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SECTIONALLY RESIDUATED LATTICES

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Abstract. The concept of residuum is relativized in the so-called sections of a given lattice. It is shown that such a concept still has a majority of good properties of residuum. The results correspond to previous ones involved in sectionally pseudocomplemented lattices.

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Residuated lattices were introduced by Ward and Dilworth [6] and studied by several authors. Two monographs contain a compendium on residuated lattices. They are that by Blyth and Janowitz [2] (where it is renamed as a residuated Abelian semigroup with a unit) and the book by R. Bělohlávek [1].

In this short note we will compare a certain modification of a residuated lattice with already introduced concepts (see [3, 4]).

At first, we recall the basic concept:

By a *residual lattice* is meant an algebra $\mathcal{L} = (L;\ \vee,\ \wedge,\ \otimes,\ \to,\ 0,\ 1)$ such that

(i) $(L;\ \vee,\ \wedge,\ 0,\ 1)$ is a bounded lattice,

(ii) $(L;\ \otimes,\ 1)$ is a commutative monoid,

(iii) it satisfies the so-called *adjointness property*: $x \otimes z \leq y$ if and only if $z \leq x \to y$.

Let us note (see, e. g., [1]) that $x \to y$ is the greatest element of the set $\{z;\ x \otimes z \leq y\}$. Moreover, if we consider $x \otimes y = x \wedge y$, then $x \to y$ is the relative pseudo-complement of x with respect to y, i. e., for $\otimes = \wedge$ residuated lattices are just relatively pseudo-complemented lattices.

It is well known that every relatively pseudo-complemented lattice is distributive. An extension of relative pseudo-complementation for the non-distributive case was already involved in [3, 4]:

Definition 1. A lattice $\mathcal{L} = (L;\ \vee,\ \wedge,\ 1)$ with the greatest element 1 is *sectionally pseudo-complemented* if each interval $[y,\ 1]$ is a pseudo-complemented lattice.

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From now on, denote by $x \circ y$ the pseudo-complement of $x \vee y$ in the interval $[y, 1]$. Naturally, $x \vee y \in [y, 1]$ thus $\mathcal{L} = (L; \vee, \wedge, 1)$ is sectionally pseudo-complemented if and only if "$\circ$" is an (everywhere defined) operation on L. The identities characterizing sectionally pseudo-complemented lattices are presented in [3], i. e., the class of these lattices is a variety in the signature $\{\vee, \wedge, \circ, 1\}$. We are going to apply a similar approach for the adjointness property:

Definition 2. An algebra $\mathcal{L} = (L; \vee, \wedge, \otimes, \rightarrow, 1)$ is called a *sectionally residuated lattice* if

(i) $(L; \vee, \wedge, 1)$ is a lattice with the greatest element 1;
(ii) $(L; \otimes, 1)$ is a commutative monoid;
(iii) it satisfies the *sectional adjointness property*: $(x \vee y) \otimes z = y$ if and only if $y \leq z \leq x \rightarrow y$.

Example 1. Consider the lattice N_5 on Figure 1. Then it is not relatively pseudo-complemented since N_5 is not distributive. e. g. a relative pseudo-complement of c with respect to a does not exist. Thus, considering $\otimes = \wedge$, N_5 is not a residual lattice. On the contrary, for $\otimes = \wedge$ it is a sectionally residuated lattice where the operation $\rightarrow$

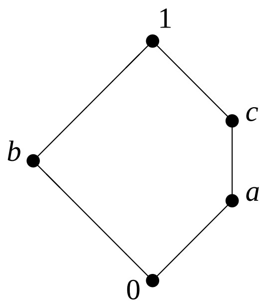

FIGURE 1

is determined by the table given below. The reader can verify the sectional adjointness property.

$\rightarrow$	0	a	c	b	1
0	1	1	1	1	1
a	b	1	1	b	1
c	b	a	1	b	1
b	c	a	c	1	1
1	0	a	c	b	1

The following result follows directly by Definition 2:

Lemma 1. *Let $\mathcal{L} = (L;\ \vee,\ \wedge,\ \otimes,\ \rightarrow,\ 1)$ be a sectionally residuated lattice. Then $x \rightarrow y$ is the greatest element of the set $\{z;\ (x \vee y) \otimes z = y\}$.*

This immediately yields the following facts:

$$(x \vee y) \otimes (x \rightarrow y) = y, \tag{1}$$

$$(x \vee y) \otimes y = y, \tag{2}$$

$$y \leq x \rightarrow y. \tag{3}$$

Lemma 2. *Let $\mathcal{L} = (L;\ \vee,\ \wedge,\ \otimes,\ \rightarrow,\ 1)$ be a sectionally residuated lattice. Then $x \leq y$ if and only if $x \rightarrow y = 1$.*

Proof. Suppose $x \leq y$. Then $x \vee y = y$ and, by Lemma 1, $x \rightarrow y$ is the greatest element of the set $\{z; y \otimes z = y\}$. By Definition 2, $y \otimes 1 = 1$ thus $x \rightarrow y = 1$. Conversely, suppose $x \rightarrow y = 1$. Then, by (1), we have $y = (x \vee y) \otimes (x \rightarrow y) = (x \vee y) \otimes 1 = x \vee y$ whence $x \leq y$. □

Lemma 3. *In a sectionally residuated lattice, the following identities are satisfied: $x \rightarrow x = 1$, $x \rightarrow 1 = 1$, $0 \rightarrow x = 1$, and $1 \rightarrow x = x$.*

Proof. The first three identities follow directly by Lemma 2. Further, by Lemma 1, $1 \rightarrow x$ is the greatest element of the set $\{z; 1 \otimes z = x\} = \{x\}$ thus $1 \rightarrow x = x$. □

Lemma 4. *In a sectionally residuated lattice, $a \otimes b = a$ if and only if $a \leq b$.*

Proof. Putting $x = y = a$ and $z = b$ in the sectional adjointness property, the assumption $a \otimes b = a$ yields $(a \vee a) \otimes b = a$ iff $a \leq b \leq a \rightarrow a = 1$ thus $a \leq b$. Conversely, $a \leq b$ implies by Lemma 3 $a \leq b \leq 1 = a \rightarrow a$ and, by sectional adjointness, $a \otimes b = (a \vee a) \otimes b = a$. □

Applying Lemma 2 and Lemma 4, we get

Corollary 1. *In a sectionally residuated lattice,*

(a) *$x \otimes y = x$ if and only if $x \rightarrow y = 1$;*
(b) *$x \otimes x = x$.*

Lemma 5. *In a sectionally residuated lattice, $x \wedge y \leq x \otimes y$.*

Proof. By (3) we have $x \wedge y \leq x \rightarrow (x \wedge y)$. Applying sectional adjointness, we infer $x \otimes (x \wedge y) = (x \vee (x \wedge y)) \otimes (x \wedge y) = x \wedge y$ and, analogously, $y \otimes (x \wedge y) = x \wedge y$. Hence, by Corollary 1 (b),

$$x \otimes y \otimes (x \wedge y) = x \otimes (x \wedge y) \otimes y \otimes (x \wedge y) = (x \wedge y) \otimes (x \wedge y) = x \wedge y$$

and, by Lemma 4, $x \wedge y \leq x \otimes y$. □

Theorem 1. *Let $\mathcal{L} = (L;\ \vee,\ \wedge,\ \otimes,\ \rightarrow,\ 1)$ be a sectionally residuated lattice. Then $\otimes = \wedge$ and $\rightarrow = \circ$, i. e., it is a sectionally pseudo-complemented lattice.*

PROOF. Replacing y by $x \wedge y$ in the sectional adjointness property, we obtain

$$x \otimes z = x \wedge y \quad \text{iff} \quad x \wedge y \leq z \leq x \to (x \wedge y).$$

However, $x \to (x \wedge y)$ is the greatest element of the set $\{t; (x \vee (x \wedge y)) \otimes t = x \wedge y\} = \{t; x \otimes t = x \wedge y\}$. By Lemma 5, $x \wedge t \leq x \otimes t = x \wedge y$, thus the greatest t of this property satisfies $t \geq y$.

Thus $y \leq x \to (x \wedge y)$, i. e.,

$$x \wedge y \leq y \leq x \to (x \wedge y)$$

and, by the sectional adjointness,

$$x \otimes y = (x \wedge (x \vee y)) \otimes y = x \wedge y.$$

Hence, $x \to y$ is the pseudocomplement of $x \vee y$ in the interval $[y, 1]$, i. e., $x \to y = x \circ y$. □

Recall that a lattice $\mathcal{L} = (L; \vee, \wedge)$ is $\wedge$-*semidistributive* if $a \wedge b_1 = a \wedge b_2$ implies $a \wedge b_1 = a \wedge (b_1 \vee b_2)$.

Corollary 2. *An algebraic lattice $\mathcal{L}$ is sectionally residuated if and only if $\mathcal{L}$ is $\wedge$-semidistributive.*

PROOF. By Proposition 2.4 from [5], an algebraic lattice $\mathcal{L}$ is $\wedge$-semidistributive if and only if $\mathcal{L}$ is sectionally pseudocomplemented. Hence, if $\mathcal{L}$ is sectionally residuated and algebraic, then, by this and Theorem 1, $\mathcal{L}$ is $\wedge$-semidistributive. Conversely, if $\mathcal{L}$ is algebraic and $\wedge$-semidistributive, then $\mathcal{L}$ is sectionally pseudocomplemented and hence sectionally residuated for $\otimes = \wedge$ and $\to = \circ$. □

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