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ON STRONG COMMUTATIVITY PRESERVING LIKE MAPS IN RINGS WITH INVOLUTION

SHAKIR ALI, NADEEM AHMAD DAR, AND ABDUL NADIM KHAN

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Abstract. The main purpose of this paper is to prove the following result: Let R be a prime ring with involution of the second kind and with $char(R) \neq 2$. If R admits a nonzero derivation $d : R \to R$ such that $[d(x), d(x^*)] = [x, x^*]$ for all $x \in R$, then R is commutative. We also provide an example which shows that the above result does not holds in case the involution is of the first kind. Moreover, a related result has also been obtained.

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1. INTRODUCTION

Throughout this article, R will represent an associative ring with centre $Z(R)$. We denote $[x, y] = xy - yx$, the commutator of x and y and $x \circ y = xy + yx$, the anti-commutator of x and y. However, given two subsets A and B of R, $[A, B]$ will denote the additive subgroup of R generated by all elements of the form $[x, y]$ where $x \in A$ and $y \in B$ and $A \circ B$ is defined similarly. Further, $\overline{A}$ will be the subring of R generated by A. A ring R is said to be 2-torsion free if $2a = 0$ (where $a \in R$) implies $a = 0$. A ring R is called a prime ring if $aRb = (0)$ (where $a, b \in R$) implies $a = 0$ or $b = 0$, and is called a semiprime ring in case $aRa = (0)$ implies $a = 0$. An additive map $x \mapsto x^*$ of R into itself is called an involution if (i) $(xy)^* = y^*x^*$ and (ii) $(x^*)^* = x$ hold for all $x, y \in R$. An element x in a ring with involution $*$ is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. A ring equipped with an involution is known as ring with involution or $*$-ring. The involution is said to be of the first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the later case $S(R) \cap Z(R) \neq (0)$. If R is 2-torsion free then every $x \in R$ can be uniquely represented in the form $2x = h + k$, where $h \in H(R)$ and $k \in S(R)$. Note that in this case x is normal i.e., $xx^* = x^*x$, if and only if h and k commute. If all elements in R are normal, then R is called a normal ring. An example is the ring of quaternions. A description of such rings can be found in [13], where further references can be found.

An additive mapping $d : R \to R$ is said to be a derivation of R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. A derivation d is said to be inner if there exists $a \in R$ such that $d(x) = ax - xa$ for all $x \in R$. Over the last 30 years, several authors have investigated the relationship between the commutativity of the ring R and certain special types of maps on R. The first result in this direction is due to Divinsky[12], who proved that a simple artinian ring is commutative if it has a commuting non-trivial automorphim. Two years later, Posner [21] proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Over the last few decades, several authors have subsequently refined and extended these results in various directions (viz., [3, 5, 6], where further references can be found).

We say that a map $f : R \to R$ preserves commutativity if $[f(x), f(y)] = 0$ whenever $[x, y] = 0$ for all $x, y \in R$. The study of commutativity preserving mappings has been an active research area in matrix theory, operator theory and ring theory (see [8, 10] for references). Following [7], let S be a subset of R, a map $f : R \to R$ is said to be strong commutativity preserving (SCP) on S if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$. In [4], Bell and Daif investigated the commutativity in rings admitting a derivation which is SCP on a nonzero right ideal. Precisely, they proved that if a semiprime ring R admits a derivation d satisfying $[d(x), d(y)] = [x, y]$ for all x, y in a right ideal I of R, then $I \subseteq Z(R)$. In particular, R is commutative if $I = R$. Later, Deng and Ashraf [11] proved that if there exist a derivation d of a semiprime ring R and a map $f : I \to R$ defined on a nonzero ideal I of R such that $[f(x), d(y)] = [x, y]$ for all $x, y \in I$, then R contains a nonzero central ideal. In particular, they showed that R is commutative if $I = R$. Further, Ali and Huang [2], showed that if R is a 2-torsion free semiprime ring and d is a derivation of R satisfying $[d(x), d(y)] + [x, y] = 0$ for all x, y in a nonzero ideal I of R, then R contains a nonzero central ideal. Many related generalizations of these results can be found in the literature (see for instance [9, 15–20, 22, 23]).

The main purpose of the present paper is to initiate the study of a more general concept than SCP mappings. More precisely, we consider an additive mapping $f : R \to R$ satisfying $[f(x), f(x^*)] = [x, x^*]$ for all $x \in R$. In fact, we investigate the commutativity of a prime ring with involution, when the mapping f is assumed to be a derivation of R. Moreover, a related result is obtained by replacing the commutator by anti-commutator.

2. RESULTS

We begin with the following lemma, which is essentially proved in [1, Lemma 2.1].

Lemma 1. *Let R be a prime ring with involution $*$ such that $char(R) \neq 2$. If $S(R) \cap Z(R) \neq (0)$ and R is normal, then R is commutative.*

Theorem 1. *Let R be a prime ring with involution $*$ of the second kind and with $char(R) \neq 2$. Let d be a nonzero derivation of R such that $[d(x), d(x^*)] = [x, x^*]$ for all $x \in R$. Then R is commutative.*

Proof. By the assumption, we have

$$[d(x), d(x^*)] = [x, x^*] \tag{2.1}$$

for all $x \in R$. A lineralization of (2.1) yields that

$$[d(x), d(y^*)] + [d(y), d(x^*)] = [x, y^*] + [y, x^*] \tag{2.2}$$

for all $x, y \in R$. Replacing y by xx^* in (2.2) and making use of (2.1), we arrive at

$$d(x)[d(x), x^*] + [d(x), x]d(x^*) + d(x)[x^*, d(x^*)] + [x, d(x^*)]d(x^*) = 0 \tag{2.3}$$

for all $x \in R$. Replacing x by $x + h^{'}$, where $h^{'} \in H(R) \cap Z(R)$, we obtain

$$d(h^{'})[d(x), x^*] + [d(x), x]d(h^{'}) + d(h^{'})[x^*, d(x^*)] + [x, d(x^*)]d(h^{'}) = 0.$$

This can be further written as

$$d(h^{'})([d(x), x^*] + [d(x), x] + [x^*, d(x^*)] + [x, d(x^*)]) = 0$$

for all $h^{'} \in H(R) \cap Z(R)$ and $x \in R$. Since the centre of a prime ring is free from zero divisors we get either $d(h^{'}) = 0$ for all $h^{'} \in H(R) \cap Z(R)$ or $[d(x), x^*] + [d(x), x] + [x^*, d(x^*)] + [x, d(x^*)] = 0$ for all $x \in R$. Suppose

$$d(h^{'}) = 0 \text{ for all } h^{'} \in H(R) \cap Z(R). \tag{2.4}$$

Replacing $h^{'}$ by $(k^{'})^2$ in (2.4), where $k^{'} \in S(R) \cap Z(R)$, we get

$$0 = d(h^{'}) = d((k^{'})^2) = d(k^{'})k^{'} + k^{'}d(k^{'}) = 2d(k^{'})k^{'}.$$

Since $char(R) \neq 2$, we arrive at

$$d(k^{'})k^{'} = 0 \text{ for all } k^{'} \in S(R) \cap Z(R).$$

For each $k^{'} \in S(R) \cap Z(R)$, the last expression yields that either $d(k^{'}) = 0$ or $k^{'} = 0$. Since $k^{'} = 0$ implies $d(k^{'}) = 0$, we may write

$$d(k^{'}) = 0 \text{ for all } k^{'} \in S(R) \cap Z(R). \tag{2.5}$$

Let $x \in Z(R)$. Since $char(R) \neq 2$, every $x \in Z(R)$ can be represented as $2x = h + k$, where $h \in H(R) \cap Z(R)$ and $k \in S(R) \cap Z(R)$. This implies that $2d(x) = d(2x) = d(h + k) = d(h) + d(k) = 0$. Since $char(R) \neq 2$, we get

$$d(x) = 0 \text{ for all } x \in Z(R). \tag{2.6}$$

Replacing y by $k^{'}y$ in (2.2), where $k^{'} \in S(R) \cap Z(R)$ and using (2.6), we arrive at

$$k^{'}(-[d(x), d(y^*)] + [d(y), d(x^*)] + [x, y^*] - [y, x^*]) = 0$$

for all $k^{'} \in S(R) \cap Z(R)$ and $x, y \in R$. Using the primeness of R and the fact that $S(R) \cap Z(R) \neq (0)$, we get

$$-[d(x), d(y^*)] + [d(y), d(x^*)] = -[x, y^*] + [y, x^*] \tag{2.7}$$

for all $x, y \in R$. On comparing (2.2) and (2.7), we obtain $2[d(x), d(y^*)] = 2[x, y^*]$ for all $x, y \in R$. Replacing y by y^* and using the fact that $char(R) \neq 2$, we conclude that $[d(x), d(y)] = [x, y]$ for all $x, y \in R$. Therefore in view of [4, Theorem 1], R is commutative. Now we consider the case

$$[d(x), x^*] + [d(x), x] + [x^*, d(x^*)] + [x, d(x^*)] = 0$$

for all $x \in R$. Replacing x by $h + k$, where $h \in H(R)$ and $k \in S(R)$, we obtain $4[d(k), h)] = 0$. Since $char(R) \neq 2$, we are force to conclude that

$$[d(k), h] = 0 \text{ for all } h \in H(R) \; and \; k \in S(R). \tag{2.8}$$

Replacing h by $k_0 k^{'}$, where $k_0 \in S(R)$ and $k^{'} \in S(R) \cap Z(R)$, we arrive at $([d(k), k_0])k^{'} = 0$. Since R is prime and $S(R) \cap Z(R) \neq (0)$, we get

$$[d(k), k_0] = 0 \; for \; all \; k, k_0 \in S(R). \tag{2.9}$$

Now since $char(R) \neq 2$, every $x \in R$ can be represented as $2x = h + k$, where $h \in H(R), k \in S(R)$, so in view of equations (2.8) and (2.9), we are force to conclude that

$$[d(k), x] = 0 \; for \; all \; k \in S(R) \; and \; x \in R. \tag{2.10}$$

for all $k \in S(R)$ and $x \in R$. That is, $d(k) \in Z(R)$ for all $k \in S(R)$. First we assume that $d(S(R)) = (0)$. Then, we have $d(x - x^*) = 0$ for all $x \in R$. That is, $d(x) = d(x^*)$ for all $x \in R$. Thus, we have $0 = [d(x), d(x^*)] = [x, x^*]$ for all $x \in R$. Hence R is normal and we are done by Lemma 1. Now suppose that $d(S(R)) \neq (0)$. For $k_o \in S(R)$ with $d(k_o) \neq 0$ and $k \in [S(R), S(R)]$, we have $d(k_o k k_o) \in Z(R)$. The last expression can be written as $d(k_o) k k_o + k_o k d(k_o) \in Z(R)$, since $d([S(R), S(R)]) = (0)$. Thus $d(k_o)(k_o k + k k_o) \in Z(R)$ and hence $k_o k + k k_o \in Z(R)$ for all $k \in [S(R), S(R)]$. This implies that $d(k_o k + k k_o) \in Z(R)$ and hence $2d(k_o)k \in Z(R)$. Since $char(R) \neq 2$ and R is prime, the above relation yields that $k \in Z(R)$. That is, $[S(R), S(R)] \subseteq Z(R)$. Suppose $[S(R), S(R)] \neq (0)$ and $k, k_o \in S(R)$ such that $[k, k_o] \neq 0$. Since $k k_o k \in S(R)$, we have $[k, k k_o k] = k[k, k_o]k = k^2[k, k_o] \in Z(R)$. This implies that $k^2 \in Z(R)$ and hence $k \in Z(R)$ for all $k \in S(R)$ as proved earlier. Therefore, R is commutative in view of Lemma 1. Now suppose $[S(R), S(R)] = (0)$. Since $\overline{S(R)^2}$ is both a Lie ideal and a commutative subring of R, by [13, Theorem 2.1.2], $k^2 \in Z(R)$ for all $k \in S(R)$ and hence $k \in Z(R)$ for all $k \in S(R)$. Thus, R is normal and hence R is commutative by Lemma 1. This completes the proof of the theorem. □

If we replace commutator by anti-commutator in Theorem 1, the corresponding result also holds.

Theorem 2. *Let R be a prime ring with involution $*$ of the second kind and with $char(R) \neq 2$. Let d be a nonzero derivation of R such that $d(x) \circ d(x^*) = x \circ x^*$ for all $x \in R$. Then R is commutative.*

Proof. By the given hypothesis, we have $d(x) \circ d(x^*) = x \circ x^*$ for all $x \in R$. This can be further written as

$$d(x)d(x^*) + d(x^*)d(x) = xx^* + x^*x \tag{2.11}$$

for all $x \in R$. A lineralization of (2.11) yields that

$$d(x)d(y^*) + d(y)d(x^*) + d(x^*)d(y) + d(y^*)d(x) = xy^* + yx^* + x^*y + y^*x \tag{2.12}$$

for all $x, y \in R$. Replacing y by $h^{'}x$, $(h^{'} \in H(R) \cap Z(R))$ in (2.12) and using (2.11), we get

$$d(h^{'})d(x)x^* + d(h^{'})xd(x^*) + d(h^{'})d(x^*)x + d(h^{'})x^*d(x) = 0.$$

That is, $d(h^{'})d(x \circ x^*) = 0$ for all $h^{'} \in H(R) \cap Z(R)$ and $x \in R$. Since the centre of a prime ring is free from zero divisors, we have either $d(h^{'}) = 0$ for all $h^{'} \in H(R) \cap Z(R)$ or $d(x \circ x^*) = 0$ for all $x \in R$. Suppose $d(h^{'}) = 0$ for all $h^{'} \in H(R) \cap Z(R)$. This further implies that $d(x) = 0$ for all $x \in Z(R)$. Replacing y by $k^{'}y$, $(k^{'} \in S(R) \cap Z(R))$ in (2.12) and using the fact $d(x) = 0$ for all $x \in Z(R)$, we obtain

$$\begin{aligned} k^{'}(-d(x)d(y^*) + d(y)d(x^*) + d(x^*)d(y) - d(y^*)d(x)) = \\ = k^{'}(-xy^* + yx^* + x^*y - y^*x). \end{aligned}$$

Again using the primeness of R and since $S(R) \cap Z(R) \neq (0)$, we arrive at

$$\begin{aligned} -d(x)d(y^*) + d(y)d(x^*) + d(x^*)d(y) - d(y^*)d(x) \\ = -xy^* + yx^* + x^*y - y^*x \end{aligned} \tag{2.13}$$

for all $x, y \in R$. Comparing (2.12) and (2), we get $2(d(x)d(y^*) + d(y^*)d(x)) = 2(xy^* + y^*x)$ for all $x, y \in R$. Since $char(R) \neq 2$ and replace y by y^* to get $d(x) \circ d(y) = x \circ y$ for all $x, y \in R$. Hence, R is commutative in view of [3, Theorem 4.4]. On the other hand, suppose $d(x \circ x^*) = 0$ for all $x \in R$. The above equation can be further written as

$$d(x)x^* + xd(x^*) + d(x^*)x + x^*d(x) = 0 \tag{2.14}$$

for all $x \in R$. Replacing x by $h \in H(R) \cap Z(R)$ in (2.14), and using the fact that $char(R) \neq 2$, we obtain

$$d(h)h = 0 \text{ for all } h \in H(R) \cap Z(R).$$

Now since the centre of a prime ring is free from zero divisors, we get for each $h \in H(R) \cap Z(R)$ either $d(h) = 0$ or $h = 0$. Since $h = 0$ implies $d(h) = 0$, we may write $d(h) = 0$ for all $h \in H(R) \cap Z(R)$ and hence $d(x) = 0$ for all $x \in Z(R)$. Linearizing (2.14), we obtain

$$d(x)y^* + d(y)x^* + xd(y^*) + yd(x^*) + d(x^*)y + d(y^*)x + x^*d(y) + y^*d(x) = 0 \tag{2.15}$$

for all $x, y \in R$. Replacing y by $y_o \in Z(R)$ in (2.15) and using the fact that $d(x) = 0$ for all $x \in Z(R)$, we get

$$d(x)y_o^* + y_o d(x^*) + d(x^*)y_o + y_o^* d(x) = 0 \tag{2.16}$$

for all $y_o \in Z(R)$ and $x \in R$. In particular, taking $y_o = h_o \in H(R) \cap Z(R)$ in (2.16), we get $2(d(x)h_o + d(x^*)h_o) = 0$. Since $char(R) \neq 2$, we obtain $d(x)h_o + d(x^*)h_o = 0$. This can be further written as

$$d(x + x^*)h_o = 0 \tag{2.17}$$

for all $h_o \in H(R) \cap Z(R)$ and $x \in R$. Using the primeness of R, we get either $d(x + x^*) = 0$ or $H(R) \cap Z(R) = (0)$. But $H(R) \cap Z(R) = (0)$ implies that $S(R) \cap Z(R) = (0)$, which gives a contradiction since we have assumed $S(R) \cap Z(R) \neq (0)$. Therefore, we are left with the case $d(x + x^*) = 0$ for all $x \in R$. Replacing x by $h + k$ in the above equation, we get $2d(h) = 0$. This implies that $d(h) = 0$ for all $h \in H(R)$. Further $d(x + x^*) = 0$ implies that $d(x) = -d(x^*)$ for all $x \in R$. Replacing x by xh, where $h \in H(R)$ in the last expression we get $d(x)h = -hd(x^*)$, since $d(h) = 0$. This further implies that $d(x)h = hd(x)$ for all $x \in R$. Therefore in view of the theorem of [14], we conclude that $h \in Z(R)$ for all $h \in H(R)$. Hence R is commutative in view of Lemma 1. Thereby completing the proof of the theorem. □

At the end, let us write an example which shows that the restriction of second kind involution in Theorem 1 is not superfluous.

Example 1. Let $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in Z \right\}$. Of course R with matrix addition and matrix multiplication is a prime ring. Define mappings $d : R \longrightarrow R$, and $* : R \longrightarrow R$ such that $d\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix}$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Obviously, $Z(R) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \middle| \, a \in Z \right\}$. Then $x^* = x$ for all $x \in Z(R)$, and hence $Z(R) \subseteq H(R)$, which shows that the involution $*$ is of the first kind. Moreover, d is nonzero and the following condition $[d(x), d(x^*)] = [x, x^*]$ for all $x \in R$, is satisfied. However, R is not commutative. Hence, in Theorem 1, the hypothesis of second kind involution is crucial.

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Authors' addresses

Shakir Ali
Department of Mathematics, Faculty of Science, Rabigh, King Abdulaziz University, Jeddah-21589, Saudi Arabia, KSA
E-mail address: shakir.ali.mm@amu.ac.in

Nadeem Ahmad Dar
Department of Mathematics, Aligarh Muslim University, Aligarh-202002, INDIA
E-mail address: ndmdarlajurah@gmail.com

Abdul Nadim Khan
Department of Mathematics, Aligarh Muslim University, Aligarh-202002, INDIA
E-mail address: abdulnadimkhan@gmail.com