Miskolc Mathematical Notes
Vol. 16 (2015), No. 2, pp. 1233–1242
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2015.1102

ON THE PRIME SPECTRUM OF MODULES

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Received 18 January, 2014

Abstract. Let R be a commutative ring and let M be an R-module. Let us denote the set of all prime submodules of M by $Spec(M)$. In this article, we explore more properties of strongly top modules and investigate some conditions under which $Spec(M)$ is a spectral space.

2010 *Mathematics Subject Classification:* 13C13; 13C99

Keywords: prime spectrum, Zariski topology, spectral space

1. INTRODUCTION, ETC

Throughout this article, all rings are commutative with identity elements, and all modules are unital left modules. $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ will denote respectively the natural numbers, the ring of integers and the field of quotients of $\mathbb{Z}$. If N is a subset of an R-module M, then $N \leq M$ denotes N is an submodule of M.

Let M be an R-module. For any submodule N of M, we denote the annihilator of M/N by $(N : M)$, i.e. $(N : M) = \{r \in R | rM \subseteq N\}$. A submodule P of M is called *prime* if $P \neq M$ and whenever $r \in R$ and $e \in M$ satisfy $re \in P$, then $r \in (P : M)$ or $e \in P$.

The set of all prime submodule of M is denoted by $Spec(M)$ (or X). For any ideal I of R containing $Ann(M)$, $\overline{I}$ and $\overline{R}$ will denote $I/Ann(M)$ and $R/Ann(M)$, respectively. Also the map $\psi : Spec(M) \rightarrow Spec(\overline{R})$ given by $P \mapsto \overline{(P : M)}$ is called the *natural map* of X. M is called *primeful* (resp. X*-injective*) if either $M = \mathbf{0}$ or $M \neq \mathbf{0}$ and the natural map ψ is surjective (resp. if either $X = \varnothing$ or $X \neq \varnothing$ and natural map ψ is injective). (See [3, 11] and [13].)

The *Zariski topology* on X is the topology τ described by taking the set $\Omega = \{V(N) | N \text{ is a submodule of } M\}$ as the set of closed sets of X, where $V(N) = \{P \in X | (P : M) \supseteq (N : M)\}$ [11].

The *quasi-Zariski topology* on X is described as follows: put $V^*(N) = \{P \in X | P \supseteq N\}$ and $\Omega^* = \{V^*(N) | N \text{ is a submodule of } M\}$. Then there exists a topology τ^* on X having Ω^* as the set of it's closed subsets if and only if Ω^* is closed under the finite union. When this is the case, τ^* is called a quasi-Zariski topology on X and M is called a *top* R*-module* [15].

Let Y be a topological space. Y is irreducible if $Y \neq \varnothing$ and for every decomposition $Y = A_1 \cup A_2$ with closed subsets $A_i \subseteq Y, i = 1, 2$, we have $A_1 = Y$ or $A_2 = Y$. A subset T of Y is irreducible if T is irreducible as a space with the relative topology. For this to be so, it is necessary and sufficient that, for every pair of sets F , G which are closed in Y and satisfy $T \subseteq F \cup G$, $T \subseteq F$ or $T \subseteq G$. Let F be a closed subset of Y. An element $y \in Y$ is called a generic point of Y if $Y = cl(\{y\})$ (here for a subset Z of Y, $cl(Z)$ denotes the topological closure of Z).

A topological space X is a *spectral space* if X is homeomorphic to $Spec(S)$ with the Zariski topology for some ring S. This concept plays an important role in studying of algebraic properties of an R-module M when we a have a related topology. For an example, when $Spec(M)$ is homeomorphic to $Spec(S)$, where S is a commutative ring, we can transfer some of known topological properties of $Spec(S)$ to $Spec(M)$ and then by using these properties explore some of algebraic properties of M.

Spectral spaces have been characterized by M. Hochster as quasi-compact T_0-spaces X having a quasi-compact open base closed under finite intersection and each irreducible closed subset of X has a generic point [9, p. 52, Proposition 4].

The concept of strongly top modules was introduced in [2] and some of its properties have been studied. In this article, we get more information about this class of modules and explore some conditions under which $Spec(M)$ is a spectral space for its Zariski or quasi-Zariski topology.

In the rest of this article, X will denote $Spec(M)$. Also the set of all maximal submodules of M is denoted by $Max(M)$.

2. Main results

Definition 1 (Definition 3.1 in [1])**.** Let M be an R-module. M is called a *strongly top* module if for every submodule N of M there exists an ideal I of R such that $V^*(N) = V^*(IM)$.

Definition 2 (Definition 3.1 in [2])**.** Let M be an R-module. M is called a *strongly top* module if M is a top module and $\tau^* = \tau$.

Remark 1. Definition 1 and Definition 2 are equivalent. This follows from the fact that if N is a submodule of M, then by [11, Result 3], we have

$$V(N) = V((N : M)M) = V^*((N : M)M).$$

Remark 2 (Theorem 6.1 in [11]). Let M be an R-module. Then the following are equivalent:

(a) (X, τ) is a T_0 space;
(b) The natural map of X is injective;
(c) $V(P) = V(Q)$, that is, $(P : M) = (Q : M)$ implies that $P = Q$ for any $P, Q \in X$;

(d) $|Spec_p(M)| \leq 1$ for every $p \in Spec(R)$.

Remark 3. (a) Let M be an R-module and $p \in Spec(R)$. The saturation of a submodule N with respect to p is the contraction of N_p in M and denoted by $S_p(N)$. It is known that

$$S_p(N) = N^{ec} = \{x \in M \,|\, tx \in N \text{ for some } t \in R \setminus p\}.$$

(b) Let M be an R-module and $N \leq M$. The radical of N, denoted by $rad(N)$, is the intersection of all prime submodules of M containing N; that is, $rad(N) = \bigcap_{P \in V^*(N)} P$ ([14]).

(c) A topological space X is Noetherian provided that the open (respectively, closed) subsets of X satisfy the ascending (respectively, descending) chain condition ([4, p. 79, Exercises 5-12]).

Proposition 1. *Let M be an strongly top module and ψ be the natural map of X. Then*

(a) $(X,\tau) = (X,\tau^*) \cong Im\psi$.

(b) *If X is Noetherian, then X is a spectral space.*

Proof. (a) By [15, Theorem 3.5] and Remark 2, $\psi|_{Im\psi}$ is bijective. Also we have

$$\psi(V(N)) = \{\overline{(P:M)} | P \in X, (P:M) \supseteq (N:M)\}.$$

Now by [11, Proposition 3.1] and the above arguments, ψ is continuous and a closed map. Consequently we have $(X,\tau) = (X,\tau^*) \cong Im\psi$.

(b) Let $Y = V^*(N)$ be an irreducible closed subset of X. Now by [6, Theorem 3.4], we have

$$V^*(N) = V^*(rad(N)) = cl(\{rad(N)\}).$$

Hence Y has a generic point. Also X is Noetherian and it is a T_0-space by [6, Proposition 3.8 (i)]. Hence it is a spectral space by [9, Pages 57 and 58]. □

An R- module M is said to be a *weak multiplication module* if either $X = Spec(M) = \varnothing$ or $X \neq \varnothing$ and for every prime submodule P of M, we have $P = IM$ for some ideal I of R (see [5]).

The following theorem extends [1, Proposition 3.5], [1, Corollary 3.6], [1, Theorem 3.9 (1)], and [1, Theorem 3.9 (7)]. In fact, in part (a) of this theorem, we withdraw the restrictions of finiteness and Noetherian property from [1, Proposition 3.5] and [1, Corollary 3.6], respectively. In part (b), we remove the conditions " M is primeful " and " R is a Noetherian ring " in [1, Theorem 3.9 (1)] and instead of them, we put the weaker conditions " $Im(\psi)$ is closed in $Spec(\overline{R})$ " and " $Spec(\overline{R})$ is a Noetherian space ". In part (c), we withdraw the condition " R has Noetherian spectrum " from [1, Theorem 3.9 (7)] and put the weaker condition " the intersection of every infinite family of maximal ideals of R is zero ".

Theorem 1. *Let M be an R-module. Then we have the following.*

(a) *Let* $(M_i)_{i\in I}$ *be a family of* R*-modules and let* $M = \oplus_{i\in I} M_i$*. If* M *is an strongly top* R*-module, then each* M_i *is an strongly top* R*-module.*

(b) *If* M *be an strongly top* R*-module and* ψ *be the natural map of* X*, then we have*

(i) *If* $Im(\psi)$ *is closed in* $Spec(\overline{R})$*, then* $(X,\tau) = (X,\tau^*)$ *is a spectral space.*

(ii) *If* $Spec(\overline{R})$ *is Noetherian, then* $(X,\tau) = (X,\tau^*)$ *is a spectral space.*

(c) *Suppose* R *is a one dimensional integral domain such that the intersection of every infinite family of its maximal ideals is zero. If* M *is a weak multiplication* R*-module, then* M *is a top module.*

Proof. (a) Each M_i is a homomorphic image of M, hence it is strongly top by [1, Proposition 3.3].

(b) (i) By Proposition 1, we have $(X,\tau) = (X,\tau^*) \cong Im(\psi)$. Now the claim follows by [11, Theorem 6.7].

(ii) As $Spec(\overline{R})$ is Noetherian, $Im(\psi)$ is also Noetherian. Now the claim follows from Proposition 1.

(c) Use the technique of [3, Theorem 3.18].

□

The following theorem extends [1, Theorem 3.9(3)].

Theorem 2. *Suppose* R *is a one dimensional integral domain such that the intersection of every infinite family of its maximal ideals is zero. If* M *is* X*-injective with* $S_0(\mathbf{0}) \subseteq rad(\mathbf{0})$*, then* M *is a top module.*

Proof. If $S_0(\mathbf{0}) = M$, then $X = \varnothing$ and there is nothing to prove. Otherwise, by [12, Corollary 3.7], $S_0(\mathbf{0})$ is a prime submodule so that $S_0(\mathbf{0}) = rad(\mathbf{0})$. Hence the natural map $f : Spec(M/S_0(\mathbf{0})) \to Spec(M)$ is a homeomorphism by [7, Proposition 1.4]. But by [3, Theorem 3.7 (a)] and [3, Theorem 3.15 (e)], $M/S_0(\mathbf{0})$ is a weak multiplication module. Now the result follows because by Theorem 1 (c), $M/S_0(\mathbf{0})$ is a top module. □

Let M be an R-module. Then M is called a *content* module if for every $x \in M$, $x \in c(x)M$, where $c(x) = \bigcap\{I \mid I$ is an ideal of R such that $x \in IM\}$ (see [13, p. 140]).

In below we generalize [1, Theorem 3.9(4)].

Theorem 3. *Suppose* R *is a one dimensional integral domain and let* M *be a content* R*-module. Then we have the following.*

(a) *If* M *is* X*-injective , then* M *is a top module.*

(b) *If* M *is* X*-injective and* $S_0(\mathbf{0}) \subseteq rad(\mathbf{0})$*, then* M *is an strongly top module. Furthermore, if* $Spec(\overline{R})$ *is Noetherian, then* (X,τ^*) *is spectral.*

Proof. (a) By [3, Theorem 3.21], we have

$$Spec(M) = \{S_p(pM) | p \in V(Ann(M)), S_p(pM) \neq M\} = \{S_0(\mathbf{0})\} \cup Max(M),$$

where

$$Max(M) = \{pM | p \in Max(R), pM \neq M\}.$$

Let $N \leq M$ and let $N \nsubseteq S_0(\mathbf{0})$. Then

$$rad(N) = \bigcap_{N \subseteq P \in Spec(M)} P = \bigcap_{N \subseteq P \in Max(M)} P.$$

So by the above arguments, there is an index set I such that $rad(N) = \bigcap_{i \in I}(p_i M)$. Since M is content module,

$$V^*(N) = V^*(rad(N)) = V^*(\bigcap_{i \in I}(p_i M)) = V((\bigcap_{i \in I} p_i M).$$

Now if $N \subseteq S_0(\mathbf{0})$, then by [10, Lemma 2],

$$\begin{aligned} V^*(N) = V^*(rad(N)) &= V^*(S_0(\mathbf{0}) \cap (\bigcap_{i \in I}(p_i M))) \\ &= V^*(S_0(\mathbf{0}) \cap ((\bigcap_{i \in I} p_i)M)) \\ &= V^*(S_0(\mathbf{0})) \cup V^*((\bigcap_{i \in I} p_i)M) \\ &= V^*(S_0(\mathbf{0})) \cup V((\bigcap_{i \in I} p_i)M). \end{aligned}$$

By the above arguments, it follows that M is a top module.

(b) By [3, Theorem 3.21],

$$Spec(M) = \{S_0(\mathbf{0})\} \cup Max(M) \; and \; Max(M) = \{pM | p \in Max(R), pM \neq M\}.$$

Let $N \leq M$. If $N \subseteq S_0(\mathbf{0})$, then $V^*(N) = V^*(\mathbf{0}) = X$. Otherwise, we have $rad(N) = \bigcap_{i \in I}(p_i M)$ by [3, Theorem 3.21]. Since M is content, by [11, Result 3] we have

$$V^*(N) = V^*(rad(N)) = V^*(\bigcap_{i \in I}(p_i M)) = V((\bigcap_{i \in I} p_i)M).$$

Hence M is an strongly top module. The second assertion follows from Theorem 1 (b). □

Theorem 4. *If M is content weak multiplication, then M is an strongly top module. Moreover, if $Spec(R)$ is Noetherian, then (X, τ^*) is a spectral space.*

Proof. Let $N \leq M$. Then we have

$$V^*(N) = V^*(rad(N)) = V^*(\bigcap_{N \leq P} P).$$

Since M is a weak multiplication module, for each prime submodule P of M containing N, there exists an ideal I_P of R such that $P = I_P M$. Hence since M is a content module,

$$V^*(N) = V^*(\bigcap_{N \leq P} (I_P M)) = V^*((\bigcap_{N \leq P} I_P)M).$$

This implies that M is an strongly top module. Since $Spec(R)$ is Noetherian, so is $Spec(\overline{R})$. Hence by Theorem 1 (b), (X, τ^*) is a spectral space. □

Theorem 5. *Let R be a one-dimensional integral domain and let M be an X-injective R-module such that $S_0(\mathbf{0}) \subseteq rad(\mathbf{0})$. If the intersection of every infinite number of maximal submodules of M is zero, then M is strongly top and (X, τ^*) is a spectral space.*

Proof. If $S_0(\mathbf{0}) = M$, then $X = \varnothing$ and there is nothing to prove. Otherwise, by [3, Theorem 3.21], we have $Spec(M) = \{S_0(\mathbf{0})\} \cup Max(M)$ and $Max(M) = \{pM \mid p \in Max(R), pM \neq M\}$. Now let $N \leq M$. If $N = \mathbf{0}$, then claim clear because $V^*(N) = V^*(\mathbf{0}) = V^*(0M) = Spec(M)$. So we assume that $N \neq \mathbf{0}$. We consider two cases.

(1) $N \subseteq S_0(\mathbf{0})$. In this case, we have $V^*(N) = V^*(\mathbf{0}) = V^*(0M) = Spec(M)$.

(2) $N \not\subseteq S_0(\mathbf{0})$. Then since $N \neq 0$ and the intersection of every infinite number of maximal submodules of M is zero, $rad(N) = \bigcap_{i=1}^{n}(p_i M)$, where $p_i M \in Max(M)$ for each i $(1 \leq i \leq n)$. Hence we have

$$V^*(N) = V^*(rad(N)) = V^*(\bigcap_{i=1}^{n}(p_i M)).$$

Now we show that $V^*(\bigcap_{i=1}^{n}(p_i M)) = V^*((\bigcap_{i=1}^{n} p_i)M)$. Clearly, $V^*(\bigcap_{i=1}^{n}(p_i M)) \subseteq V^*((\bigcap_{i=1}^{n} p_i)M)$. Too see this reverse inclusion, let $P \in V^*((\bigcap_{i=1}^{n} p_i)M)$. If $P = S_0(\mathbf{0})$, then $(\bigcap_{i=1}^{n} p_i)M \subseteq S_0(\mathbf{0})$ implies that $\bigcap_{i=1}^{n} p_i \subseteq ((\bigcap_{i=1}^{n} p_i)M : M) \subseteq (S_0(\mathbf{0}) : M) = 0$. Thus, there exists j $(1 \leq j \leq n)$ such that $p_j = 0$, a contradiction. Hence we must have $P = qM$, where $q \in Max(R)$. Then, similar the above arguments, there exists j $(1 \leq j \leq n)$ such that $q = p_j$. Therefore, $P = qM = p_j M \in V^*(\bigcap_{i=1}^{n}(p_i M))$. So we have

$$V^*(N) = V^*(\bigcap_{i=1}^{n}(p_i M)) = V^*((\bigcap_{i=1}^{n} p_i)M).$$

Hence M is strongly top so that $\tau = \tau^*$. On the other hand, $\tau = \tau^*$ is a subset of a finite complement topology. This implies that (X, τ^*) is Noetherian. Now by Proposition 1, $(X, \tau^*) = (X, \tau)$ is spectral. □

Theorem 6. *If for each submodule N of M, $rad(N) = \sqrt{(N : M)}M$, then M is an strongly top module. Moreover, if $Spec(R)$ is Noetherian , then (X, τ^*) is spectral.*

Proof. Let $N \leq M$. Then we have

$$V^*(N) = V^*(rad(N)) = V^*(\sqrt{(N : M)}M) \\ = V(\sqrt{(N : M)}M) = V(rad(N)) = V(N).$$

Hence M is an strongly top module. Now the result follows by using similar arguments as in the proof of Theorem 4. □

Remark 4. Theorems 4, 5, and 6 improve respectively [1, Theorem 3.9(5)], [1, Theorem 3.9(8)], and [1, Theorem 3.9(6)]. They show that the notion of " top modules " can be replaced by " strongly top modules " and the proofs can be shortened considerably.

In below we generalize [1, Theorem 3.36].

Theorem 7. *Let M be a primeful R-module. Then we have the following.*

(a) *If (X, τ) is discrete, then $Spec(M) = Max(M)$.*
(b) *If R is Noetherian and $Spec(M) = Max(M)$, then (X, τ) is a finite discrete space.*

Proof. (a) Since (X, τ) is discrete, it is a T_1-space. Now by [3, Theorem 4.3], we have $Spec(M) = Max(M)$.

(b) By [3, Theorem 4.3], $Spec(\overline{R}) = Max(\overline{R})$. Hence $\overline{R}$ is Artinian. Now by [3, Theorem 4.3], (X, τ) is a T_0-space. Thus by Remark 2, M is X-injective. But M is a cyclic $\overline{R}$-module and hence a cyclic R-module by [3, Remark 3.13] and [3, Theorem 3.15]. Also $(Spec(M), \tau)$ is homoeomorphic to $Spec(\overline{R})$ by [11, Theorem 6.5(5)]. Hence X is a finite discrete space by [4, Chapter 8, Exe 2].

□

It is well known that if R is a PID and $Max(R)$ is not finite, then the intersection every infinite number of maximal ideals of R is zero. Now it is natural to ask the following question: Is the same true when R is a one dimensional integral domain with infinite maximal ideals? In below, we show that this true when $Spec(R)$ is a Noetherian space. Although this is not a simple fact, it used by some authors without giving any proof.

Theorem 8. (a) *Let I be an ideal of R and let $k, n \in \mathbb{N}$. Then $(\sqrt{I} : a^k) = (\sqrt{I} : a^n)$.*

(b) *Let I be an ideal of R and let $a \in R$, $n \in \mathbb{N}$. Then $\sqrt{I} = \sqrt{(\sqrt{I} : a^n)} \cap \sqrt{\langle \sqrt{I}, a^n \rangle}$.*

(c) *Suppose $Spec(R)$ is a Noetherian topological space. Then for every ideal I of R, $\sqrt{I}$ has a primary decomposition.*

(d) *Suppose R is a one dimensional integral domain and $Spec(R)$ is a Noetherian topological space. Then the intersection of every infinite number of maximal ideals is zero.*

Proof. (a) It is clear.

(b) Let $f \in \sqrt{(\sqrt{I}:a^n)} \cap \sqrt{\langle \sqrt{I}, a^n \rangle}$. Then there is $m \in \mathbb{N}$ such that $f^m \in (\sqrt{I} : a^n) \cap \langle \sqrt{I}, a^n \rangle$. It follows that $f^m = g + xa^n$ for some $g \in \sqrt{I}$ and $x \in R$ and we also get $a^n f^m \in \sqrt{I}$. Hence $a^n f^m = a^n g + xa^{2n}$. This implies that $xa^{2n} \in \sqrt{I}$ and so $x \in (\sqrt{I} : a^n)$ by part (a). Thus $xa^n \in \sqrt{I}$. It follows that $f \in \sqrt{I}$. The reverse inclusion is clear.

(c) Set $\Sigma =$

$$\{\sqrt{I} \mid I \textit{ is a proper ideals of } R \textit{ and } \sqrt{I} \textit{ doesn't have any primary decomposition}\}.$$

Since $Spec(R)$ is Noetherian, the radicals of ideals satisfy the a.c.c. condition. So Σ has a maximal member, $\sqrt{I_0}$ say. Thus $\sqrt{I_0} \notin Spec(R)$. In other words,

$$\exists a, b \in R \textit{ s.t. } ab \in \sqrt{I_0} \textit{ and } a \notin \sqrt{I_0} \textit{ and } b \notin \sqrt{I_0}.$$

By part (b) we have $\sqrt{I_0} = \sqrt{(\sqrt{I_0} : b)} \cap \sqrt{\langle \sqrt{I_0}, b \rangle}$. Further, $\sqrt{I_0} \subsetneq \sqrt{(\sqrt{I_0} : b)}$ and $\sqrt{I_0} \subsetneq \sqrt{\langle \sqrt{I_0}, b \rangle}$. Since $\sqrt{(\sqrt{I_0} : b)}$ and $\sqrt{\langle \sqrt{I_0}, b \rangle}$ have primary decompositions by hypothesis, $\sqrt{I_0}$ has a primary decomposition, a contradiction.

(d) Since R is one dimensional integral domain, $Spec(R) = \{0\} \cup Max(R)$. Suppose $\{m_i\}_{i \in I}$ is an infinite family of maximal ideals of R such that $\bigcap_{i \in I} m_i \neq 0$. By part (c), $\sqrt{\bigcap_{i \in I} m_i}$ has a primary decomposition. Hence

$$\sqrt{\bigcap_{i \in I} m_i} = \bigcap_{j=1}^{n} m'_j, \quad m'_j \in Max(R).$$

This implies that $\{m_i\}_{i \in I}$ is a finite family, a contradiction. So the proof is compeleted. □

Example 1. We show that $\mathbb{Z}[i\sqrt{5}]$ is a one dimensional Noetherian integral domain which has infinite number of maximal ideals and it is not a PID. To see this, let $\phi : \mathbb{Z}[X] \to \mathbb{Z}[i\sqrt{5}]$ be the natural epimorphism given by $p(x) \mapsto p(i\sqrt{5})$. by using [8] or [16], one can see that

$$Spec(\mathbb{Z}[X]) = \{\langle p \rangle, \langle f \rangle, \langle q, g \rangle \mid p \textit{ and } q \textit{ are prime numbers, } f \textit{ is a primary irreducible polynomial in } \mathbb{Q}[X], \textit{ and } g \textit{ is an irreducible polynomial in } \mathbb{Z}_q[X]\}.$$

Now we have $ker\phi = \langle X^2 + 5 \rangle$. A simple verification shows that

$$Spec(\mathbb{Z}[i\sqrt{5}]) = \{0\} \cup Max(\mathbb{Z}[i\sqrt{5}])$$

$$= \{0\} \cup \{\langle q, g(\sqrt{-5})\rangle \mid \langle q, g\rangle \in Spec(\mathbb{Z}[X]) \text{ and } X^2+5 \in \langle q, g\rangle\}.$$

Further $\mathbb{Z}[i\sqrt{5}]$ contains a finite number elements which are invertible by [17, p. 38]. So $\mathbb{Z}[i\sqrt{5}]$ is a Noetherian one dimensional integral domain with infinite number of maximal ideals. Hence the intersection of every infinite number of maximal ideals of $\mathbb{Z}[i\sqrt{5}]$ is zero by Theorem 8 (c). Note that $\mathbb{Z}[i\sqrt{5}]$ is not a PID by [17, p. 38].

ACKNOWLEDGEMENT

The authors would like to thank the referee for the careful reading of our manuscript and valuable comments.

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