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A NOTE ON A SERIES CONTAINING THE LAGUERRE POLYNOMIALS

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Abstract. Expressions for the summation of the series involving the Laguerre polynomials

$$S(\pm\nu,\pm j)\equiv e^{-x}\sum_{n=0}^{\infty}\frac{x^n\,L_n^{(\nu)}(x)}{(1\pm\nu\pm j)_n}$$

for any non-negative integer j are obtained in terms of generalized hypergeometric functions. These results provide alternative, and in some cases simpler expressions to those recently obtained in the literature.

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1. INTRODUCTION

The generalized Laguerre polynomials $L_n^{(\nu)}(x)$ are encountered in many branches of pure and applied mathematics. They form an orthogonal set on $[0,\infty)$ with the weight function $x^\nu e^{-x}$, with the first three polynomials given by

$$L_0^{(\nu)}(x)=1,$$
$$L_1^{(\nu)}(x)=1-x+\nu,$$
$$L_2^{(\nu)}(x)=\frac{1}{2}x^2-(\nu+2)x+\frac{1}{2}(\nu+1)(\nu+2).$$

In general, $L_n^{(\nu)}(x)$ can be represented as a terminating confluent hypergeometric function ${}_1F_1$ in the form

$$L_n^{(\nu)}(x)=\frac{(\nu+1)_n}{n!}\,{}_1F_1(-n;\nu+1;x).$$

Here $(a)_n$ denotes the Pochhammer symbol, or rising factorial, defined by $(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}$.

In [2], Kim *et al.* obtained summation formulas for the two series involving the generalized Laguerre polynomial $L_n^{(\nu)}(x)$ given by

$$\sum_{n=0}^{\infty} \frac{x^n L_n^{(\nu)}(x)}{(1\pm\nu+j)_n}$$

for integer j, where $-5 \le j \le 5$. Recently, Brychkov [1] has extended these results for any integer j. The aim of this note is to derive alternative expressions for the summation of the series

$$S(\pm\nu,\pm j) \equiv e^{-x}\sum_{n=0}^{\infty} \frac{x^n L_n^{(\nu)}(x)}{(1\pm\nu\pm j)_n}$$

for any non-negative integer j. Our results are different from, and in some cases simpler, than those obtained in [1].

2. THE SERIES $S(\nu,\pm j)$

We start with the transformation [2, (3.5)]

$$e^{-x}\sum_{n=0}^{\infty} \frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}(-xy)^n L_n^{(\nu)}(x) = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\,{}_{p+2}F_q\left[\begin{matrix}-n,-n-\nu,a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix};y\right], \tag{2.1}$$

where p and q are non-negative integers and ${}_pF_q$ denotes the generalized hypergeometric function. In this, if we take $p=0$, $q=1$, $b_1=1+\nu+j$ and $y=-1$, then

$$e^{-x}\sum_{n=0}^{\infty} \frac{x^n L_n^{(\nu)}(x)}{(1+\nu+j)_n} = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\,{}_2F_1\left[\begin{matrix}-n,-n-\nu\\ 1+\nu+j\end{matrix};-1\right]. \tag{2.2}$$

The ${}_2F_1$ series on the right-hand side of (2.2) can be evaluated with the help of the generalized Kummer summation theorem [3]

$${}_2F_1\left[\begin{matrix}a,b\\ 1+a-b+j\end{matrix};-1\right] = \frac{2^{-a}\Gamma(\frac12)\Gamma(b-j)\Gamma(1+a-b+j)}{\Gamma(b)\Gamma(\frac12 a-b+\frac12 j+\frac12)\Gamma(\frac12 a-b+\frac12 j+1)} \times\sum_{r=0}^{j}(-1)^r\binom{j}{r}\frac{\Gamma(\frac12 a-b+\frac12 j+\frac12 r+\frac12)}{\Gamma(\frac12 a-\frac12 j+\frac12 r+\frac12)} \tag{2.3}$$

for $j=0,1,2,\dots$.
After some straightforward simplification, we obtain

$$S(\nu,j) \equiv e^{-x}\sum_{n=0}^{\infty}\frac{x^n L_n^{(\nu)}(x)}{(1+\nu+j)_n}$$

$$= \frac{(-1)^j 2^{2\nu+j}\Gamma(1+\nu)}{\Gamma(1+2\nu+j)} \sum_{r=0}^{j} (-1)^r \binom{j}{r} \Bigg\{ \frac{\Gamma(\nu+\frac{1}{2}j+\frac{1}{2}r+\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}j+\frac{1}{2}r)}$$

$$\times {}_4F_5\left[\begin{matrix} \frac{1}{2}+\frac{1}{2}\nu,\ 1+\frac{1}{2}\nu,\ \frac{1}{2}+\nu+\frac{1}{2}j+\frac{1}{2}r,\ \frac{1}{2}+\frac{1}{2}j-\frac{1}{2}r \\ \frac{1}{2},\ \frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}j,\ 1+\frac{1}{2}\nu+\frac{1}{2}j,\ \frac{1}{2}+\nu+\frac{1}{2}j,\ 1+\nu+\frac{1}{2}j \end{matrix} ;-x^2 \right]$$

$$- \frac{4x(1+\nu)}{(1+\nu+j)(1+2\nu+j)} \frac{\Gamma(\nu+\frac{1}{2}j+\frac{1}{2}r+1)}{\Gamma(\frac{1}{2}r-\frac{1}{2}j)}$$

$$\times {}_4F_5\left[\begin{matrix} 1+\frac{1}{2}\nu, \frac{3}{2}+\frac{1}{2}\nu, 1+\nu+\frac{1}{2}j+\frac{1}{2}r, 1+\frac{1}{2}j-\frac{1}{2}r \\ \frac{3}{2}, 1+\frac{1}{2}\nu+\frac{1}{2}j, \frac{3}{2}+\frac{1}{2}\nu+\frac{1}{2}j, 1+\nu+\frac{1}{2}j, \frac{3}{2}+\nu+\frac{1}{2}j \end{matrix} ;-x^2 \right] \Bigg\} \tag{2.4}$$

for $j = 0, 1, 2, \ldots$.

Again, in (2.1), if we take $p = 0$, $q = 1$, $b_1 = 1+\nu-j$ and $y = -1$, then

$$e^{-x} \sum_{n=0}^{\infty} \frac{x^n L_n^{(\nu)}(x)}{(1+\nu-j)_n} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} {}_2F_1\left[\begin{matrix} -n, -n-\nu \\ 1+\nu-j \end{matrix} ;-1 \right]. \tag{2.5}$$

The ${}_2F_1$ series on the right-hand side of (2.5) can be evaluated with the help of the known result [3]

$${}_2F_1\left[\begin{matrix} a, b \\ 1+a-b-j \end{matrix} ;-1 \right] = \frac{2^{-a}\Gamma(\frac{1}{2})\Gamma(1+a-b-j)}{\Gamma(\frac{1}{2}a-b-\frac{1}{2}j+\frac{1}{2})\Gamma(\frac{1}{2}a-b-\frac{1}{2}j+1)}$$

$$\times \sum_{r=0}^{j} \binom{j}{r} \frac{\Gamma(\frac{1}{2}a-b-\frac{1}{2}j+\frac{1}{2}r+\frac{1}{2})}{\Gamma(\frac{1}{2}a-\frac{1}{2}j+\frac{1}{2}r+\frac{1}{2})} \tag{2.6}$$

for $j = 0, 1, 2, \ldots$ and, after some simplification, we obtain

$$S(\nu, -j) \equiv e^{-x} \sum_{n=0}^{\infty} \frac{x^n L_n^{(\nu)}(x)}{(1+\nu-j)_n}$$

$$= \frac{2^{2\nu-j}\Gamma(1+\nu-j)}{\Gamma(1+2\nu-j)} \sum_{r=0}^{j} \binom{j}{r} \Bigg\{ \frac{\Gamma(\nu-\frac{1}{2}j+\frac{1}{2}r+\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}j+\frac{1}{2}r)}$$

$$\times {}_2F_3\left[\begin{matrix} \frac{1}{2}+\nu-\frac{1}{2}j+\frac{1}{2}r, \frac{1}{2}+\frac{1}{2}j-\frac{1}{2}r \\ \frac{1}{2}, \frac{1}{2}+\nu-\frac{1}{2}j, 1+\nu-\frac{1}{2}j \end{matrix} ;-x^2 \right]$$

$$- \frac{4x\Gamma(\nu-\frac{1}{2}j+\frac{1}{2}r+1)}{(1+2\nu-j)\Gamma(\frac{1}{2}r-\frac{1}{2}j)} {}_2F_3\left[\begin{matrix} 1+\nu-\frac{1}{2}j+\frac{1}{2}r, 1+\frac{1}{2}j-\frac{1}{2}r \\ \frac{3}{2}, 1+\nu-\frac{1}{2}j, \frac{3}{2}+\nu-\frac{1}{2}j \end{matrix} ;-x^2 \right] \Bigg\} \tag{2.7}$$

for $j = 0, 1, 2, \ldots$.

3. THE SERIES $S(-\nu, \pm j)$

Further, if we take $p = 0$, $q = 1$, $b_1 = 1-\nu+j$ and $y = -1$ in (2.1), we find

$$e^{-x}\sum_{n=0}^{\infty}\frac{x^n L_n^{(\nu)}(x)}{(1-\nu+j)_n}=\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\,{}_2F_1\left[\begin{matrix}-n,-n-\nu\\1-\nu+j\end{matrix};-1\right]. \tag{3.1}$$

The ${}_2F_1$ series on the right-hand side of (3.1) can be evaluated by (2.3) to produce the result after some simplification

$$\begin{aligned}S(-\nu,j)&\equiv e^{-x}\sum_{n=0}^{\infty}\frac{x^n L_n^{(\nu)}(x)}{(1-\nu+j)_n}\\&=\frac{(-2)^j}{j!}\sum_{r=0}^{j}(-1)^r\binom{j}{r}\left\{\frac{\Gamma(-\frac{1}{2}\nu+\frac{1}{2}j+\frac{1}{2}r+\frac{1}{2})}{\Gamma(-\frac{1}{2}\nu-\frac{1}{2}j+\frac{1}{2}r+\frac{1}{2})}\right.\\&\times{}_3F_4\left[\begin{matrix}1,\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}j-\frac{1}{2}r,\frac{1}{2}-\frac{1}{2}\nu+\frac{1}{2}j+\frac{1}{2}r\\\frac{1}{2}+\frac{1}{2}j,1+\frac{1}{2}j,\frac{1}{2}-\frac{1}{2}\nu+\frac{1}{2}j,1-\frac{1}{2}\nu+\frac{1}{2}j\end{matrix};-x^2\right]\\&-\frac{4x}{(j+1)(1-\nu+j)}\,\frac{\Gamma(-\frac{1}{2}\nu+\frac{1}{2}j+\frac{1}{2}r+1)}{\Gamma(-\frac{1}{2}\nu-\frac{1}{2}j+\frac{1}{2}r)}\\&\left.\times{}_3F_4\left[\begin{matrix}1,\ 1+\frac{1}{2}\nu+\frac{1}{2}j-\frac{1}{2}r,\ 1-\frac{1}{2}\nu+\frac{1}{2}j+\frac{1}{2}r\\1+\frac{1}{2}j,\ \frac{3}{2}+\frac{1}{2}j,\ 1-\frac{1}{2}\nu+\frac{1}{2}j,\ \frac{3}{2}-\frac{1}{2}\nu+\frac{1}{2}j\end{matrix};-x^2\right]\right\}\end{aligned} \tag{3.2}$$

for $j = 0, 1, 2, \ldots$.

Finally, if we take $p = 0$, $q = 1$, $b_1 = 1-\nu-j$ and $y = -1$ in (2.1), we find

$$e^{-x}\sum_{n=0}^{\infty}\frac{x^n L_n^{(\nu)}(x)}{(1-\nu-j)_n}=\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\,{}_2F_1\left[\begin{matrix}-n,-n-\nu\\1-\nu-j\end{matrix};-1\right]. \tag{3.3}$$

The ${}_2F_1$ series on the right-hand side of (3.3) can be evaluated by (2.6) to produce the result after some simplification

$$\begin{aligned}S(-\nu,-j)&\equiv e^{-x}\sum_{n=0}^{\infty}\frac{x^n L_n^{(\nu)}(x)}{(1-\nu-j)_n}\\&=2^{-j}\sum_{r=0}^{j}\binom{j}{r}\left\{{}_2F_3\left[\begin{matrix}\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}j+\frac{1}{2}r,\frac{1}{2}+\frac{1}{2}\nu+\frac{1}{2}j-\frac{1}{2}r\\\frac{1}{2},\frac{1}{2}-\frac{1}{2}\nu-\frac{1}{2}j,1-\frac{1}{2}\nu-\frac{1}{2}j\end{matrix};-x^2\right]\right.\\&\left.-\frac{2x(\nu+j-r)}{\nu+j-1}\,{}_2F_3\left[\begin{matrix}1-\frac{1}{2}\nu-\frac{1}{2}j+\frac{1}{2}r,\ 1+\frac{1}{2}\nu+\frac{1}{2}j-\frac{1}{2}r\\\frac{3}{2},\ 1-\frac{1}{2}\nu-\frac{1}{2}j,\ \frac{3}{2}-\frac{1}{2}\nu-\frac{1}{2}j\end{matrix};-x^2\right]\right\}\end{aligned} \tag{3.4}$$

for $j = 0, 1, 2, \ldots$.

4. Concluding remarks

To conclude we make a brief comparison of the results (2.4), (2.7), (3.2) and (3.4) with those obtained in [1]. The summations $S(\nu, \pm j)$ derived by Brychkov were expressed respectively in terms of finite sums of ${}_2F_3(-x^2)$ functions and Bessel functions of the first kind. The summations $S(-\nu, \pm j)$ were expressed respectively in terms of finite sums of four ${}_4F_3(-x^2)$ functions and four ${}_6F_7(-x^2)$ functions, including the Jacobi polynomials of zero argument. Our expressions in (3.2) and (3.4) involve simpler finite sums of two ${}_3F_4(-x^2)$ and two ${}_2F_3(-x^2)$ functions, respectively.

Finally, we mention that the summations $S(\pm\nu, \pm j)$ have been verified numerically with the help of *Mathematica*.

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