = 0$. In particular, if R is a prime ring, then $d = 0$ and F is a commuting left multiplier mapping of R.*

In particular, if R is a prime ring, then F is identity mapping of R.

Proof. We have the relation

$$F(x^n y^m) = F(x^n)F(y^m) \tag{4.1}$$

for all $x, y \in R$. In particular, when $x = 1$, we have from above that

$$F(y^m) = F(1)F(y^m) \tag{4.2}$$

for all $y \in R$. Now replacing x by $x + k1$ in (4.1), where k is any positive integer, we get

$$F((x+k1)^n y^m) = F((x+k1)^n)F(y^m)$$

for all $x, y \in R$. Expanding the power values of $(x + k1)$, we have

$$\begin{aligned} F\left(\left\{x^n + \binom{n}{1}kx^{n-1} + \binom{n}{2}k^2x^{n-2} + \cdots + \binom{n}{n-1}k^{n-1}x + k^n 1\right\} y^m\right) \\ = F\left(\left\{x^n + \binom{n}{1}kx^{n-1} + \binom{n}{2}k^2x^{n-2} + \right.\right. \\ \left.\left. \cdots + \binom{n}{n-1}k^{n-1}x + k^n 1\right\}\right) F(y^m) \end{aligned} \tag{4.3}$$

for all $x, y \in R$. Using relation (4.1) and (4.2), this can be written as

$$kf_1(x,y) + k^2 f_2(x,y) + \cdots + k^{n-1} f_{n-1}(x,y) = 0 \tag{4.4}$$

for all $x, y \in R$. Now, replacing k by $1, 2, 3, \ldots, n-1$ in turn, and considering the resulting system of $n-1$ homogeneous equations, we see that the coefficient matrix of the system is a Van der Monde matrix

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 2 & 2^2 & 2^3 & \cdots & 2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ (n-1) & (n-1)^2 & (n-1)^3 & \cdots & (n-1)^{n-1} \end{pmatrix}.$$

Since the determinant of the matrix is equal to a product of positive integers, each of which is less than $n-1$, and since R is $(n-1)!$-torsion free, it follows immediately that

$$f_1(x,y) = f_2(x,y) = \cdots = f_{n-1}(x,y) = 0$$

for all $x \in R$. Now, $f_{n-1}(x,y) = 0$ implies that

$$F\left(\binom{n}{n-1} x y^m\right) = F\left(\binom{n}{n-1} x\right) F(y^m) \tag{4.5}$$

for all $x, y \in R$. Which gives

$$nF(xy^m) = nF(x)F(y^m) \tag{4.6}$$

for all $x, y \in R$. Since R is n-torsion free, we have

$$F(xy^m) = F(x)F(y^m) \tag{4.7}$$

for all $x, y \in R$. Again, since R is $m!$-torsion free, by applying the same argument for y as above for x, we can write that

$$F(xy) = F(x)F(y) \tag{4.8}$$

for all $x, y \in R$. Then by Lemma 10, $d = 0$ and F is a commuting left multiplier mapping of R.

In particular, if R is a prime ring, then F is identity mapping of R. □

By the similar proof of Theorem 5, following theorem is straight forward by using Lemma 11.

Theorem 6. *Let R be a $(m \vee n)!$-torsion free semiprime ring with identity 1, where m and n are two fixed positive integers and F a nonzero generalized derivation of R associated with a derivation d. If $F(x^n y^m) = F(y^n)F(x^m)$ for all $x, y \in R$, then $d = 0$ or R contains a nonzero central ideal.*

In particular, if R is a prime ring, then R is commutative and and F is left multiplier mapping of R.

5. Some examples

This section contains two examples which shows that the main results are not true in the case of arbitrary rings.

Example 1. Let $\mathbb{Z}$ be the ring of integers. Consider

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{Z} \right\} \quad \text{and} \quad U = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} : b \in \mathbb{Z} \right\}.$$

Clearly, R is a ring with identity under the natural operations which is not prime. Define the maps on R as follows

$$F\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 2b \\ 0 & 0 \end{pmatrix}, \quad d\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix};$$

$$G\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & b+a \\ 0 & 0 \end{pmatrix}, \quad \delta\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & a-c \\ 0 & 0 \end{pmatrix};$$

$$H\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix}, \quad h\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}.$$

Then, it is easy to see that U is a nonzero square closed Lie ideal of R and F, G, and H are generalized derivations associated with nonzero derivations d, δ, and h of R respectively. Moreover, F, G and H satisfies the requirements of Theorems 1, 2, 3, and 4, but $U \nsubseteq Z(R)$. Hence, the hypothesis of primeness is crucial.

Example 2. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$. Clearly, R is a ring with identity which is not semiprime as $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} R \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = (0)$ for $b \neq 0$. Define $F, d : R \to R$ such that $F\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, and $d\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}$ for all $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R$. Then, it is easy to see that F is a generalized derivation associated with derivation d of R. Further, for any $x, y \in R$ the following conditions: $F(x^n y^m) = F(x^n)F(y^m)$, $F(x^n y^m) = F(y^n)F(x^m)$ are satisfied, where m, n are positive integers. However, $d \neq 0$. Hence, in Theorems 5 and 6, the hypothesis of semiprimeness can not be omitted.

ACKNOWLEDGEMENT

The authors wishes to thank the referee for his/her valuable comments and suggestions.

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