Miskolc Mathematical Notes
Vol. 6 (2005), No 2, pp. 173-183

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2005.97

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HU ISSN 1787-2413
electronic version

ON THE DIRECTIONAL DERIVATIVE AND DIRECTIONAL CONTINUITY OF SET VALUED MAPS

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[Received: June 16, 2004]

ABSTRACT. In this study, the directional lower and upper derivative sets of the set valued map the values of which have piecewise smooth boundary are investigated and obtained. Moreover, the connection between the directional continuity and the directional derivative sets of set-valued maps is investigated.

Mathematics Subject Classification: 26E25, 49J53, 58C20

Keywords: set-valued map, derivative sets, directional continuity

1. Introduction

The concepts of the directional lower and upper derivative sets of set valued maps and continuity of set-valued maps are studied in many papers (see, e. g., [2, 3, 5–7, 9, 12–21]).

The concepts of the directional lower and upper derivative sets of the set valued maps are based on the concepts of the lower and upper Bouligand cones [5, 6]. In this paper, when the value of set-valued map has a piecewise smooth boundary, the directional lower and upper derivative sets are investigated. Furthermore, the connection between the directional continuity and the directional derivative sets of the set-valued maps is investigated.

In what follows, $\mathrm{cl}\,(\mathbb{R}^m)$ (resp., $\mathrm{comp}\,(\mathbb{R}^m)$) denotes the set of all nonempty closed (resp., compact) subsets in $\mathbb{R}^m$.

Let $a(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ be a set valued map and let $x_0 \in \mathbb{R}^n$. It is said that $a(\cdot)$ is upper semi-continuous at x_0 if for all open neighbourhoods $N(a(x_0))$ of the set $a(x_0)$, there exists a neighbourhood $N(x_0)$ of x_0 such that $a(x) \subset N(a(x_0))$ for all $x \in N(x_0)$.

It is said that $a(\cdot)$ is lower semi-continuous at x_0 if for all $y_0 \in a(x_0)$ and for all open neighbourhoods $N(y_0)$ of y_0, there exists a neighbourhood $N(x_0)$ of x_0 such that $a(x) \cap N(y_0) \neq \varnothing$ for all $x \in N(x_0)$ [3, 18].

It is well-known that a set-valued map $a(\cdot) : \mathbb{R}^n \to \mathrm{comp}\,(\mathbb{R}^m)$ is upper semi-continuous at x_0 if and only if for all $\varepsilon > 0$, there exists a $\delta(\varepsilon, x_0) > 0$ such that $\|x - x_0\| < \delta(\varepsilon, x_0) \Rightarrow a(x) \subset a(x_0) + \varepsilon B$, where $B = \{y \in \mathbb{R}^m : \|y\| < 1\}$.

Furthermore, $a(\cdot) : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^m)$ is lower semi-continuous at x_0 if and only if for all $\varepsilon > 0$, there exists a $\delta(\varepsilon, x_0) > 0$ such that $\|x - x_0\| < \delta(\varepsilon, x_0) \Rightarrow a(x_0) \subset a(x) + \varepsilon B$, where $B = \{y \in \mathbb{R}^m : \|y\| < 1\}$ [1, 4].

It is said that the set-valued map $a(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is locally bounded at $x_0 \in \mathbb{R}^n$ if $a(\cdot)$ is bounded in a neighbourhood of x_0.

2. Directional derivative sets of set-valued maps

Let $a(\cdot) : \mathbb{R}^n \to \operatorname{cl}(\mathbb{R}^m)$ be an upper semi-continuous set-valued map. Let us consider the following sets. For $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ and vector $f \in \mathbb{R}^n$, we set

$$Da(x, y) \mid (f) = \left\{ d \in \mathbb{R}^m : \liminf_{\delta \to +0} \frac{1}{\delta} \operatorname{dist}(y + \delta d, a(x + \delta f)) = 0 \right\},$$

$$D^* a(x, y) \mid (f) = \left\{ v \in \mathbb{R}^m : \lim_{\delta \to +0} \frac{1}{\delta} \operatorname{dist}(y + \delta d, a(x + \delta f)) = 0 \right\}.$$

Here for $x \in \mathbb{R}^n$, $D \subset \mathbb{R}^n$, $\operatorname{dist}(x, D) = \inf_{d \in D} \|x - d\|$. $Da(x, y) \mid (f)$ $(D^* a(x, y) \mid (f))$ is called the upper (lower) derivative set of the set valued map $a(\cdot)$ at (x, y) in the direction f.

Note that the directional upper (lower) derivative set of the set-valued map $a(\cdot)$ is closed and there is a connection between the upper (lower) derivative set of a set-valued map and the upper (lower) contingent cone which is used to investigate several problems in nonsmooth analysis (see, e. g., [3, 8, 15]).

It is obvious that $D^* a(x, y) \mid (f) \subset Da(x, y) \mid (f)$.

$$A = \operatorname{graph} a(\cdot) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y \in a(x)\}$$

denotes the graph of the set-valued map $a(\cdot)$. Since $a(\cdot)$ is upper semicontinuous, $\operatorname{graph} a(\cdot)$ is a closed set. It is possible to show that $Da(x, y) \mid (f) = D^* a(x, y) \mid (f) = \varnothing$ if $(x, y) \notin \operatorname{graph} a(\cdot)$, $Da(x, y) \mid (f) = D^* a(x, y) \mid (f) = \mathbb{R}^m$ if $(x, y) \in \operatorname{int}(\operatorname{graph} a(\cdot))$ where $\operatorname{int}(\operatorname{graph} a(\cdot))$ denotes the interior of $\operatorname{graph} a(\cdot)$.

Suppose that the set-valued map $a(\cdot)$ is given as

$$a(x) = \{y \in \mathbb{R}^m : c(x, y) \geq 0\} \tag{2.1}$$

where $c(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{R}^m \to R$ is a continuous function on $\mathbb{R}^n \times \mathbb{R}^m$ and locally Lipschitz on $\mathbb{R}^m$. The lower and upper derivative of $c(\cdot, \cdot)$ at the point (x, y) in the direction (f, d) is denoted by $\frac{\partial^- c(x,y)}{\partial(f,d)}$ and $\frac{\partial^+ c(x,y)}{\partial(f,d)}$, respectively, and defined by

$$\frac{\partial^- c(x, y)}{\partial(f, d)} = \liminf_{\delta \to +0} \left[c(x + \delta f, y + \delta d) - c(x, y) \right] \delta^{-1},$$

$$\frac{\partial^+ c(x, y)}{\partial(f, d)} = \limsup_{\delta \to +0} \left[c(x + \delta f, y + \delta d) - c(x, y) \right] \delta^{-1},$$

respectively. If

$$\frac{\partial c(x,y)}{\partial(f,d)} = \lim_{\delta\to+0}\left[c(x+\delta f, y+\delta d) - c(x,y)\right]\delta^{-1}$$

exists and is finite, then $c(\cdot,\cdot)$ is called differentiable at the point (x,y) in the direction (f,d) and $\frac{\partial c(x,y)}{\partial(f,d)}$ denotes the derivative of $c(\cdot,\cdot)$ at the point (x,y) in the direction (f,d).

We introduce the sets

$$H_*^-(x,y) \mid (f) = \left\{d \in \mathbb{R}^m : \frac{\partial^+ c(x,y)}{\partial(f,d)} > 0\right\},$$
$$H_*(x,y) \mid (f) = \left\{d \in \mathbb{R}^m : \frac{\partial^+ c(x,y)}{\partial(f,d)} \geq 0\right\},$$
$$E_*^-(x,y) \mid (f) = \left\{d \in \mathbb{R}^m : \frac{\partial^- c(x,y)}{\partial(f,d)} > 0\right\},$$
$$E_*(x,y) \mid (f) = \left\{d \in \mathbb{R}^m : \frac{\partial^- c(x,y)}{\partial(f,d)} \geq 0\right\}.$$

Proposition 1. *Let the set-valued map $a(\cdot)$ be in the form* (2.1). *Then for all $(x,y) \in \partial A$ and $f \in \mathbb{R}^n$,*

$$\operatorname{cl} H_*^-(x,y) \mid (f) \subset Da(x,y) \mid (f) \subset H_*(x,y) \mid (f),$$
$$\operatorname{cl} E_*^-(x,y) \mid (f) \subset D^*a(x,y) \mid (f) \subset E_*(x,y) \mid (f),$$

where ∂A denotes the boudary of A and cl *A denotes the closure of A.*

Proposition 2. *Let $(x,y) \in \partial A$, $c(\cdot,\cdot)$ be differentiable at (x,y) and $\frac{\partial c(x,y)}{\partial y} \neq 0$. Then it is possible to show that*

$$\begin{aligned} Da(x,y) \mid (f) &= D^*a(x,y) \mid (f) \\ &= \left\{d \in \mathbb{R}^m : \left\langle \frac{\partial c(x,y)}{\partial x}, f\right\rangle + \left\langle \frac{\partial c(x,y)}{\partial y}, d\right\rangle \geq 0\right\}, \end{aligned}$$

where the symbol $\langle\cdot,\cdot\rangle$ denotes the inner product.

Remark 1. Now suppose that the set-valued map $a(\cdot) : \mathbb{R}^n \to \operatorname{cl}(\mathbb{R}^m)$ is given by the relation

$$a(x) = \{y \in \mathbb{R}^m : \max_{i\in I} c_i(x,y) \geq 0\} \tag{2.2}$$

where I is a finite set and $c_i(\cdot,\cdot)$ is a continuous differentiable function for all $i \in I$. Then (see [10, 11]) $b(x,y) = \max_{i\in I} c_i(x,y)$ is a directional derivable function and

$$\frac{\partial c(x,y)}{\partial(f,d)} = \max_{i\in I_*(x,y)}\left[\left\langle \frac{\partial c_i(x,y)}{\partial x}, f\right\rangle + \left\langle \frac{\partial c_i(x,y)}{\partial y}, d\right\rangle\right],$$

where

$$I_*(x,y) = \{i_* \in I : c_{i_*}(x,y) = \max_{i\in I} c_i(x,y)\}.$$

In that case, it follows herefrom that

$$\begin{aligned} E_*^-(x,y) \mid (f) &= H_*^-(x,y) \mid (f) \\ &= \left\{ d \in \mathbb{R}^m : \max_{i \in I_*(x,y)} \left[\left\langle \frac{\partial c_i(x,y)}{\partial x}, f \right\rangle + \left\langle \frac{\partial c_i(x,y)}{\partial y}, d \right\rangle \right] > 0 \right\}, \\ E_*(x,y) \mid (f) &= H_*(x,y) \mid (f) \\ &= \left\{ d \in \mathbb{R}^m : \max_{i \in I_*(x,y)} \left[\left\langle \frac{\partial c_i(x,y)}{\partial x}, f \right\rangle + \left\langle \frac{\partial c_i(x,y)}{\partial y}, d \right\rangle \right] \geq 0 \right\}. \end{aligned}$$

Theorem 1. *Let the set-valued map $a(\cdot) : \mathbb{R}^n \to \mathrm{cl}\,(\mathbb{R}^m)$ be of form* (2.2)*, $(x,y) \in \partial A$, $f \in \mathbb{R}^n$ and $H_*^-(x,y) \mid (f) \neq \varnothing$. Then*

$$Da(x,y) \mid (f) = D^* a(x,y) \mid (f) = H_*(x,y) \mid (f).$$

Proof. It is obtained by using the preceding propositions and the remark. □

Remark 2. The above theorem is not true when $H_*^-(x,y) \mid (f) = \varnothing$ for $(x,y) \in \partial A$ and for $f \in \mathbb{R}^n$.

Example 1. Let us take the set-valued map $a(\cdot) : [0,1] \to \mathrm{cl}\,(\mathbb{R}^2)$, $x \mapsto a(x) = \{(y_1, y_2) \in \mathbb{R}^2 : -y_1^2 - y_2^2 \geq 0\}$. We know that $a(x) = \{(0,0)\}$ for all $x \in [0,1]$ and $b(\cdot,\cdot,\cdot) : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$, $(x, y_1, y_2) \mapsto b(x, y_1, y_2) = y_1^2 + y_2^2$ is a differentiable function. Then we obtain $H_*(x,0,0) \mid (1) = \mathbb{R}^2$, $H_*^-(x,0,0) \mid (1) = \varnothing$ and $Da(x,0,0) \mid (1) = \{(0,0)\}$ for $(x,0,0) \in \partial A$.

3. Directional continuity of set-valued maps

Theorem 2. *Let $a(\cdot) : \mathbb{R}^n \to \mathrm{comp}\,(\mathbb{R}^m)$ be a set valued map. If* graph $a(\cdot)$ *is closed and $a(\cdot)$ is locally bounded at $x_0 \in \mathbb{R}^n$, then $a(\cdot)$ is upper semi continuous at x_0* [4, 18].

Theorem 3. *Let $a(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ be a set valued map and let $x_0 \in \mathbb{R}^n$. $a(\cdot)$ be lower semi continuous at x_0 if and only if for all $\{x_n\}_{n=1}^\infty$ where $x_n \to x_0$ as $n \to \infty$ and for all $y_0 \in a(x_0)$, there exists a $\{y_n\}_{n=1}^\infty$ such that $y_n \in a(x_n)$ and $y_n \to y_0$ as $n \to \infty$* [3].

Definition 1. Let $x_0 \in \mathbb{R}^n$, $f \in \mathbb{R}^n$ and let $a(\cdot) : \mathbb{R}^n \to \mathrm{comp}\,(\mathbb{R}^m)$ be a set-valued map. If

$$\lim_{\delta \to 0+} a(x_0 + \delta f) = a(x_0),$$

then it is said that $a(\cdot)$ is continuous at x_0 in the direction f. If

$$\lim_{\delta \to 0+} a(x + \delta f) = a(x)$$

for all $x \in \mathbb{R}^n$, then it is said that $a(\cdot)$ is continuous on $\mathbb{R}^n$ in the direction f.

Proposition 3. *Let $x_0 \in \mathbb{R}^n$, $f \in \mathbb{R}^n$ and let $a(\cdot) : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^m)$ be a set-valued map. The set valued map $\varphi(\cdot) : \mathbb{R} \to \operatorname{comp}(\mathbb{R}^m)$, $\delta \mapsto \varphi(\delta) = a(x_0 + \delta f)$, is continuous on the right at $\delta = 0$ if and only if $a(\cdot)$ is continuous at x_0 in the direction f.*

Proposition 4. *Let $x_0 \in \mathbb{R}^n$, $f \in \mathbb{R}^n$ and $a(\cdot) : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^m)$ be a set-valued map. If the set valued map $a(\cdot)$ is upper continuous (resp., lower continuous) at x_0, then the set-valued map $\varphi(\cdot) : \mathbb{R} \to \operatorname{comp}(\mathbb{R}^m)$,*

$$\delta \longmapsto \varphi(\delta) = a(x_0 + \delta f),$$

is upper continuous (resp., lower continuous) at $\delta = 0$.

Hence, if the set-valued map $a(\cdot)$ is continuous at x_0, then the set-valued map $\varphi(\cdot) : \mathbb{R} \to \operatorname{comp}(\mathbb{R}^m)$, $\delta \mapsto \varphi(\delta) = a(x_0 + \delta f)$, is continuous at $\delta = 0$.

Proof. Let $f \in \mathbb{R}^n$ and let $a(\cdot)$ be continuous at x_0. We will show that there exists a $\delta(\varepsilon) > 0$ such that

$$|\delta| < \delta(\varepsilon) \quad \Longrightarrow \quad \varrho_H(\varphi(\delta), \varphi(0)) = \varrho_H(a(x_0 + \delta f), a(x_0)) < \varepsilon$$

for all $\varepsilon > 0$, where ϱ_H denotes Hausdorff distance. Then the set-valued map $\varphi(\cdot)$ will be continuous at $\delta = 0$.

Let $\varepsilon > 0$. Since the set-valued map $a(\cdot)$ is continuous at x_0, then there exists a $\delta_1 > 0$ such that

$$\|x - x_0\| < \delta_1 \quad \Longrightarrow \quad \varrho_H(a(x), a(x_0)) < \varepsilon. \tag{3.1}$$

Put $\delta_* = \delta_1 \|f\|^{-1}$. Then for all $\delta \in (-\delta_*, \delta_*)$,

$$\|x_0 + \delta f - x_0\| = |\delta|\, \|f\| < \|f\|. \frac{\delta_1}{\|f\|} = \delta_1.$$

It follows that from (3.1), for all $\delta \in (-\delta_*, \delta_*)$,

$$\varrho_H(a(x_0 + \delta f), a(x_0)) < \varepsilon.$$

Hence, we obtain that the set-valued map $\varphi(\cdot)$ is continuous at $\delta = 0$. □

Theorem 4. *Let $x_0 \in \mathbb{R}^n$ and let $a(\cdot) : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^m)$ be a set-valued map. If the set-valued map $a(\cdot)$ is continuous at x_0, then $a(\cdot)$ is continuous at x_0 in the direction f for any $f \in \mathbb{R}^n$.*

Proof. Since the set-valued map $a(\cdot)$ is continuous at x_0, then the set valued map $\varphi(\cdot)$ is continuous at $\delta = 0$ from Proposition 4. From Proposition 3, the set-valued map $a(\cdot)$ is continuous at x_0 in the direction f for any $f \in \mathbb{R}^n$. □

Remark 3. The converse of the above theorem is not true. The following example will show this.

Example 2. Let $f(\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be the function defined by

$$(x, y) \longmapsto f(x, y) = \begin{cases} xy/(x^{4/3} + y^6) & \text{for } x \neq 0,\ y \neq 0, \\ 0 & \text{for } x = 0,\ y = 0. \end{cases}$$

We will show that $f(\cdot,\cdot)$ is continuous at $(0, 0)$ in the direction (f_1, f_2) where $f_1 \neq 0$ or $f_2 \neq 0$, but $f(\cdot,\cdot)$ is not continuous at $(0, 0)$.

Let $(x_0, y_0) = (0, 0)$. Let $(f_1, f_2) \in \mathbb{R}\times\mathbb{R}$ such that $f_1 \neq 0$ or $f_2 \neq 0$. Then

$$\begin{aligned} f((x_0, y_0) + \delta(f_1, f_2)) &= f(x_0 + \delta f_1, y_0 + \delta f_2) \\ &= f(\delta f_1, \delta f_2) \\ &= \delta^2 f_1 f_2/(\delta^{4/3} f_1^{4/3} + \delta^6 f_2^6) \\ &= \delta^{2/3} f_1 f_2/(f_1^{4/3} + \delta^{4\frac{2}{3}} f_2^6) \end{aligned}$$

and hence, as $\delta \to +0$,

$$\frac{\delta^{2/3} f_1 f_2}{f_1^{4/3} + \delta^{4\frac{2}{3}} f_2^6} \to 0 = f(0, 0).$$

Therefore, $f(\cdot,\cdot)$ is continuous at $(0, 0)$ in the direction (f_1, f_2).

Take $\{(x_n, y_n)\}_{n=1}^{\infty} = \{(\delta_n, \delta_n^{1/3})\}_{n=1}^{\infty}$, where $\delta_n \to 0$ as $n \to \infty$. Then $(x_n, y_n) \to (0, 0)$. Since

$$f(x_n, y_n) = \frac{\delta_n^{4/3}}{\delta_n^{4/3} + \delta_n^2} = \frac{1}{1 + \delta_n^{2/3}}$$

and $\left(1 + \delta_n^{2/3}\right)^{-1} \to 1$ as $n \to \infty$, then it follows that

$$\lim_{(x,y)\to(0,0)} f(x, y) \neq f(0, 0).$$

Hence, $f(\cdot,\cdot)$ is not continuous at $(0, 0)$.

Theorem 5. *If the set-valued map $a(\cdot) : \mathbb{R}^n \to \mathrm{comp}(\mathbb{R}^m)$ is continuous at $x_0 = (x_0^1, x_0^2, \ldots, x_0^n)$ in all directions $f \in \mathbb{R}^n$, then the set-valued map*

$$x^i \longmapsto a(x_0^1, x_0^2, \ldots, x_0^{i-1}, x^i, x_0^{i+1}, \ldots, x_0^n)$$

is continuous at x_0^i.

Proof. Let $f_1 = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ and $f_2 = (0, 0, \ldots, 0, -1, 0, \ldots, 0)$ where 1 and -1 are the ith components. Since the set-valued map $a(\cdot)$ is continuous at x_0 in all directions $f \in \mathbb{R}^n$, we see that

$$\lim_{\delta\to 0+} a(x_0 + \delta f_1) = a(x_0),$$
$$\lim_{\delta\to 0+} a(x_0 + \delta f_2) = a(x_0).$$

Then we obtain

$$\lim_{\delta\to 0+} a(x_0^1, x_0^2, \dots, x_0^{i-1}, x_0^i + \delta, x_0^{i+1}, \dots, x_0^n) = a(x_0), \tag{3.2}$$

$$\lim_{\delta\to 0+} a(x_0^1, x_0^2, \dots, x_0^{i-1}, x_0^i - \delta, x_0^{i+1}, \dots, x_0^n) = a(x_0). \tag{3.3}$$

It follows from (3.2) and (3.3) that

$$\lim_{\Delta\to 0} a(x_0^1, x_0^2, \dots, x_0^{i-1}, x_0^i + \Delta, x_0^{i+1}, \dots, x_0^n) = a(x_0).$$

This imples that the set-valued map $x^i \mapsto a(x_0^1, x_0^2, \dots, x_0^{i-1}, x^i, x_0^{i+1}, \dots, x_0^n)$ is continuous at x_0^i. $\square$

Theorem 6. *Let $x_0 \in \mathbb{R}^n$, $f \in \mathbb{R}^n$ and let $a(\cdot) : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^m)$ be a set-valued map. Suppose that the set $\operatorname{graph}_D a(\cdot) = \{(x, y) \in D \times \mathbb{R}^m : y \in a(x)\}$ is compact for any compact set $D \subset \mathbb{R}^n$. If $D^* a(x_0, y_0) \mid (f) \neq \emptyset$ for all $y_0 \in a(x_0)$, then the set-valued map $a(\cdot)$ is continuous at x_0 in the direction f.*

Proof. First, we will show that the set-valued map $a(\cdot)$ is upper semi continuous at x_0 in the direction f.

Since the set $\overline{B}(x_0, 1) = \{x \in \mathbb{R}^n : \|x - x_0\| \leq 1\}$ is compact, it follows that the set $\operatorname{graph}_{\overline{B}(x_0,1)} a(\cdot)$ is compact. Then, we obtain that the set-valued map $a(\cdot) : \overline{B}(x_0, 1) \to \operatorname{comp}(\mathbb{R}^m)$ is locally bounded at x_0 and its graph is closed. From Theorem 2, the set-valued map $a(\cdot) : \overline{B}(x_0, 1) \to \operatorname{comp}(\mathbb{R}^m)$ is upper semi-continuous at x_0. Hence, $a(\cdot)$ is upper semi-continuous at x_0 in the direction f and then the set valued map $\delta \to a(x_0 + \delta f)$ is upper semi-continuous on the right at $\delta = 0$.

Now we will show that the set-valued map $\delta \mapsto a(x_0 + \delta f)$ is lower semi-continuous on the right at $\delta = 0$, i. e., we will show that for all $y_0 \in a(x_0)$ and for all $\{\delta_k\}_{k=1}^{\infty}$ where $\delta_k \to +0$ as $k \to \infty$, there exists $y_k \in a(x_0 + \delta_k f)$ such that $y_k \to y_0$ as $k \to \infty$.

Take $y_0 \in a(x_0)$. From the hypothesis of the theorem, $D^* a(x_0, y_0) \mid (f) \neq \emptyset$. Take $d_* \in D^* a(x_0, y_0) \mid (f)$. From the definition of $D^* a(x_0, y_0) \mid (f)$, there exists a $\delta_* > 0$ such that for all $\delta \in [0, \delta_*]$,

$$y(\delta) = y_0 + \delta d_* + \delta s(\delta) \in a(x_0 + \delta f)$$

where $s(\delta) \to 0$ as $\delta \to +0$. For any sequence $\{\delta_k\}_{k=1}^{\infty}$ where $\delta_k \to +0$ as $k \to \infty$, there exists a $k_0 \in \mathbb{N}$ such that $\delta_k \in [0, \delta_*]$ for all $k \geq k_0$. Then for all $k \geq k_0$,

$$\begin{aligned} y_k &= y(\delta_k) \\ &= y_0 + \delta_k d_* + \delta_k s(\delta_k) \in a(x_0 + \delta_k f). \end{aligned}$$

Hence, we obtain that $y_k \in a(x_0 + \delta_k f)$ and $y_k \to y_0$ as $k \to \infty$ and hence the set-valued map $\delta \to a(x_0 + \delta f)$ is lower semi-continuous on the right at $\delta = 0$.

Consequently, we obtain that the set-valued map $\varphi(\cdot) : \mathbb{R} \to \operatorname{comp}(\mathbb{R}^m)$, $\delta \mapsto \varphi(\delta) = a(x_0 + \delta f)$, is upper and lower semi-continuous on the right at $\delta = 0$ and then it is continuous on the right. Therefore, $a(\cdot)$ is continuous at x_0 in the direction f. $\square$

Corollary 1. *Let* $[a,b] \subset \mathbb{R}$, $x_0 \in (a,b)$ *and let* $a(\cdot) : [a,b] \to \operatorname{comp}(\mathbb{R}^m)$ *be a set-valued map and let* graph $a(\cdot)$ *be a compact set. If* $D^*a(x_0,y) \mid (1) \neq \emptyset$ *and* $D^*a(x_0,y) \mid (-1) \neq \emptyset$ *for all* $y \in a(x_0)$*, then* $a(\cdot)$ *is continuous at* x_0*.*

Proof. Since $D^*a(x_0,y) \mid (1) \neq \emptyset$ for all $y \in a(x_0)$, by Theorem 6,

$$\lim_{\delta\to 0+} a(x_0+\delta) = a(x_0) \tag{3.4}$$

and since $D^*a(x_0,y) \mid (-1) \neq \emptyset$ for all $y \in a(x_0)$, from Theorem 6,

$$\lim_{\delta\to 0+} a(x_0-\delta) = a(x_0). \tag{3.5}$$

Then from (3.4) and (3.5),

$$\lim_{\delta\to 0} a(x_0+\delta) = a(x_0).$$

Hence, $a(\cdot)$ is continuous at x_0. □

Remark 4. The following example shows that Theorem 6 and Corollary 1 are not true when the condition $D^*a(x,y) \mid (f) \neq \emptyset$ is replaced by the condition $Da(x,y) \mid (f) \neq \emptyset$.

Example 3. Let us set, for $x \in [-1,1]$,

$$a(x) = \begin{cases} [-1,1] & \text{for } x \in [-1,0], \\ \sin\frac{1}{x} & \text{for } x \in (0,1]. \end{cases}$$

One can show that the set-valued map $x \mapsto a(x)$ is not continuous at $x_0 = 0$ in the direction $f = 1$. However,

$$Da(0,y) \mid (1) = \begin{cases} (-\infty,0] & \text{for } y = 1, \\ (-\infty,+\infty) & \text{for } y \in (-1,1), \\ [0,+\infty) & \text{for } y = -1, \end{cases}$$

$$Da(0,y) \mid (-1) = \begin{cases} (-\infty,0] & \text{for } y = 1, \\ (-\infty,+\infty) & \text{for } y \in (-1,1), \\ [0,+\infty) & \text{for } y = -1, \end{cases}$$

and then for all $y \in [-1,1] = a(0)$, $Da(0,y) \mid (1) \neq \emptyset$, $Da(0,y) \mid (-1) \neq \emptyset$. On the other hand,

$$D^*a(0,y) \mid (-1) = \begin{cases} l(-\infty,0] & \text{for } y = 1, \\ (-\infty,+\infty) & \text{for } y \in (-1,1), \\ [0,+\infty) & \text{for } y = -1, \end{cases}$$

and $D^*a(0,y) \mid (1) = \emptyset$ for all $y \in [-1,1]$.

We obtain the following corollary from Theorem 6 and Theorem 5.

Let $f_i^+ = (0,0,\dots,0,1,0,\dots,0)$ and $f_i^- = (0,0,\dots,0,-1,0,\dots,0)$ where $i = 1,2,\dots,n$ and 1 and -1 are ith coordinates of the vectors f_i^+ and f_i^-, respectively.

Corollary 2. *Let* $a(\cdot) : \mathbb{R}^n \to \operatorname{comp}(\mathbb{R}^m)$ *be a set valued map. Suppose that* $x_0 \in \mathbb{R}^n$ *and the set*

$$\operatorname{graph}_D a(\cdot) = \{(x, y) \in D \times \mathbb{R}^m : y \in a(x)\}$$

is compact for any compact set $D \subset \mathbb{R}^n$. *If* $D^* a(x_0, y) \mid (f_i^+) \neq \emptyset$ *and* $D^* a(x_0, y) \mid (f_i^-) \neq \emptyset$ *for all* $y_0 \in a(x_0)$ *and for all* $i = 1, 2, \ldots, n$, *then the set-valued map* $x \mapsto a(x)$ *is continuous according to all coordinates.*

Remark 5. We will show that the converse of Theorem 6 is not true in general.

Example 4. Take the set-valued map $a(\cdot) : [-1, 1] \to \operatorname{comp}(\mathbb{R})$,

$$x \longmapsto a(x) = \{y \in \mathbb{R} : \sqrt[3]{x} \leq y \leq 2\}.$$

One can show that $a(\cdot)$ is continuous at $x_0 = 0$ and then $a(\cdot)$ is continuous at $x_0 = 0$ in any direction f.

We know that $(0, 0) \in \operatorname{graph} a(\cdot)$ and

$$Da(0,0) \mid (1) = \Big\{v \in \mathbb{R} : \ \exists \delta_k > 0 \text{ such that } \delta_k \to 0+ \ \text{ as } k \to \infty, \\ \exists y_k \in a(\delta_k) \ \ni \lim_{k\to\infty} y_k/\delta_k = v\Big\}.$$

Take a sequence $\{y_k\}_{k=1}^{\infty}$ such that $y_k \in a(\delta_k)$ and $\delta_k \to 0+$ as $k \to \infty$. Since $y_k \in a(\delta_k)$, we have $y_k \geq \sqrt[3]{\delta_k}$ and then

$$\lim_{k\to\infty} \frac{y_k}{\delta_k} \geq \lim_{k\to\infty} \frac{\sqrt[3]{\delta_k}}{\delta_k} = \lim_{k\to\infty} \frac{1}{\sqrt[3]{\delta_k^2}} \to \infty.$$

Then $Da(0, 0) \mid (1) = \emptyset$ and since $D^* a(0, 0) \mid (1) \subset Da(0, 0) \mid (1)$, then it follows that

$$D^* a(0, 0) \mid (1) = \emptyset.$$

Remark 6. The hyphothesis of the above theorem does not imply the continuity of $a(\cdot)$.

Example 5. Let us put, for all $(x, y) \in \mathbb{R} \times \mathbb{R}$,

$$f(x, y) = \begin{cases} 1 & \text{if } y = x^3, \ x > 0, \\ 0 & \text{in the other cases,} \end{cases}$$

and take the set-valued map $a(\cdot, \cdot) : \mathbb{R} \times \mathbb{R} \to \operatorname{comp}(\mathbb{R})$, $(x, y) \mapsto a(x, y) = \{f(x, y)\}$. One can show that the function $f(\cdot, \cdot)$ is not continuous at $(0, 0)$.

We have

$$\frac{\partial f(0,0)}{\partial(f_1, f_2)} = \lim_{\delta\to 0+} \frac{f(\delta f_1, \delta f_2) - f(0,0)}{\delta} = 0$$

where $f_1 \neq 0$ or $f_2 \neq 0$. Then we get

$$D^* a(0, 0, 0) \mid (f_1, f_2) = \left\{ \frac{\partial f(0,0)}{\partial(f_1, f_2)} \right\} = \{0\} \neq \emptyset.$$

4. Conclusion

The concepts of directional lower and upper sets of set-valued maps and continuity of the set-valued maps are basic concepts in set-valued analysis. In this study, the directional lower and upper derivative sets of the set valued map the values of which have piecewise smooth boundary are suggested and investigated. Moreover, the connection between the directional continuity and the directional derivative sets of set-valued maps is investigated.

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