A DOUBLE INEQUALITY INVOLVING ERDŐS-BORWEIN CONSTANTS

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Abstract. In this note, the author proves a double inequality by introducing an approximation of Erdős-Borwein constants.

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1. Introduction

Recently, after an experiment, we stumbled upon the following double inequality

\[
\left( \frac{x - 1}{x + 1} \right)^{-1} < \sum_{n=1}^{\infty} \frac{1}{1 + x + x^2 + \cdots + x^n} < \left( \frac{x - 1}{x} \right)^{-1}, \quad x > 1.
\]  

(1.1)

In this paper, we present a proof of this inequality using several tools from number theory.

Recall that a Mersenne number is a positive integer that is one less than a power of two:

\[ 2^n - 1. \]

The sum of the reciprocals of all Mersenne numbers, namely

\[ E = \sum_{n=1}^{\infty} \frac{1}{2^n - 1} = 1.6066951524152917637833\ldots \]

is known as the Erdős-Borwein constant. This mysterious number is known to be irrational, as shown by Erdős [4] in 1948.

More recently, using Padé approximant techniques, P. Borwein [2] established the irrationality of more general numbers

\[ EB(x) = \sum_{n=1}^{\infty} \frac{1}{x^n - 1} \]
for $x$ a positive integer, $x > 1$. We remark that the numbers $EB(x)$ can be written as

$$EB(x) = \frac{1}{x-1} + \frac{1}{x-1} \sum_{n=1}^{\infty} \frac{1}{1+x+x^2+\ldots+x^n}, \quad x > 1.$$ 

We have the following approximation of these irrational numbers.

**Theorem 1.** Let $x$ be a real number such that $x > 1$. Then

$$\frac{1}{x} + \frac{1}{x^4} \cdot \frac{x^2+1}{x+1} < \sum_{n=1}^{\infty} \frac{1}{1+x+x^2+\ldots+x^n} \leq \frac{x}{x^2-1} - \frac{x}{x^3-1} + \frac{x}{x^4-1} - \frac{x}{x^7-1}.$$ 

It is clear that this series has an asymptotic behaviour, i.e.,

$$\sum_{n=1}^{\infty} \frac{1}{1+x+x^2+\ldots+x^n} \sim \frac{1}{x} \quad (x \to \infty).$$

The case $x = 2$ of this theorem can be written as

$$\frac{29}{48} < E - 1 < \frac{153320}{248031}.$$ 

In other words, we have

$$1.60416\ldots < E < 1.61814\ldots$$

**2. Proof of Theorem 1**

Let $n$ be a positive integer. The divisors function $\tau(n)$ is defined as the number of divisors of $n$, unity and $n$ itself included, i.e.,

$$\tau(n) = \sum_{d|n} 1.$$ 

In number theory, the inequality

$$\tau(n) \leq \lfloor 2\sqrt{n} \rfloor$$

is well-known. It is an easy exercise to show that

$$\lfloor 2\sqrt{n} \rfloor < 1 + \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{6} \right\rfloor - \left\lfloor \frac{n}{7} \right\rfloor, \quad n \geq 36$$

and

$$\tau(n) \leq 1 + \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{6} \right\rfloor - \left\lfloor \frac{n}{7} \right\rfloor, \quad n < 36.$$ 

Clearly,

$$\tau(n) \leq 1 + \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{6} \right\rfloor - \left\lfloor \frac{n}{7} \right\rfloor,$$

for any positive integer $n$. 
To prove the inequality we consider: the generating function of $\tau(n)$ [1, s. 24.3.3],
\[
\sum_{n=1}^{\infty} \tau(n)q^n = \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}, \quad |q| < 1,
\]
the generating function of 1,
\[
\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}, \quad |q| < 1,
\]
and the generating function of $[n/k]$,
\[
\sum_{n=0}^{\infty} \binom{n}{k} q^n = \frac{q^k}{(1-q)(1-q^k)}, \quad |q| < 1,
\]
where $k$ is a positive integer.

For $0 < q < 1$, we can write
\[
\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} < \sum_{n=1}^{\infty} \left(1 + \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{6} \right\rfloor - \left\lfloor \frac{n}{7} \right\rfloor\right) q^n
\]
\[
= \frac{q}{1-q} + \frac{q^2}{(1-q)(1-q^2)} - \frac{q^5}{(1-q)(1-q^5)}
\]
\[
+ \frac{q^6}{(1-q)(1-q^6)} - \frac{q^7}{(1-q)(1-q^7)}.
\]
Thus we deduce that
\[
\sum_{n=2}^{\infty} \frac{q^n}{1-q^n} < \frac{q^2}{(1-q)(1-q^2)} - \frac{q^5}{(1-q)(1-q^5)}
\]
\[
+ \frac{q^6}{(1-q)(1-q^6)} - \frac{q^7}{(1-q)(1-q^7)}, \quad 0 < q < 1
\]

or
\[
\sum_{n=1}^{\infty} \frac{q^n}{1+q+q^2+\cdots+q^n} < \frac{q}{1-q^2} - \frac{q^4}{1-q^5} + \frac{q^5}{1-q^6} - \frac{q^6}{1-q^7}, \quad 0 < q < 1.
\]

Replacing $q$ by $1/x$ in this inequality, we get
\[
\sum_{n=1}^{\infty} \frac{1}{1+x+x^2+\cdots+x^n} \leq \frac{x}{x^2-1} - \frac{x}{x^3-1} + \frac{x}{x^6-1} - \frac{x}{x^7-1}, \quad x > 1.
\]

On the other hand, due to Clausen’s [3], we have the following identity:
\[
\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} = \sum_{n=1}^{\infty} q^{n+1} \frac{1+q^n}{1-q^n}, \quad |q| < 1.
\]
For $x > 1$, we obtain
\[
\sum_{n=1}^{\infty} \frac{1}{1 + x + x^2 + \cdots + x^n} = (x - 1)EB(x) - 1
\]
\[
= (x - 1) \sum_{n=1}^{\infty} x^{-n^2} \frac{x^n + 1}{x^n - 1} - 1
\]
\[
> (x - 1) \left( \frac{1}{x} \cdot \frac{x + 1}{x - 1} + \frac{1}{x^4} \cdot \frac{x^2 + 1}{x^2 - 1} \right) - 1
\]
\[
= \frac{1}{x} + \frac{1}{x^3} \cdot \frac{x^2 + 1}{x + 1}
\]
and Theorem 1 is proved.

3. CONCLUDING REMARKS

Lower and upper bounds of Erdős-Borwein constants has been introduced in the paper. For $x > 1$, it is an easy exercise to prove that
\[
\frac{x + 1}{x^2 + x - 1} < \frac{1}{x} + \frac{1}{x^4} \cdot \frac{x^2 + 1}{x + 1}
\]
and
\[
\frac{x}{x^2 - 1} - \frac{x}{x^5 - 1} + \frac{x}{x^6 - 1} - \frac{x}{x^7 - 1} < \frac{x}{x^2 - 1}.
\]
So by Theorem 1, we derive the double inequality (1.1) that can be written in terms of Erdős-Borwein constants as follows

**Corollary 1.** For $x > 1$,
\[
\left( x - \frac{1}{x + 1} \right)^{-1} < (x - 1)EB(x) - 1 < \left( x - \frac{1}{x} \right)^{-1}.
\]

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REFERENCES


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