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OPTIMALITY CONDITIONS AND DUALITY IN MULTIOBJECTIVE FRACTIONAL PROGRAMMING INVOLVING RIGHT UPPER-DINI-DERIVATIVE FUNCTIONS

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Abstract. In this paper, we introduce ρ-generalized pseudo and ρ-genera-lized quasi with the tool right upper-Dini-derivative and illustrated these by non-trivial examples. Necessary and sufficient optimality conditions are obtained for a nonlinear multiobjective fractional programming problem involving some classes of generalized convexities with the tool-right upper-Dini-derivative. Furthermore, usual duality theorems are proved for a general dual problem using the concept of generalized convex functions. Our results generalize and extend the several results appeared in the literature.

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1. INTRODUCTION

The weak minimum (weakly efficient, weak Pareto) solution is an important concept in mathematical models, economics, decision theory, optimal control and game theory. During the last decades the analysis of duality in multiobjective theory has been a focal issue. For brief accounts of the evolution of convex theory, with and without differentiability assumptions, and in the framework of multiobjective theory, the reader may consult [1–3, 5–8, 10–12, 16, 19] and the references cited therein.

In the past few years extensive literature relative to the other families of more general functions to substitute the convex functions in the mathematical programming has grown immensely. Such functions are called generalized convex functions. In fact the concept of invexity existed earlier in the literature, e.g., Hanson [9] used it to give Kuhn-Tucker sufficient optimality conditions in nonlinear programming. For the most part, the study of invexity has been in the context of differentiable functions. But the corresponding conclusions cannot be obtained for nondifferentiable programming with the help of invex because the derivative is required in the definition of invex. However, in the recent years, the concept of invexity, previously introduced for differentiable functions, was generalized to the case of nondifferentiable functions.

Ye [17] defined a generalized convexity, called d-invexity, and discussed some theoretical problems for nondifferentiable programming. Nahak and Mohapatra [13] introduced the notation of d-ρ-(η,θ)-invexity and obtained optimality conditions and duality results for a multiobjective programming problem. Jayswal et al. [11] established sufficient conditions and duality theorems for a multiobjective fractional programming problem under (p,r)-ρ-(η,θ)-invex functions. In [16], Sharma and Ahmad used the concept of (F,ρ,σ)-type I functions to establish Karush-Kuhn-Tucker type sufficient optimality conditions for a nonsmooth multiobjective programming problem.

Making use of arcwise connected set, as defined by Ortega and Rheinboldt [15], Avriel and Zang [4] extended the concept of convex functions to the corresponding forms of arcwise connected functions and presented some interrelations between them. Zhang [20] introduced B-arcwise connected and strictly B-arcwise connected functions based on arcwise connected functions. Optimality and duality results are also obtained for a nonlinear semi-infinite programming problem in [20]. Recently, Yuan and Liu [18] introduced some new generalized convexity notations using right-upper Dini derivative, and established optimality conditions and duality theorems for two types of dual programming.

Inspired and motivated by above works, the purpose of this paper is to investigate the following multiobjective nonlinear fractional programming problem involving generalized convex functions, in terms of the right upper-Dini-derivative:

$$\text{(VFP)} \qquad \min\left(\frac{f_1(x)}{g_1(x)},\frac{f_2(x)}{g_2(x)},\dots,\frac{f_p(x)}{g_p(x)}\right)$$

$$\text{subject to}\begin{cases} h_j(x)\leqq 0, \quad j=1,2,\dots,m,\\ x\in X_0,\end{cases}$$

where X_0 be a nonempty subset of R^n and $g_i(x)>0$ for all $x\in X_0$ and each $i=1,2,\dots,p$. Let $f=(f_1,f_2,\dots,f_p)$, $g=(g_1,g_2,\dots,g_p)$, $h=(h_1,h_2,\dots,h_m)$. We denote by $X=\{x\in X_0 : h_j(x)\leqq 0,\ j=1,2,\dots,m\}$ the set of all feasible solutions to problem (VFP). For $\bar{x}\in X$ we denote $M(\bar{x})=\{j\in\{1,2,\dots,m\} : h_j(\bar{x})=0\}$, $h^0=(h_j)_{j\in M(\bar{x})}$ and $N(\bar{x})=\{1,2,\dots,m\}\setminus M(\bar{x})$.

This paper is divided into five sections. Section 2 recalls some definitions and related results which will be used in later sections. Non-trivial examples are also discussed to support our functions. In Section 3, necessary and sufficient optimality conditions are established for a nonlinear fractional multiobjective programming problem involving generalized convex functions. In Section 4, we establish appropriate duality theorems for a general dual problem. Finally, conclusion and further developments are given in Section 5.

2. Preliminaries

Let R^n be the n-dimensional Euclidean space and R^n_+ its non-negative orthant. If $x, y \in R^n$ then $x < y \Leftrightarrow x_i < y_i, i = 1, 2, \ldots, n$; $x \leqq y \Leftrightarrow x_i \leqq y_i, i = 1, 2, \ldots, n$ and $x \leq y \Leftrightarrow x_i \leqq y_i, i = 1, 2, \ldots, n$ and $x \neq y$.
In this section, we recall some well known results and concepts.

Definition 1. A point $\bar{x} \in X$ is said to be a weak minimum to (VFP) if there exists no other feasible point x such that $\frac{f(\bar{x})}{g(\bar{x})} > \frac{f(x)}{g(x)}$.

Definition 2 ([20]). A set $C \subset R^n$ is said to be an arcwise connected set if, for every $x_1 \in C$, $x_2 \in C$, there exists a continuous vector-valued function $H_{x_1,x_2} : [0,1] \to C$, called an arc, such that

$$H_{x_1,x_2}(0) = x_1, \quad H_{x_1,x_2}(1) = x_2.$$

Definition 3 ([18]). Let φ be a real-valued function defined on an arcwise connected set $C \subset R^n$. Let $x_1, x_2 \in C$ and H_{x_1,x_2} be the arc connecting x_1 and x_2 in C. The right upper-Dini-derivative of φ with respect to $H_{x_2,x_1}(t)$ at $t = 0$ is defined as follows:

$$(d\varphi)^+(H_{x_2,x_1}(0^+)) = \lim_{t \to 0^+} \sup \frac{\varphi(H_{x_2,x_1}(t)) - \varphi(x_2)}{t}. \tag{2.1}$$

Using this upper-Dini-derivative concept, Yuan and Liu [18] introduced a class of functions, which called (α, ρ)-right upper-Dini-derivative function. For convenience, we recall the following definitions.

Definition 4 ([18]). A set $X_0 \subset R^n$ is said to be locally arcwise connected at $\bar{x}$ if for any $x \in X_0$ and $x \neq \bar{x}$ there exist a positive number $a(x, \bar{x})$, with $0 < a(x, \bar{x}) \leqq 1$, and a continuous arc $H_{\bar{x},x}$ such that $H_{\bar{x},x}(t) \in X_0$ for any $t \in (0, a(x, \bar{x}))$.
The set X_0 is locally arcwise connected on X_0 if X_0 is locally arcwise connected at any $x \in X_0$.

Definition 5 ([18]). Let $X_0 \subset R^n$ be a locally arcwise connected set and $\varphi : X_0 \to R$ be a real function defined on X_0. The function φ is said to be (α, ρ)-right upper-Dini-derivative locally arcwise connected with respect to H at $\bar{x}$, if there exist real functions $\alpha : X_0 \times X_0 \to R$, $\rho : X_0 \times X_0 \to R$ such that

$$\varphi(x) - \varphi(\bar{x}) \geqq \alpha(x, \bar{x})(d\varphi)^+(H_{\bar{x},x}(0^+)) + \rho(x, \bar{x}), \ \forall x \in X_0.$$

If φ is (α, ρ)-right upper-Dini-derivative locally arcwise connected with respect to H at $\bar{x}$ for any $\bar{x} \in X_0$, then φ is called (α, ρ)-right upper-Dini-derivative locally arcwise connected with respect to H on X_0.

Remark 1. It revealed by an example given in [18] that there exists a function, which is (α, ρ)-right upper-Dini-derivative locally arcwise connected but neither d-ρ-(η, θ)-invex [13] nor d-invex [17], nor directional differentially B-arcwise connected [20].

Now we define the notions of ρ-generalized-pseudo-right upper-Dini-derivative locally arcwise connected, strictly ρ-generalized-pseudo-right upper-Dini-deriv-ative locally arcwise connected and ρ-generalized-quasi-right upper-Dini-deriva-tive locally arcwise connected functions.

Definition 6. The function $\varphi : X_0 \to R$ is said to be ρ-generalized-pseudo-right upper-Dini-derivative locally arcwise connected (with respect to H) at $\bar{x}$, if there exists a real function $\rho : X_0 \times X_0 \to R$ such that

$$(d\varphi)^+(H_{\bar{x},x}(0^+)) \geqq -\rho(x,\bar{x}) \Rightarrow \varphi(x) \geqq \varphi(\bar{x}),\ \forall x \in X_0,$$

equivalently

$$\varphi(x) < \varphi(\bar{x}) \Rightarrow (d\varphi)^+(H_{\bar{x},x}(0^+)) < -\rho(x,\bar{x}),\ \forall x \in X_0.$$

The function $\varphi : X_0 \to R$ is said to be ρ-generalized-pseudo-right upper-Dini-derivative locally arcwise connected (with respect to H) on X_0 if it is ρ-generalized-pseudo-right upper-Dini-derivative locally arcwise connected (with respect to H) at any $\bar{x} \in X_0$.

It is noted that, not every ρ-generalized-pseudo-right upper-Dini-derivative locally arcwise connected with respect to H is (α,ρ)-right upper-Dini-derivative locally arcwise connected with respect to H [18]. We have the following counter-example, which shows that the function φ is ρ-generalized-pseudo-right upper-Dini-derivative locally arcwise connected with respect to H but not (α,ρ)-right upper-Dini-derivative locally arcwise connected with respect to H.

Example 1. Let $\varphi : R \to R$ be defined by

$$\varphi(x) = \begin{cases} x \cos^2 \dfrac{1}{x}; & \text{if } x > 0 \\ 0; & \text{if } x \leq 0. \end{cases}$$

For any, $x, y \in R$, defining the arc $H : [0,1] \to R$ by

$$H_{y,x}(t) = tx + (1-t)y, t \in [0,1].$$

Now, by the definition of right upper-Dini-derivative defined by (2.1), we have

$$(d\varphi)^+(H_{0,x}(0^+)) = \lim_{t\to 0^+} \sup \frac{\varphi(H_{0,x}(t)) - \varphi(0)}{t} = \lim_{t\to 0^+} \sup \frac{\varphi(tx)}{t}$$

$$= \lim_{t\to 0^+} \sup \frac{tx \cos^2 \dfrac{1}{tx}}{t} = x.$$

Consider the function $\rho : R \times R \to R$ to be defined by

$$\rho(x,y) = \begin{cases} -x + 3x \cos^2 \frac{1}{x}; & \text{if } x > 0 \\ 0; & \text{if } x \leq 0. \end{cases}$$

For $\bar{x} = 0$, we have

$$\varphi(x) - \varphi(\bar{x}) = \begin{cases} x\cos^2\frac{1}{x}; & \text{if } x > 0 \\ 0; & \text{if } x \leq 0, \end{cases}$$

while

$$(d\varphi)^+(H_{\bar{x},x}(0^+)) + \rho(x,\bar{x}) = \begin{cases} 3x\cos^2\frac{1}{x}; & \text{if } x > 0 \\ 0; & \text{if } x \leq 0. \end{cases}$$

So, clearly φ is ρ-generalized-pseudo-right upper-Dini-derivative locally arcwise connected (with respect to H) at $\bar{x} = 0$. But, φ is not a (α, ρ)-right upper-Dini-derivative locally arcwise connected with respect to same arc H and ρ at $\bar{x} = 0$ because for $x = 0.3$ and $\alpha(x, \bar{x}) = 1$, we get

$$\varphi(x) - \varphi(\bar{x}) - \alpha(x,\bar{x})(d\varphi)^+(H_{\bar{x},x}(0^+)) - \rho(x,\bar{x}) = -0.5782 < 0.$$

Definition 7. The function $\varphi : X_0 \to R$ is said to be strictly ρ-generalized-pseudo-right upper-Dini-derivative locally arcwise connected (with respect to H) at $\bar{x}$, if there exists a real function $\rho : X_0 \times X_0 \to R$ such that

$$(d\varphi)^+(H_{\bar{x},x}(0^+)) \geqq -\rho(x,\bar{x}) \Rightarrow \varphi(x) > \varphi(\bar{x}),\ \forall x \in X_0, x \neq \bar{x},$$

equivalently

$$\varphi(x) \leqq \varphi(\bar{x}) \Rightarrow (d\varphi)^+(H_{\bar{x},x}(0^+)) < -\rho(x,\bar{x}),\ \forall x \in X_0, x \neq \bar{x}.$$

The function $\varphi : X_0 \to R$ is said to be strictly ρ-generalized-pseudo-right upper-Dini-derivative locally arcwise connected (with respect to H) on X_0 if it is strictly ρ-generalized-pseudo-right upper-Dini-derivative locally arcwise connected (with respect to H) at any $\bar{x} \in X_0$.

Definition 8. The function $\varphi : X_0 \to R$ is said to be ρ-generalized-quasi-right upper-Dini-derivative locally arcwise connected with respect to H at $\bar{x}$, if there exists a real function $\rho : X_0 \times X_0 \to R$ such that

$$\varphi(x) \leqq \varphi(\bar{x}) \Rightarrow (d\varphi)^+(H_{\bar{x},x}(0^+)) \leqq -\bar{\rho}(x,\bar{x}),\ \forall x \in X_0,$$

equivalently

$$(d\varphi)^+(H_{\bar{x},x}(0^+)) > -\bar{\rho}(x,\bar{x}) \Rightarrow \varphi(x) > \varphi(\bar{x}),\ \forall x \in X_0.$$

The function $\varphi : X_0 \to R$ is said to be ρ-generalized-quasi-right upper-Dini-derivative locally arcwise connected (with respect to H) on X_0 if it is ρ-generalized-quasi-right upper-Dini-derivative locally arcwise connected (with respect to H) at any $\bar{x} \in X_0$.

Now, we present an example which shows that there exists a function which is ρ-generalized-quasi-right upper-Dini-derivative locally arcwise connected but neither (α, ρ)-right upper-Dini-derivative locally arcwise connected nor ρ-generalized-pseudo-right upper-Dini-derivative locally arcwise connected with respect to the arc H.

Example 2. Let $\varphi : R \to R$ be a function defined by

$$\varphi(x) = \begin{cases} x \sin\dfrac{1}{x} - 3x; & \text{if } 0 < x < 1 \\ 0; & \text{otherwise }. \end{cases}$$

For any, $x, y \in R$, defining the arc $H : [0,1] \to R$ by

$$H_{y,x}(t) = tx + (1-t)y, t \in [0,1].$$

Clearly, $(d\varphi)^+(H_{0,x}(0^+)) = -2x$.
Let $\rho : R \times R \to R$ be defined by

$$\rho(x,y) = \begin{cases} 2x; & \text{if } 0 < x < 1 \\ 0; & \text{otherwise }. \end{cases}$$

Then, it can be easily seen that φ is ρ-generalized-quasi-right upper-Dini-derivative locally arcwise connected (with respect to H) at $\bar{x} = 0$. However, for $0 < x < 1, \alpha(x,\bar{x}) = 1$ and $\bar{x} = 0$, we have

$$\varphi(x) - \varphi(\bar{x}) - \alpha(x,\bar{x})(d\varphi)^+(H_{\bar{x},x}(0^+)) - \rho(x,\bar{x}) < 0,$$

and

$$\varphi(x) - \varphi(\bar{x}) < 0, \text{ but } (d\varphi)^+(H_{\bar{x},x}(0^+)) + \bar{\rho}(x,\bar{x}) = 0.$$

Hence, φ is neither (α,ρ)-right upper-Dini-derivative locally arcwise connected nor ρ-generalized-pseudo-right upper-Dini-derivative locally arcwise connected with respect to same arc H and ρ at $\bar{x} = 0$.

Definition 9 ([18])**.** A function $f : X_0 \to R^p$ is called preinvex (with respect to $\eta : X_0 \times X_0 \to R^n$) on X_0 if there exists a vector valued function η such that,

$$f(u + t\eta(x,u)) \leqq tf(x) + (1-t)f(u)$$

holds for all $x, u \in X_0$ and any $t \in [0,1]$.

Definition 10 ([18])**.** A function $f : X_0 \to R^p$ is said to be convexlike if for any $x, y \in X_0$ and $0 \leqq \lambda \leqq 1$, there is $z \in X_0$ such that

$$f(z) \leqq \lambda f(x) + (1-\lambda) f(y).$$

Remark 2. The convex and the preinvex functions are convexlike functions.

Lemma 1 ([18])**.** *Let S be a nonempty set in R^n and $\psi : S \to R^p$, a convexlike function. Then either*

$\psi(x) < 0$ *has a solution* $x \in S$

or

$\lambda^T \psi(x) \geqq 0$ *for all* $x \in S$, *for some* $\lambda \in R^p, \lambda \geq 0$,

but both alternatives are never true simultaneously, and the symbol T denotes the transpose of a matrix.

In the next section we will use the following version of Theorem 3.2 from [18].

Lemma 2. *Let $\bar{x} \in X$ be a (local) weak minimum solution for the following problem:*

$$min\ (\varphi_1(x), \varphi_2(x), \ldots, \varphi_p(x)),$$

$$subject\ to \begin{cases} h_j(x) \leqq 0, & j \in \{1,2,...,m\}, \\ x \in X_0, \end{cases}$$

where $\varphi = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_p(x)) : X_0 \to R^p$ and $h_1, h_2, \ldots, h_m$ are right upper-Dini-derivative functions of x with respect to the arc $H_{\bar{x},x}(0^+)$. Assume that h_j ($j \in N(\bar{x})$) is a upper semi-continuous function at $\bar{x}$ and $(d\varphi)^+$ $(H_{\bar{x},x}(0^+))$ and $(dh^0)^+(H_{\bar{x},x}(0^+))$ are convexlike functions of x on X_0. If $(dh^0)^+(H_{\bar{x},x}(0^+))$ satisfies a regularity condition at $\bar{x}$ [18], then there exist $\lambda^0 \in R^p_+$, $u^0 \in R^m_+$ such that

$$\lambda^{0^T}(d\varphi)^+(H_{\bar{x},x}(0^+)) + u^{0^T}(dh)^+(H_{\bar{x},x}(0^+)) \geqq 0,\ for\ all\ x \in X_0,$$

$$u^{0^T} h(\bar{x}) = 0,$$

$$\sum_{i=1}^{p} \lambda_i^0 = 1.$$

3. Optimality conditions

In this section, we first derive Fritz-John type and Karush-Kuhn-Tucker type necessary conditions for (VFP) and then using the concept of (local) weak optimality, we also establish some sufficient optimality conditions.

Theorem 1 (Fritz-John type necessary conditions)**.** *Assume that $\bar{x}$ is a (local) weak minimum solution to* (VFP)*. Let $(df)^+(H_{\bar{x},x}(0^+))$, $(dg)^+(H_{\bar{x},x}(0^+))$ and $(dh^0)^+(H_{\bar{x},x}(0^+))$ be convexlike functions of x on X_0 and also let h_j be upper semi-continuous at $\bar{x}$ for $j \in N(\bar{x})$. Then there exist $\lambda^0 \in R^p_+$, $u^0 \in R^p_+$ and $v^0 \in R^m_+$, $(\lambda^0, u^0, v^0) \neq 0$ such that*

$$\lambda^{0^T}(df)^+(H_{\bar{x},x}(0^+)) - u^{0^T}(dg)^+(H_{\bar{x},x}(0^+)) + v^{0^T}(dh)^+(H_{\bar{x},x}(0^+)) \geqq 0,$$

$$for\ all\ x \in X_0,$$

$$v^{0^T} h(\bar{x}) = 0.$$

Proof. Let $\bar{x}$ be a (local) weak minimum solution for (VFP) and suppose there exists $x^* \in X_0$ such that

$$(df)^+(H_{\bar{x},x^*}(0^+)) < 0, \tag{3.1}$$

$$(dg)^+(H_{\bar{x},x^*}(0^+)) > 0, \tag{3.2}$$

$$(dh^0)^+(H_{\bar{x},x^*}(0^+)) < 0. \tag{3.3}$$

By the relation (3.1) we have

$$\lim_{t \to 0^+} \sup \frac{f_i(H_{\bar{x},x^*}(t)) - f_i(\bar{x})}{t} < 0, \forall\ i \in \{1,2,...,p\},$$

which implies that there exists $\delta_i > 0$ such that

$$f_i(H_{\bar{x},x^*}(t)) < f_i(\bar{x}), \text{ for } t \in (0, \delta_i).$$

Similarly (by the relations (3.2), (3.3)), for each $i \in \{1,2,...,p\}$, $j \in M(\bar{x})$ there exist $\xi_i > 0, \sigma_j > 0$ such that

$$g_i(H_{\bar{x},x^*}(t)) > g_i(\bar{x}), \text{ for } t \in (0, \xi_i),$$

$$h^0(H_{\bar{x},x^*}(t)) < h^0(\bar{x}) = 0, \text{ for } t \in (0, \sigma_j).$$

Now, for $j \in N(\bar{x})$, $h_j(\bar{x}) < 0$ and h_j is semi-continuous at $\bar{x}$, then $h_j(H_{\bar{x},x^*}(t))$ is semi-continuous at $t = 0$. Hence, for $\epsilon = \frac{1}{2}h_j(\bar{x}) > 0$, there exists σ_j such that

$$h_j(H_{\bar{x},x^*}(t)) < h_j(\bar{x}) + \epsilon = \frac{1}{2}h_j(\bar{x}), \text{ for } t \in (0, \sigma_j).$$

Let $\delta^* = \min\{\delta_i, \xi_i, \sigma_j\}$, then for $t \in (0, \delta^*)$, we have

$$f_i(H_{\bar{x},x^*}(t)) < f_i(\bar{x}), \text{ for } i \in \{1,2,...,p\}, \tag{3.4}$$

$$g_i(H_{\bar{x},x^*}(t)) > g_i(\bar{x}), \text{ for } i \in \{1,2,...,p\}, \tag{3.5}$$

$$h_j(H_{\bar{x},x^*}(t)) < h_j(\bar{x}) = 0, \text{ for } j \in \{1,2,...,m\}. \tag{3.6}$$

Using (3.4) and (3.5), for $\frac{f(x)}{g(x)} = \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, ..., \frac{f_p(x)}{g_p(x)}\right)$ we get $\frac{f(\bar{x})}{g(\bar{x})} > \frac{f(H_{\bar{x},x^*}(t))}{g(H_{\bar{x},x^*}(t))}$, which contradicts the assumption that $\bar{x}$ is a local weak minimum solution of (VFP). By Lemma 1 and the hypothesis that $(df)^+, (dg)^+$ and $(dh^0)^+$ are convexlike functions of x on X_0, we obtain the required result. □

For each $u = (u_1, u_2, \ldots, u_p)^T \in R^p_+$, where R^p_+ denotes the positive orthant of R^p, we consider the parametric problem

$$(\text{VFP}_u) \quad \text{minimize } \left(f_1(x) - u_1 g_1(x), f_2(x) - u_2 g_2(x), \ldots, f_p(x) - u_p g_p(x)\right),$$

$$\text{subject to} \begin{cases} h_j(x) \leqq 0, & j \in \{1,2,...,m\}, \\ x \in X_0. \end{cases}$$

Lemma 3 ([14])**.** *If $\bar{x}$ is a (local) weak minimum to* (VFP)*, then $\bar{x}$ is a (local) weak minimum to* (VFP_{u^0})*, where $u^0 = \frac{f(\bar{x})}{g(\bar{x})}$.*

Using this lemma we can derive a Karush-Kuhn-Tucker type necessary optimality criterion for the problem (VFP).

Theorem 2 (Karush-Kuhn-Tucker type necessary conditions)**.** *Assume that $\bar{x}$ is a (local) weak minimum solution to* (VFP)*. Let $(df)^+(H_{\bar{x},x}(0^+))$, $(dg)^+(H_{\bar{x},x}(0^+))$ and $(dh^0)^+(H_{\bar{x},x}(0^+))$ be convexlike functions of x on X_0 and also let h_j be upper semi-continuous at $\bar{x}$ for $j \in N(\bar{x})$, further $(dg)^+(H_{\bar{x},x}(0^+))$ satisfies the slater* *constraint qualification* (*i.e., there exists* $\hat{x} \in X$ *such that* $(dg)^+(H_{\bar{x},\hat{x}}(0^+)) < 0$). *Then there exist* $\lambda^0 \in R^p_+$, $u^0 \in R^p_+$ *and* $v^0 \in R^m_+$ *such that*

$$\sum_{i=1}^{p} \lambda_i^0 \left((df_i)^+(H_{\bar{x},x}(0^+)) - u_i^0 (dg_i)^+(H_{\bar{x},x}(0^+))\right) + v^{0^T} (dh)^+(H_{\bar{x},x}(0^+)) \geqq 0,$$

$$\text{for all } x \in X_0,$$

$$v^{0^T} h(\bar{x}) = 0,$$

$$\sum_{i=1}^{p} \lambda_i^0 = 1.$$

Proof. Let $\bar{x}$ be a (local) weak minimum solution for (VFP). According to Lemma 3, we have that $\bar{x}$ is a (local) weak minimum solution for (VFP_{u^0}), where $u^0 = (u_1^0, u_2^0, \ldots, u_p^0)$, $u_i^0 = f_i(\bar{x})/g_i(\bar{x}), i \in \{1, 2, \ldots, p\}$. Now, applying Lemma 2 to problem (VFP_{u^0}), we get that there exist $\lambda^0 \in R^p_+$, $v^0 \in R^m_+$ such that

$$\sum_{i=1}^{p} \lambda_i^0 \left((df_i)^+(H_{\bar{x},x}(0^+)) - u_i^0 (dg_i)^+(H_{\bar{x},x}(0^+))\right) + v^{0^T} (dh)^+(H_{\bar{x},x}(0^+)) \geqq 0,$$

$$\text{for all } x \in X_0,$$

$$v^{0^T} h(\bar{x}) = 0,$$

$$\sum_{i=1}^{p} \lambda_i^0 = 1,$$

and the theorem is proved. □

Remark 3. In the above theorem we can suppose, for any $i = 1, 2, \ldots, p$, that $(df_i)^+(H_{\bar{x},x}(0^+)) - u_i^0 (dg_i)^+(H_{\bar{x},x}(0^+))$ is convexlike on X_0, where $u_i^0 = \frac{f_i(\bar{x})}{g_i(\bar{x})}$ instead of considering that $(df_i)^+(H_{\bar{x},x}(0^+))$ and $(dg_i)^+(H_{\bar{x},x}(0^+))$ are convexlike on X_0, for any $i = 1, 2, \ldots, p$.

Now, we establish some sufficient optimality conditions for the problem (VFP) using the concept of (local) weak minimum. As we go on weakening the assumptions of convexity, we get a weaker conclusion of the (weak) efficient solution of (VFP). Let $\alpha : X_0 \times X_0 \to R$, $\bar{\rho}, \hat{\rho} : X_0 \times X_0 \to R^p$ and $\breve{\rho} : X_0 \times X_0 \to R^m$.

Theorem 3 (Karush-Kuhn-Tucker type sufficient conditions)**.** *Let* $\bar{x} \in X$ *and* f *be* $(\alpha, \bar{\rho})$*-right upper-Dini-derivative locally arcwise connected (with respect to* H*) at* $\bar{x}$*,* $-g$ *be* $(\alpha, \hat{\rho})$*-right upper-Dini-derivative locally arcwise connected (with respect to* H*) at* $\bar{x}$*, and* h *be* $(\alpha, \breve{\rho})$*-right upper-Dini-derivative locally arcwise connected (with* *respect to* H*) at* $\bar{x}$*. Assume also that there exist* $\lambda^0 \in R^p$*,* $u^0 \in R^p$ *and* $v^0 \in R^m$ *such that* $\alpha(\tilde{x},\bar{x}) > 0$*,* $\lambda^{0^T}\bar{\rho}(\tilde{x},\bar{x}) + v^{0^T}\breve{\rho}(\tilde{x},\bar{x}) \geqq 0$*,* $\hat{\rho}(\tilde{x},\bar{x}) \geqq 0$ *and*

$$\sum_{i=1}^{p}\lambda_i^0\big((df_i)^+(H_{\bar{x},x}(0^+))\big) + \sum_{j=1}^{m} v_j^0(dh_j)^+(H_{\bar{x},x}(0^+)) \geqq 0, \; for\ all\ x \in X, \tag{3.7}$$

$$(dg_i)^+(H_{\bar{x},x}(0^+)) \leqq 0, \; \forall x \in X, \; i = 1,2,...,p, \tag{3.8}$$

$$\sum_{j=1}^{m} v_j^0 h_j(\bar{x}) = 0, \tag{3.9}$$

$$\sum_{i=1}^{p}\lambda_i^0 = 1, \tag{3.10}$$

$$\lambda^0 \geqq 0, u^0 \geqq 0, v^0 \geqq 0. \tag{3.11}$$

Then $\bar{x}$ *is a weak minimum solution to* (VFP).

Proof. Suppose contrary to the result. Hence there exists $\tilde{x} \in X$ such that

$$\frac{f_i(\tilde{x})}{g_i(\tilde{x})} < \frac{f_i(\bar{x})}{g_i(\bar{x})}, \quad i = 1,2,...,p. \tag{3.12}$$

Since f is $(\alpha,\bar{\rho})$-right upper-Dini-derivative locally arcwise connected (with respect to H) at $\bar{x}$, and h is $(\alpha,\breve{\rho})$-right upper-Dini-derivative locally arcwise connected (with respect to H) at $\bar{x}$, we get

$$f_i(\tilde{x}) - f_i(\bar{x}) \geqq \alpha(\tilde{x},\bar{x})(df_i)^+(H_{\bar{x},\tilde{x}}(0^+)) + \bar{\rho}_i(\tilde{x},\bar{x}), \; i = 1,2,...,p, \tag{3.13}$$

$$h_j(\tilde{x}) - h_j(\bar{x}) \geqq \alpha(\tilde{x},\bar{x})(dh_j)^+(H_{\bar{x},\tilde{x}}(0^+)) + \breve{\rho}_j(\tilde{x},\bar{x}), \; j = 1,2,...,m. \tag{3.14}$$

Multiplying (3.13) by $\lambda_i^0 \geqq 0$, $i \in P$, $\lambda^0 \in R_+^p$, (3.14) by $v_j^0 \geqq 0$, $j = 1,2,...,m$, and then summing the obtained relations, we get

$$\left[\sum_{i=1}^{p}\lambda_i^0(f_i(\tilde{x}) - f_i(\bar{x}))\right] + \left[\sum_{j=1}^{m} v_j^0(h_j(\tilde{x}) - h_j(\bar{x}))\right]$$

$$\geqq \alpha(\tilde{x},\bar{x})\sum_{i=1}^{p}\lambda_i^0(df_i)^+(H_{\bar{x},\tilde{x}}(0^+)) + \alpha(\tilde{x},\bar{x})\sum_{j=1}^{m} v_j^0(dh_j)^+(H_{\bar{x},\tilde{x}}(0^+))$$

$$+\lambda^{0^T}\bar{\rho}(\tilde{x},\bar{x}) + v^{0^T}\breve{\rho}(\tilde{x},\bar{x}) \geqq 0,$$

where the last inequality is according to (3.7), $\alpha(\tilde{x},\bar{x}) > 0$ and

$$\lambda^{0^T}\bar{\rho}(\tilde{x},\bar{x}) + v^{0^T}\breve{\rho}(\tilde{x},\bar{x}) \geqq 0.$$

Hence

$$\left[\sum_{i=1}^{p}\lambda_i^0(f_i(\tilde{x}) - f_i(\bar{x}))\right] + \left[\sum_{j=1}^{m} v_j^0(h_j(\tilde{x}) - h_j(\bar{x}))\right] \geqq 0. \tag{3.15}$$

Since $x \in X$, $v^0 \geqq 0$, by (3.9) and (3.15), we get

$$\sum_{i=1}^{p} \lambda_i^0 (f_i(\tilde{x}) - f_i(\bar{x})) \geqq 0. \tag{3.16}$$

Using (3.11) and (3.16), we obtain that there exists $i_0 \in \{1, 2, ..., p\}$ such that

$$f_{i_0}(\tilde{x}) \geqq f_{i_0}(\bar{x}). \tag{3.17}$$

Since $-g$ is $(\alpha, \hat{\rho})$-right upper-Dini-derivative locally arcwise connected (with respect to H) at $\bar{x}$, we get

$$-g_i(\tilde{x}) + g_i(\bar{x}) \geqq -\alpha(\tilde{x}, \bar{x})(dg_i)^+(H_{\bar{x},\tilde{x}}(0^+)) + \hat{\rho}_i(\tilde{x}, \bar{x}),\ i = 1, 2, ..., p,$$

equivalently,

$$g_i(\tilde{x}) - g_i(\bar{x}) \leqq \alpha(\tilde{x}, \bar{x})(dg_i)^+(H_{\bar{x},\tilde{x}}(0^+)) - \hat{\rho}_i(\tilde{x}, \bar{x}),\ i = 1, 2, ..., p. \tag{3.18}$$

By (3.8), (3.18) and using $\alpha(\tilde{x}, \bar{x}) > 0$, $\hat{\rho}(\tilde{x}, \bar{x}) \geqq 0$, it follows

$$g_i(\tilde{x}) \leqq g_i(\bar{x}), \quad i = 1, 2, ..., p. \tag{3.19}$$

Now, using (3.17), (3.19), $f \geqq 0$ and $g > 0$, we obtain

$$\frac{f_{i_0}(\tilde{x})}{g_{i_0}(\tilde{x})} \geqq \frac{f_{i_0}(\bar{x})}{g_{i_0}(\bar{x})},$$

which contradicts (3.12). Thus $\bar{x}$ is a weak minimum solution to (VFP). □

Theorem 4 (Karush-Kuhn-Tucker type sufficient conditions)**.** *Let $\bar{x} \in X$ and let $\sum_{i=1}^{p} \lambda_i^0 [f_i(.) - u_i^0 g_i(.)] + \sum_{j=1}^{m} v_j^0 h_j(.)$ be (α, ρ)-right upper-Dini-derivative locally arcwise connected (with respect to H) at $\bar{x}$. Assume also that there exist $\lambda^0 \in R^p$, $u_i^0 = \frac{f_i(\bar{x})}{g_i(\bar{x})}$, $i = 1, 2, ..., p$, and $v^0 \in R^m$ such that $\alpha(x, \bar{x}) > 0$, $\rho(x, \bar{x}) \geqq 0$,*

$$\sum_{i=1}^{p} \lambda_i^0 \big((df_i)^+(H_{\bar{x},x}(0^+)) - u_i^0 (dg_i)^+(H_{\bar{x},x}(0^+))\big)$$

$$+ \sum_{j=1}^{m} v_j^0 (dh_j)^+(H_{\bar{x},x}(0^+)) \geqq 0, \quad \forall x \in X, \tag{3.20}$$

$$\sum_{j=1}^{m} v_j^0 h_j(\bar{x}) = 0, \tag{3.21}$$

$$\sum_{i=1}^{p} \lambda_i^0 = 1, \tag{3.22}$$

$$\lambda^0 \geqq 0, u^0 \geqq 0, v^0 \geqq 0. \tag{3.23}$$

Then $\bar{x}$ is a weak minimum solution to (VFP).

Proof. Suppose contrary to the result. Hence there exists a $x \in X$ such that

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(\bar{x})}{g_i(\bar{x})}, \text{ for any } i = 1,2,...,p.$$

That is,

$$f_i(x) < u_i^0 g_i(x), \text{ for any } i = 1,2,...,p. \tag{3.24}$$

From the (α,ρ)-right upper-Dini-derivative locally arcwise connected (with respect to H) of $\sum_{i=1}^{p} \lambda_i^0[f_i(.) - u_i^0 g_i(.)] + \sum_{j=1}^{m} v_j^0 h_j(.)$ at $\bar{x}$, we have

$$\left[\sum_{i=1}^{p} \lambda_i^0 (f_i(x) - f_i(\bar{x}))\right] - \left[\sum_{i=1}^{p} \lambda_i^0 u_i^0 (g_i(x) - g_i(\bar{x}))\right] + \left[\sum_{j=1}^{m} v_j^0 (h_j(x) - h_j(\bar{x}))\right]$$
$$\geqq \alpha(x,\bar{x})\left[\sum_{i=1}^{p} \lambda_i^0 \Big((df_i)^+(H_{\bar{x},x}(0^+)) - u_i^0 (dg_i)^+(H_{\bar{x},x}(0^+))\Big)\right.$$
$$\left. + \sum_{j=1}^{m} v_j^0 (dh_j)^+(H_{\bar{x},x}(0^+))\right] + \rho(x,\bar{x}) \geqq 0,$$

where the last inequality is according to (3.20), $\alpha(\tilde{x},\bar{x}) > 0$ and $\rho(x,\bar{x}) \geqq 0$. Hence,

$$\left[\sum_{i=1}^{p} \lambda_i^0 (f_i(x) - f_i(\bar{x}))\right] - \left[\sum_{i=1}^{p} \lambda_i^0 u_i^0 (g_i(x) - g_i(\bar{x}))\right]$$
$$+ \left[\sum_{j=1}^{m} v_j^0 (h_j(x) - h_j(\bar{x}))\right] \geqq 0. \tag{3.25}$$

Now, $x \in X$, (3.21) and (3.25), gives

$$\sum_{i=1}^{p} \lambda_i^0 (f_i(x) - f_i(\bar{x})) - \sum_{i=1}^{p} \lambda_i^0 u_i^0 (g_i(x) - g_i(\bar{x})) \geqq 0.$$

That is,

$$\sum_{i=1}^{p} \lambda_i^0 ((f_i(x) - u_i^0 g_i(x)) - (f_i(\bar{x}) - u_i^0 g_i(\bar{x}))) \geqq 0.$$

Since $u_i^0 = \frac{f_i(\bar{x})}{g_i(\bar{x})}$, $i = 1,2,...,p$, we obtain

$$\sum_{i=1}^{p} \lambda_i^0 (f_i(x) - u_i^0 g_i(x)) \geqq 0.$$

From $\lambda^0 \geqq 0$, $\sum_{i=1}^{p} \lambda_i^0 = 1$, we obtain that there exists $i_0 \in \{1,2,...,p\}$ such that

$$f_{i_0}(x) - u_{i_0}^0 g_{i_0}(x) \geqq 0,$$

That is,

$$f_{i_0}(x) \geqq u_{i_0}^0 g_{i_0}(x), \text{ for some } i_0 \in \{1,2,...,p\},$$

which contradicts (3.24). Thus $\bar{x}$ is a weak minimum solution for (VFP). □

Theorem 5 (Karush-Kuhn-Tucker type sufficient conditions)**.** *Let* $\bar{x} \in X$, $\lambda^0 \in R^p$, $u_i^0 = \frac{f_i(\bar{x})}{g_i(\bar{x})}$, $i = 1,2,...,p$ *and* $v^0 \in R^m$ *such that the conditions (3.20)-(3.23) of Theorem 3.4 hold. Assume also that for any* $i = 1,2,...,p$, $f_i(.) - u_i^0 g_i(.)$ *is* $\bar{\rho}$*-generalized-pseudo-right upper-Dini-derivative locally arcwise connected (with respect to* H*) at* $\bar{x}$*, and for any* $j \in M(\bar{x})$, $h_j(.)$ *is* $\breve{\rho}$*-generalized-quasi-right upper-Dini-derivative locally arcwise connected (with respect to* H*) at* $\bar{x}$*, further the inequality* $\sum_{i=1}^p \lambda_i^0 \bar{\rho}_i(x,\bar{x}) + \sum_{j=1}^m v_j^0 \breve{\rho}_j(x,\bar{x}) \geqq 0$ *holds. Then* $\bar{x}$ *is a weak minimum solution to* (VFP).

Proof. Suppose contrary to the result. Hence there exists $x \in X$ such that

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(\bar{x})}{g_i(\bar{x})}, \text{ for any } i = 1,2,...,p.$$

That is,

$$f_i(x) - u_i^0 g_i(x) < 0, \text{ for any } i = 1,2,...,p,$$

which is equivalent to

$$f_i(x) - u_i^0 g_i(x) < f_i(\bar{x}) - u_i^0 g_i(\bar{x}), \text{ for any } i = 1,2,...,p.$$

Now, by the $\bar{\rho}$-generalized-pseudo-right upper-Dini-derivative locally arcwise connected (with respect to H) of $f_i(.) - u_i^0 g_i(.)$, $i = 1,2,...,p$, at $\bar{x}$, we get

$$(df_i)^+(H_{\bar{x},x}(0^+)) - u_i^0 (dg_i)^+(H_{\bar{x},x}(0^+)) < -\bar{\rho}_i(x,\bar{x}), \text{ for any } i = 1,2,...,p.$$

Using $\lambda_i^0 \in R_+^p$, $\sum\limits_{i=1}^p \lambda_i^0 = 1$, we obtain

$$\sum_{i=1}^p \lambda_i^0 \big((df_i)^+(H_{\bar{x},x}(0^+)) - u_i^0 (dg_i)^+(H_{\bar{x},x}(0^+))\big) < -\sum_{i=1}^p \lambda_i^0 \bar{\rho}_i(x,\bar{x}). \tag{3.26}$$

For $x \in X$ we have $h_j(x) \leqq 0$. But for $j \in M(\bar{x})$, $h_j(\bar{x}) = 0$. Hence

$$h_j(x) \leqq h_j(\bar{x}), \text{ for any } j \in M(\bar{x}),$$

which by using the $\breve{\rho}$-generalized-quasi-right upper-Dini-derivative locally arcwise connected (with respect to H) of $h_j(.)$, $j \in M(\bar{x})$ at $\bar{x}$, we get

$$(dh_j)^+(H_{\bar{x},x}(0^+)) \leqq -\breve{\rho}_j(x,\bar{x}), \ \forall j \in M(\bar{x}).$$

But $v^0 \in R_+^m$ and $v_j^0 = 0$ for $j \in N(\bar{x})$, we obtain

$$\sum_{j=1}^m v_j^0 (dh_j)^+(H_{\bar{x},x}(0^+)) \leqq -\sum_{j=1}^m v_j^0 \breve{\rho}_j(x,\bar{x}). \tag{3.27}$$

On adding (3.26), (3.27) and using $\sum_{i=1}^{p}\lambda_i^0\bar{\rho}_i(x,\bar{x})+\sum_{j=1}^{m}v_j^0\breve{\rho}_j(x,\bar{x})\geqq 0$, we get

$$\sum_{i=1}^{p}\lambda_i^0\big((df_i)^+(H_{\bar{x},x}(0^+))-u_i^0(dg_i)^+(H_{\bar{x},x}(0^+))\big)$$
$$+\sum_{j=1}^{m}v_j^0(dh_j)^+(H_{\bar{x},x}(0^+))<-\left(\sum_{i=1}^{p}\lambda_i^0\bar{\rho}_i(x,\bar{x})+\sum_{j=1}^{m}v_j^0\breve{\rho}_j(x,\bar{x})\right)\leqq 0.$$

That is,

$$\sum_{i=1}^{p}\lambda_i^0\big((df_i)^+(H_{\bar{x},x}(0^+))-u_i^0(dg_i)^+(H_{\bar{x},x}(0^+))\big)$$
$$+\sum_{j=1}^{m}v_j^0(dh_j)^+(H_{\bar{x},x}(0^+))<0,$$

which contradicts (3.20). Thus $\bar{x}$ is a weak minimum to (VFP). □

4. Duality

We consider, for (VFP), a general Mond-Weir dual (FMWD) as

$$\max\ \Psi(y,\lambda,u,v)=u-v_{I_0}^Th_{I_0}(y)e,$$

subject to

$$\sum_{i=1}^{p}\lambda_i\big((df_i)^+(H_{y,x}(0^+))-u_i(dg_i)^+(H_{y,x}(0^+))\big)+v^T(dh)^+(H_{y,x}(0^+))\geqq 0,$$
$$\text{for all } x\in X, \tag{4.1}$$
$$f_i(y)-u_ig_i(y)\geqq 0,\ \text{for any } i=1,2,...,p, \tag{4.2}$$
$$v_{I_S}^Th_{I_S}(y)\geqq 0,\ (1\leqq s\leqq \gamma), \tag{4.3}$$
$$\lambda^Te=1,\ \lambda\geqq 0,\ \lambda\in R^p, \tag{4.4}$$
$$u\geqq 0,\ u\in R^p,\ v\geqq 0,\ y\in X_0, \tag{4.5}$$

where e is p-tuple of $1'$s, $\gamma\geqq 1$, $I_s\cap I_t=\emptyset$ for $s\neq t$ and $\bigcup_{s=0}^{\gamma}I_s=\{1,2,...,m\}$, $v_{I_s}=(v_j)_{j\in I_s}$ and $h_{I_s}=(h_j)_{j\in I_s}$.

Let W denote the set of all feasible solutions to (FMWD). Also, we define the following sets

$$A=\{(\lambda,u,v)\in R^p\times R^p\times R^m:(y,\lambda,u,v)\in W \text{ for some } y\in X_0\}$$

and for $(\lambda,u,v)\in A$,

$$B(\lambda,u,v)=\big\{y\in X_0:(y,\lambda,u,v)\in W\big\}.$$

We put $B = \bigcup_{(\lambda,u,v)\in A} B(\lambda,u,v)$ and note that $B \subset X_0$. Also, we note that if $(y,\lambda,u,v) \in W$ then $(\lambda,u,v) \in A$ and $y \in B(\lambda,u,v)$.

Now we establish certain duality results between (VFP) and (FMWD). Assume that f, g and h are right upper-Dini-derivative functions of x with respect to the arc H on X.

Theorem 6 (Weak duality)**.** *Let $x \in X$ and $(y,\lambda,u,v) \in W$ be feasible solutions for* (VFP) *and* (FMWD) *respectively. Assume also that $\sum_{i=1}^{p} \lambda_i\big(f_i(.) - u_i g_i(.)\big) + v_{I_0}^T h_{I_0}(.)$ is ρ-generalized-pseudo-right upper-Dini-derivative locally arcwise connected at y on $B(\lambda,u,v)$ and for $1 \leqq s \leqq \gamma$, $v_{I_s}^T h_{I_s}(.)$ are $\breve{\rho}_s$-generalized-quasi-right upper-Dini-derivative locally arcwise connected at y on $B(\lambda,u,v)$, further $\rho(x,y) + \sum_{s=1}^{\gamma} \breve{\rho}_s(x,y) \geqq 0$. Then the following cannot hold:*

$$f_i(x) - u_i g_i(x) \leqq v_{I_0}^T h_{I_0}(y) \ for\ any\ i = 1,2,...,p, \tag{4.6}$$

and

$$f_{i_0}(x) - u_{i_0} g_{i_0}(x) < v_{I_0}^T h_{I_0}(y) \ for\ some\ i_0 \in \{1,2,...,p\}. \tag{4.7}$$

Proof. Using the feasibility of x for (VFP) and of (y,λ,u,v) for (FMWD), we have

$$v_{I_S}^T h_{I_S}(x) \leqq v_{I_S}^T h_{I_S}(y),\ (1 \leqq s \leqq \gamma),$$

which by using $\breve{\rho}_s$-generalized-quasi-right upper-Dini-derivative locally arcwise connected of $v_{I_s}^T h_{I_s}(.)$, for $1 \leqq s \leqq \gamma$ at y on $B(\lambda,u,v)$, we obtain

$$\sum_{j\in I_s} v_j (dh_j)^+(H_{y,x}(0^+)) \leqq -\breve{\rho}_s(x,y),\ 1 \leqq s \leqq \gamma. \tag{4.8}$$

Now, we suppose to the contrary of the result of the theorem that (4.6) and (4.7) holds. Then by (4.4), we get

$$\sum_{i=1}^{p} \lambda_i\big(f_i(x) - u_i g_i(x)\big) < v_{I_0}^T h_{I_0}(y). \tag{4.9}$$

On the other hand, by using (4.2), (4.4), (4.5) and the feasibility of x for (VFP), we have

$$v_{I_0}^T h_{I_0}(x) \leqq 0 \leqq \sum_{i=1}^{p} \lambda_i\big(f_i(y) - u_i g_i(y)\big). \tag{4.10}$$

Combining (4.9) and (4.10), we get

$$\sum_{i=1}^{p} \lambda_i\big(f_i(x) - u_i g_i(x)\big) + v_{I_0}^T h_{I_0}(x) < \sum_{i=1}^{p} \lambda_i\big(f_i(y) - u_i g_i(y)\big) + v_{I_0}^T h_{I_0}(y),$$

which by using ρ-generalized-pseudo-right upper-Dini-derivative locally arcwise connected of $\sum_{i=1}^{p} \lambda_i \big(f_i(.) - u_i g_i(.)\big) + v_{I_0}^T h_{I_0}(.)$ at y on $B(\lambda, u, v)$, we obtain

$$\sum_{i=1}^{p} \lambda_i \big((df_i)^+(H_{y,x}(0^+)) - u_i (dg_i)^+(H_{y,x}(0^+))\big) + \sum_{j\in I_0} v_j (dh_j)^+(H_{y,x}(0^+)) < -\rho(x,y).$$

Now, by (4.1) and $\rho(x,y) + \sum_{s=1}^{\gamma} \breve{\rho}_s(x,y) \geqq 0$, we obtain

$$\sum_{s=1}^{\gamma} \sum_{j\in I_s} v_j (dh_j)^+(H_{y,x}(0^+)) > \rho(x,y) \geqq -\sum_{s=1}^{\gamma} \breve{\rho}_s(x,y).$$

That is,

$$\sum_{s=1}^{\gamma} \sum_{j\in I_s} v_j (dh_j)^+(H_{y,x}(0^+)) > -\sum_{s=1}^{\gamma} \breve{\rho}_s(x,y),$$

which contradicts (4.8). Thus, the theorem is proved. □

Theorem 7 (Strong duality)**.** *Let $\bar{x}$ be a (local) weak minimum solution for* (VFP). *Assume that the hypotheses of Theorem 2 hold. Then, there exist $(\bar{\lambda}^0, \bar{u}^0, \bar{v}^0) \in R^p \times R^p \times R^m$ such that $(\bar{x}, \bar{\lambda}^0, \bar{u}^0, \bar{v}^0)$ is feasible for dual* (FMWD) *and the two objectives are equal. Also, if the weak duality Theorem 6 holds for all feasible solutions of the problems* (VFP) *and* (FMWD)*, then $(\bar{x}, \bar{\lambda}^0, \bar{u}^0, \bar{v}^0)$ is a (local) weak minimum solution of* (FMWD).

Proof. Since $\bar{x}$ is a (local) weak minimum solution of (VFP) and all the assumptions of Theorem 2 are satisfied, therefore, there exist $\bar{\lambda}^0 \in R^p_+$, $\bar{u}^0 \in R^p_+$ and $\bar{v}^0 \in R^m_+$ such that

$$\sum_{i=1}^{p} \bar{\lambda}_i^0 \big((df_i)^+(H_{\bar{x},x}(0^+)) - \bar{u}_i^0 (dg_i)^+(H_{\bar{x},x}(0^+))\big) + \bar{v}^{0^T} (dh)^+(H_{\bar{x},x}(0^+)) \geqq 0,$$

$$\text{for all } x \in X_0,$$

$$\bar{v}^{0^T} h(\bar{x}) = 0,$$

$$\sum_{i=1}^{p} \bar{\lambda}_i^0 = 1,$$

where $\bar{u}_i^0 = f_i(\bar{x})/g_i(\bar{x}), i = 1, 2, ..., p$. Thus, $(\bar{x}, \bar{\lambda}^0, \bar{u}^0, \bar{v}^0)$ is feasible for (FMWD) and the two objectives are equal. The (local) weak minimum solution of $(\bar{x}, \bar{\lambda}^0, \bar{u}^0, \bar{v}^0)$ for (FMWD) follows from weak duality Theorem 6. This completes the proof. □

Theorem 8 (Strict converse duality). *Let $\bar{x}$ and $(\bar{y}, \bar{\lambda}, \bar{u}, \bar{v})$ be (local) weak minimum solutions for* (VFP) *and* (FMWD)*, respectively such that $\bar{u}_i^0 = f_i(\bar{x})/g_i(\bar{x}), i = 1, 2, ..., p$ and*

$$\sum_{i=1}^{p} \bar{\lambda}_i \big(f_i(\bar{x}) - \bar{u}_i g_i(\bar{x}) \big) \leqq \bar{v}_{I_0}^T h_{I_0}(\bar{y}). \tag{4.11}$$

Assume also that $\sum_{i=1}^{p} \bar{\lambda}_i \big(f_i(.) - \bar{u}_i g_i(.) \big) + \bar{v}_{I_0}^T h_{I_0}(.)$ is strictly ρ-generalized pseudo-right upper-Dini-derivative locally arcwise connected at $\bar{y}$ on $B(\bar{\lambda}, \bar{u}, \bar{v})$ and for $1 \leqq s \leqq \gamma$, $\bar{v}_{I_s}^T h_{I_s}(.)$ are $\breve{\rho}_s$-generalized-quasi-right upper-Dini-derivative locally arcwise connected at $\bar{y}$ on $B(\bar{\lambda}, \bar{u}, \bar{v})$, further the inequality $\rho(\bar{x}, \bar{y}) + \sum_{s=1}^{\gamma} \breve{\rho}_s(\bar{x}, \bar{y}) \geqq 0$ holds. Then $\bar{x} = \bar{y}$; that is, $\bar{y}$ is an efficient solution for (VFP).

Proof. Suppose that $\bar{x} \neq \bar{y}$. Using the feasibility of $\bar{x}$ for (VFP) and of $(\bar{y}, \bar{\lambda}, \bar{u}, \bar{v})$ for (FMWD), we have

$$\bar{v}_{I_S}^T h_{I_S}(\bar{x}) \leqq \bar{v}_{I_S}^T h_{I_S}(\bar{y}), \ (1 \leqq s \leqq \gamma),$$

which by using $\breve{\rho}_s$-generalized-quasi-right upper-Dini-derivative locally arcwise connected of $\bar{v}_{I_s}^T h_{I_s}(.)$, for $1 \leqq s \leqq \gamma$ at $\bar{y}$ on $B(\bar{\lambda}, \bar{u}, \bar{v})$, we obtain

$$\sum_{j \in I_s} \bar{v}_j (dh_j)^+ (H_{\bar{y},\bar{x}}(0^+)) \leqq -\breve{\rho}_s(\bar{x}, \bar{y}), \ 1 \leqq s \leqq \gamma. \tag{4.12}$$

Now, by (4.1) and (4.12), we get

$$\begin{aligned}
0 &\leqq \sum_{i=1}^{p} \bar{\lambda}_i \big((df_i)^+ (H_{\bar{y},\bar{x}}(0^+)) - \bar{u}_i (dg_i)^+ (H_{\bar{y},\bar{x}}(0^+)) \big) + \bar{v}^T (dh)^+ (H_{\bar{y},\bar{x}}(0^+)) \\
&= \sum_{i=1}^{p} \bar{\lambda}_i \big((df_i)^+ (H_{\bar{y},\bar{x}}(0^+)) - \bar{u}_i (dg_i)^+ (H_{\bar{y},\bar{x}}(0^+)) \big) + \sum_{j \in I_0} \bar{v}_j (dh_j)^+ (H_{\bar{y},\bar{x}}(0^+)) \\
&\quad + \sum_{s=1}^{\gamma} \sum_{j \in I_s} \bar{v}_j (dh_j)^+ (H_{\bar{y},\bar{x}}(0^+)) \\
&\leqq \sum_{i=1}^{p} \bar{\lambda}_i \big((df_i)^+ (H_{\bar{y},\bar{x}}(0^+)) - \bar{u}_i (dg_i)^+ (H_{\bar{y},\bar{x}}(0^+)) \big) + \sum_{j \in I_0} \bar{v}_j (dh_j)^+ (H_{\bar{y},\bar{x}}(0^+)) \\
&\quad - \sum_{s=1}^{\gamma} \breve{\rho}_s(\bar{x}, \bar{y}).
\end{aligned}$$

Since, $\rho(\bar{x},\bar{y})+\sum_{s=1}^{\gamma}\breve{\rho}_s(\bar{x},\bar{y})\geqq 0$, we have

$$\sum_{i=1}^{p}\bar{\lambda}_i\big((df_i)^+(H_{\bar{y},\bar{x}}(0^+))-\bar{u}_i(dg_i)^+(H_{\bar{y},\bar{x}}(0^+))\big)$$
$$+\sum_{j\in I_0}\bar{v}_j(dh_j)^+(H_{\bar{y},\bar{x}}(0^+))\geqq -\rho(\bar{x},\bar{y}),$$

which by using strictly ρ-generalized pseudo-right upper-Dini-derivative locally arcwise connected of $\sum_{i=1}^{p}\bar{\lambda}_i\big(f_i(.)-\bar{u}_i g_i(.)\big)+\bar{v}_{I_0}^T h_{I_0}(.)$ at $\bar{y}$ on $B(\bar{\lambda},\bar{u},\bar{v})$, we obtain

$$\sum_{i=1}^{p}\bar{\lambda}_i\big(f_i(\bar{x})-\bar{u}_i g_i(\bar{x})\big)+\bar{v}_{I_0}^T h_{I_0}(\bar{x})>\sum_{i=1}^{p}\bar{\lambda}_i\big(f_i(\bar{y})-\bar{u}_i g_i(\bar{y})\big)+\bar{v}_{I_0}^T h_{I_0}(\bar{y}).$$

Using (4.2), (4.4), (4.5) and the feasibility of $\bar{x}$ for (VFP), the above inequality yields

$$\sum_{i=1}^{p}\bar{\lambda}_i\big(f_i(\bar{x})-\bar{u}_i g_i(\bar{x})\big)>\bar{v}_{I_0}^T h_{I_0}(\bar{y}),$$

which contradicts (4.11). Thus, the theorem is proved. □

5. CONCLUSION

In this paper, we established necessary and sufficient optimality conditions and duality results for a class of multiobjective fractional programming problem under generalized convexity using right-upper Dini derivative. The methods used here can be extended to the study of nonsmooth variational and nonsmooth control problems, which will orient the future research of the author.

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