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ON UPPER AND LOWER D_δ-SUPERCONTINUOUS MULTIFUNCTIONS

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ABSTRACT. In this paper, we define a multifunction $F : X \rightsquigarrow Y$ as an upper (lower) D_δ-supercontinuous if $F^+(V)$ ($F^-(V)$) is d_δ-open in X for every open set V of Y. We obtain some characterizations and several properties concerning upper (lower) D_δ-supercontinuous multifunctions.

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1. Introduction

Several weak and strong variants of continuity of multifunctions occur in the literature. The strong variants of continuity of multifunctions we shall be dealing with in this paper are treated in [1–4]. Certain of these strong forms of continuity of multifunctions coincide with the continuity of multifunctions if the domain or range space is suitably augmented. In 2003, J. K. Kohli [5] introduced the concept of D_δ-supercontinuous functions and gave some properties of D_δ-supercontinuous functions. In this paper we introduce a new strong form of continuity of multifunctions called the "upper (lower) D_δ-supercontinuity," which coincides with the upper (lower) continuity if the domain or range is a D_δ-completely regular space. Characterizations and basic properties of upper (lower) D_δ-supercontinuous multifunctions are elaborated in Section 3. In Section 4, we show that if the domain of a upper (lower) D_δ-supercontinuous multifunction F is retopologized in an appropriate way, then F is simply a continuous multifunction.

A multifunction F of a set X into a set Y is a set-valued function mapping set of X into $2^Y \setminus \{\varnothing\}$, where 2^Y stands for the power set of Y. Let A be a subset of a topological space (X, τ). $\mathring{A}$ and $\overline{A}$ denote the interior and closure of A respectively. A multifunction F of a set X into Y is a correspondence such that $F(x)$ is a nonempty subset of Y for each $x \in X$. We will denote such a multifunction by $F : X \rightsquigarrow Y$. For a multifunction F, the upper and lower inverse set of a set B of Y will be denoted by $F^+(B)$ and $F^-(B)$ respectively that is $F^+(B) = \{x \in X : F(x) \subseteq B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \varnothing\}$. The graph $G(F)$ of the multifunction $F : X \rightsquigarrow Y$ is strongly closed [4] if for each $(x, y) \notin G(F)$, there exist open sets U and V containing x and containing y respectively such that $(U \times \overline{V}) \cap G(F) = \emptyset$ [6]. A multifunction $F : X \rightsquigarrow Y$ is said to be upper semi continuous (briefly u. s. c.) at a point $x \in X$ if for each open set V in Y with $F(x) \subseteq V$, there exists an open set U containing x such that $F(U) \subseteq V$; lower semi continuous (briefly l. s. c.) at a point $x \in X$ if for each open set V in Y with $F(x) \cap V \neq \emptyset$, there exists an open set U containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in U$. A set G in a topological space X is said to be z-open if for each $x \in G$ there exists a cozero set H such that $x \in H \subset G$, or equivalently, if G is expressible as the union of cozero sets. The complement of a z-open set will be referred to as a z-closed set [7]. A subset H in a space X is said to be a regular G_δ-set if H is an intersection of a sequence of closed sets whose interior contain H. The complement of a regular G_δ-set is called a regular F_σ-set [8].

Throughout this paper, the spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces, and $F : X \rightsquigarrow Y$ (resp., $f : X \to Y$) is a multivalued (resp., single valued) function.

2. Preliminaries and basic properties

Definition 1. (a) A multifunction $F : X \rightsquigarrow Y$ is called upper D_δ-supercontinuous (u. D_δ-super c.) at a point $x \in X$ if for any open set $V \subset Y$ such that $F(x) \subset V$ there exists a regular F_σ-set $U \subset X$ containing x such that $F(U) \subset V$.

(b) A multifunction $F : X \rightsquigarrow Y$ is called lower D_δ-supercontinuous (l. D_δ-super c.) at a point $x \in X$ if for any open set $V \subset Y$ such that $F(x) \cap V \neq \emptyset$ there exists a regular F_σ-set $U \subset X$ containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$.

(c) F is said to be D_δ-supercontinuous (briefly D_δ-super c.) at $x \in X$, if it is both u. D_δ-super c. and l. D_δ-super c. at $x \in X$.

(d) F is said to be u. D_δ-super c. (l. D_δ-super c., D_δ-super c.) on X, if it has this property at each point $x \in X$.

Example 1. Let X denote the real line endowed by the cofinite topology, $Y = \{a, b\}$ with the topology $\tau = \{\emptyset, Y, \{a\}\}$ and we define the multifunction as follows: $F : X \rightsquigarrow Y$, $F(x) = \{a\}$ if x is irrational and $F(x) = \{b\}$ if x is rational. Then F is u. D_δ-supercontinuous (l. D_δ-super c.).

Definition 2 ([3]). A multifunction $F : X \rightsquigarrow Y$ is said to be (a) upper Z-supercontinuous (briefly, u. Z-super c.) at a point $x \in X$ if for every open set V with $F(x) \subset V$, there exists a cozero set U containing x such that $F(U) = \bigcup\{F(u) : u \in U\} \subset V$;

(b) lower Z-supercontinuous (l. Z-super c.) at a point $x \in X$ if for every open set V with $F(x) \cap V \neq \emptyset$, there exists a cozero set U containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$.

Example 2. Let X denote the real line endowed with usual topology and we define the multifunction as follows: $F : X \rightsquigarrow X$, $F(x) = \{x\}$ for each $x \in X$. Then F is u. Z-supercontinuous (l. Z-super c.).

Definition 3 ([4])**.** (a) A multifunction $F : X \rightsquigarrow Y$ is called strongly θ-upper semi continuous (s. θ-u. s. c.) at a point $x \in X$ if for any open set $V \subset Y$ such that $F(x) \subset V$ there exists an open set $U \subset X$ containing x such that $F(\overline{U}) \subset V$.

(b) A multifunction $F : X \rightsquigarrow Y$ is called strongly θ-lower semi continuous (s. θ-l. s. c.) at a point $x \in X$ if for any open set $V \subset Y$ such that $F(x) \cap V \neq \emptyset$ there exists an open set $U \subset X$ containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in \overline{U}$.

Definition 4 ([2])**.** (a) A multifunction $F : X \rightsquigarrow Y$ is called upper supercontinuous (u. super c.) at a point $x \in X$ if for any open set $V \subset Y$ such that $F(x) \subset V$ there exists an open set $U \subset X$ containing x such that $F(\overset{\circ}{\overline{U}}) \subset V$.

(b) A multifunction $F : X \rightsquigarrow Y$ is called lower supercontinuous (l. super c.) at a point $x \in X$ if for any open set $V \subset Y$ such that $F(x) \cap V \neq \emptyset$ there exists an open set $U \subset X$ containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in \overset{\circ}{\overline{U}}$.

Definition 5 ([1])**.** (a) A multifunction $F : X \rightsquigarrow Y$ is called upper D-supercontinuous (u. D-super c.) at a point $x \in X$ if for any open set $V \subset Y$ such that $F(x) \subset V$ there exists an open F_σ-set $U \subset X$ containing x such that $F(U) \subset V$.

(b) A multifunction $F : X \rightsquigarrow Y$ is called lower D-supercontinuous (l. D-super c.) at a point $x \in X$ if for any open set $V \subset Y$ such that $F(x) \cap V \neq \emptyset$ there exists an open F_σ-set $U \subset X$ containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$.

The following diagram well illustrates the relations that exist between u. D_δ-supercontinuous (l. D_δ-supercontinuous) multifunctions and other strong variants of continuity defined above.

$$\begin{array}{ccc}
\text{u. } Z\text{-super c. (l. } Z\text{-super c.)} & & \text{u. super c. (l. super c.)} \\
\Downarrow & & \Uparrow \\
\text{u. } D_\delta\text{-super c. (l. } D_\delta\text{-super c.)} & & \text{u. strongly } \theta\text{-c. (l. strongly } \theta\text{-c.)} \\
\Downarrow & & \Downarrow \\
\text{u. } D\text{-super c. (l. } D\text{-super c.)} & & \text{u. semi c. (l. semi c.)}
\end{array}$$

None of the above implications in general is reversible, as will be shown the sequel. We give Example 3, Example 4, and Example 5 to show that a u. strongly θ-c. (l. strongly θ-c.) multifunction need not be u. Z-super c. (l. Z-super c.) and that u. D-super c. (l. D-super c.) multifunction need not be u. Z-super c. (l. Z-super c.) and u. D_δ-super c. (l. D_δ-super c.) multifunction need not be u. Z-super c. (l. Z-super c.) and that u. D-super c. (l. D-super c.) multifunction.

Example 3 ([5]). Let $X = Y$ be the Mountain chain space due to Helderman [7], which is a regular space but not a D_δ-completely regular space [10]. Then the multifunction $F : X \rightsquigarrow X$, $F(x) = \{x\}$ for each $x \in X$ is a u. strongly θ-continuous (l. strongly θ-continuous) but not u. D_δ-supercontinuous (l. D_δ-supercontinuous).

Example 4. Let X denote the set of positive integers endowed with cofinite topology. Then the multifunction $F : X \rightsquigarrow X$, $F(x) = \{x\}$ for each $x \in X$ is u. D-supercontinuous (l. D-supercontinuous) but neither u. supercontinuous (l. supercontinuous) nor u. strongly θ-continuous (l. strongly θ-continuous) and hence not u. Z-supercontinuous (l. Z-supercontinuous).

Example 5. Consider the space A defined by E. Hewitt in [10, p. 504], which is a D_δ-completely regular but not completely regular. It turns out that the multifunction $F : A \rightsquigarrow A$, $F(x) = \{x\}$ for each $x \in X$ is u. D_δ-supercontinuous (l. D_δ-supercontinuous) but not u. Z-supercontinuous (l. Z-supercontinuous.).

3. Characterizations

Definition 6. A set G in a topological space X is said to be d_δ-open if for each $x \in G$ there exists a regular F_σ-set H such that $x \in H \subset G$. The complement of a d_δ-open set will be referred to as a d_δ-closed set [5].

Theorem 1. *The following statements are equivalent for a multifunction $F : X \rightsquigarrow Y$:*

(a) *F is u. D_δ-super c. (l. D_δ-super c.).*
(b) *For each open set $V \subseteq Y$, $F^+(V)$ $(F^-(V))$ is a d_δ-open set in X.*
(c) *For each closed set $K \subseteq Y$, $F^-(K)$ $(F^+(K))$ is a d_δ-closed set in X.*
(d) *For each x of X and for each open set V with $F(x) \subset V (F(x) \cap V \neq \emptyset)$, there is a d_δ-open set U containing x such that the implication $y \in U \Rightarrow F(y) \subset V$ holds $(F(y) \cap V \neq \emptyset)$.*

Proof. (a)$\Rightarrow$(b): Let V be an open set of Y and $x \in F^+(V)$. Then there exist a regular F_σ-set U containing x such that $F(U) \subset V$. Then $U \subset F^+(V)$. Since U is regular F_σ-set, we have $x \in U \subset F^+(V)$.

(b)$\Rightarrow$(c): Let K be a closed set of Y. Then $Y - K$ is an open set and $F^+(Y - K) = X - F^-(K)$ is d_δ-open. Thus, $F^-(K)$ is d_δ-closed in X.

(c)$\Rightarrow$(b): Obvious.

(b)$\Rightarrow$(a): Let V be an open set of Y containing $F(x)$. Then $F^+(V)$ is d_δ-open and $x \in F^+(V)$. Since $F^+(V)$ is a d_δ-open set there exists a regular F_σ-set U containing x such that $U \subset F^+(V)$. Thus, $F(U) \subset F(F^+(V)) \subset V$.

(a)$\Leftrightarrow$(d): Clear. □

The proof for the case where F is l. D_δ-super c. is similar.

Definition 7. Let X be a topological space and let $A \subset X$. A point $x \in X$ is said to be a d_δ-adherent point of A if every regular F_σ-set containing x intersects A. Let $[A]_{d_\delta}$ denote the set of all d_δ-adherent points of A. Clearly the set A is d_δ-closed if and only if $[A]_{d_\delta} = A$ [5].

Theorem 2. *A multifunction $F : X \rightsquigarrow Y$ is l. D_δ-super c. if and only if $F([A]_{d_\delta}) \subset \overline{F(A)}$ for every $A \subset X$.*

Proof. Suppose that F is l. D_δ-super c. Since $\overline{F(A)}$ is closed in Y, by Theorem 1, $F^+(\overline{F(A)})$ is d_δ-closed in X. Also, since $A \subset F^+(\overline{F(A)})$, $[A]_{d_\delta} \subset [F^+(\overline{F(A)})]_{d_\delta} = F^+F([A]_{d_\delta})$. Thus, $F([A]_{d_\delta}) \subset F(F^+(\overline{F(A)})) \subset \overline{F(A)}$.

Conversely, suppose $F([A]_{d_\delta}) \subset \overline{F(A)}$ for every $A \subset X$. Let K be any closed set in Y. Then $F([F^+(K)]_{d_\delta}) \subset \overline{F(F^+(K))}$ and $\overline{F(F^+(K))} \subset \overline{K} = K$. Hence, $[F^+(K)]_{d_\delta} \subset F^+(K)$ which shows that F is l. D_δ-super c. □

Theorem 3. *A multifunction F from a space X into a space Y is l. D_δ-super c. if and only if $[F^+(B)]_{d_\delta} \subset F^+(\overline{B})$ for every $B \subset Y$.*

Proof. Suppose F is l. D_δ-super c. By Theorem 1, $F^+(\overline{B})$ is d_δ-closed in X for every $B \subset Y$ and $F^+(\overline{B}) = [F^+(\overline{B})]_{d_\delta}$. Hence, $[F^+(B)]_{d_\delta} \subset F^+(\overline{B})$.

Conversely, let K be any closed set in Y. Then $[F^+(K)]_{d_\delta} \subset F^+(\overline{K}) = F^+(K) \subset [F^+(K)]_{d_\delta}$. Thus, we have $F^+(K) = [F^+(K)]_{d_\delta}$, which in turn implies that F is l. D_δ-super c. □

Definition 8. A filter base $\mathcal{F}$ is said to d_δ-converge to a point x (written as $\mathcal{F} \xrightarrow{d_\delta} x$) if for every regular F_σ-set containing x contains a member of $\mathcal{F}$ [5].

Theorem 4. *A multifunction $F : X \rightsquigarrow Y$ is l. D_δ-super c. a point x of X if and only if for each $x \in X$ and each filter base $\mathcal{F}$ that D_δ-converges to x, $F(\mathcal{F}) \to F(x)$.*

Proof. Assume that F is l. D_δ-super c. and let $\mathcal{F} \xrightarrow{d_\delta} x$. Let W be an open set containing $F(x)$. Then $x \in F^-(W)$ and $F^-(W)$ is d_δ-open. Let H be a regular F_σ-set in X such that $x \in H \subset F^-(W)$. Since $\mathcal{F} \xrightarrow{d_\delta} x$, there exists $U \in \mathcal{F}$ such that $U \subset H$ and so $F(U) \subset F(H) \subset W$. Thus, $F(\mathcal{F}) \to F(x)$.

Conversely, let W be an open subset of Y containing $F(x)$. Now, the filter $\mathcal{F}$ generated by the filterbase $\aleph_x$ consisting of all regular F_σ-sets containing x, d_δ-converges to x and so by hypothesis $F(\mathcal{F}) \to F(x)$. Hence, there exists a member $F(N)$ of $F(\aleph_x)$ such that $F(N) \subset W$. Since $N \in \aleph_x$, N is a regular F_σ-set containing x. Thus, F is l. D_δ-super c. at x. □

Theorem 5. *If $F : X \rightsquigarrow Y$ is u. D_δ-super c. (l. D_δ-super c.) and $F(X)$ is endowed with subspace topology, then $F : X \rightsquigarrow F(X)$ is u. D_δ-super c. (l. D_δ-super c.)*

Proof. Since $F : X \rightsquigarrow Y$ is u. D_δ-super c. (l. D_δ-super c.), for every open subset V of Y, $F^+(V \cap F(X)) = F^+(V) \cap F^+(F(X)) = F^+(V)(F^-(V \cap F(X)) = F^-(V) \cap F(F(X)) = F^-(V))$ is d_δ-open. Hence, $F : X \rightsquigarrow F(X)$ is u. D_δ-super c. (l. D_δ-super c.) □

Theorem 6. *If $F : X \rightsquigarrow Y$ is u. D_δ-super c. (l. D_δ-super c.) and $G : Y \rightsquigarrow Z$ u. s. c. (l. s. c.), then $G \circ F$ is u. D_δ-super c. (l. D_δ-super c.).*

Proof. Let V be an open subset of Z. Since G is u. s. c. (l. s. c.), it follows that $G^+(V)$ $(G^-(V))$ is open subset of Y, and since F is u. D_δ-super c. (l. D_δ-super c.), one has that $F^+(G^+(V))$ $(F^-(G^-(V)))$ is d_δ-open in X. Thus, $G \circ F$ is u. D_δ-super c. (l. D_δ-super c.). □

Theorem 7. *Let $\{F_\alpha : X \rightsquigarrow X_\alpha, \alpha \in \Delta\}$ be a family of multifunctions and let $F : X \rightsquigarrow \prod_{\alpha\in\Delta} X_\alpha$ be defined by $F(x) = (F_\alpha(x))$. Then F is u. D_δ-super c. if and only if each $F_\alpha : X \rightsquigarrow X_\alpha$ is u. D_δ-super c.*

Proof. Let G_{α_0} be an open set of X_{α_0}. Then $(P_{\alpha_0} \circ F)^+(G_{\alpha_0}) = F^+(P^+_{\alpha_0}(G_{\alpha_0})) = F^+(G_{\alpha_0} \times \prod_{\alpha\neq\alpha_0} X_\alpha)$. Since F is u. D_δ-super c. $F^+(G_{\alpha_0} \times \prod_{\alpha\neq\alpha_0} X_\alpha)$ is d_δ-open in X. Thus, $P_{\alpha_0} \circ F = F_\alpha$ is u. D_δ-super c. Here, P_α denotes the projection of X onto the α-coordinate space X_α.

Conversely, let us suppose that each $F_\alpha : X \rightsquigarrow X_\alpha$ is u. D_δ-super c. To show that the multifunction F is u. D_δ-super c., in view of Theorem 1, it is sufficient to show that $F^+(V)$ is d_δ-open for each open set V in the product space $\prod_{\alpha\in\Delta} X_\alpha$. Since the finite intersections and arbitrary unions of d_δ-open sets are d_δ-open, it suffices to prove that $F^+(S)$ is d_δ-open for every subbasic open set S in the product space $\prod_{\alpha\in\Delta} X_\alpha$. Let $U_\beta \times \prod_{\alpha\neq\beta} X_\alpha$ be a subbasic open set in $\prod_{\alpha\in\Delta} X_\alpha$. Then $F^+(U_\beta \times \prod_{\alpha\neq\beta} X_\alpha) = F^+(P^+_\beta(U_\beta)) = F^+_\beta(U_\beta)$ is d_δ-open. Hence, F is u. D_δ-super c. □

Theorem 8. *Let $F : X \rightsquigarrow Y$ be a multifunction and $G : X \rightsquigarrow X \times Y$ defined by $G(x) = (x, F(x))$ for each $x \in X$ be the graph function. Then G is u. D_δ-super c. if and only if F is u. D_δ-super c. and X is D_δ-completely regular space.*

Proof. To prove the necessity, suppose that G is D_δ-super c. By Theorem 6, $F = P_Y \circ G$ is D_δ-super c., where P_Y is the projection from $X \times Y$ onto Y. Let U be any open set in X and let $U \times Y$ be an open set containing $G(x)$. Since G is D_δ-super c., there exists a regular open F_σ-set W containing x such that the implication $x' \in W \Rightarrow G(x') \subset U \times Y$ holds. Thus, $x \in W \subset U$, which shows that U is d_δ-open and so X is a D_δ-completely regular space.

To prove the sufficiency, let $x \in X$ and let W be an open set containing $G(x)$. There exists open sets $U \subset X$ and $V \subset Y$ such that $(x, F(x)) \subset U \times V \subset W$. Since X is D_δ-completely regular space, there exists a regular F_σ-set G_1in X containing x such that $x \in G_1 \subset V$. Since F is D_δ-super c., there exists a cozero set G_2 in X containing x such that the implication $x' \in G_2 \Rightarrow F(x') \subset V$. Let $G_1 \cap G_2 = H$. Then H is a regular F_σ-set containing x and $G(H) \subset U \times V \subset W$ which implies that G is u. D_δ-super c. □

Definition 9. Let $F : X \rightsquigarrow Y$ be a multifunction.

(a) F is said to be upper D_δ-continuous (briefly u. D_δ-c.) at $x \in X$, if for each regular F_σ-set V with $F(x) \subset V$, there exists an open U set containing x such that the implication $x' \in U \Rightarrow F(x') \subset V$ is hold.

(b) F is said to be lower D_δ-continuous (briefly l. D_δ-c.) at $x \in X$, if for each regular F_σ-set V with $F(x) \cap V \neq \emptyset$, there exists an open set U containing x such that the implication $x' \in U \Rightarrow F(x') \cap V \neq \emptyset$ holds.

(c) F is said to be D_δ-continuous (briefly D_δ-c.) at $x \in X$, if it is both u. D_δ-c. and l. D_δ-c. at $x \in X$.

(d) F is said to be u. D_δ-c. (l. D_δ-c., D_δ-c.) on X if it has this property at every point $x \in X$.

Theorem 9. *For a multifunction $F : X \rightsquigarrow Y$, the following statements are equivalent:*

(a) *F is u. D_δ-c. (l. D_δ-c.).*
(b) *For every d_δ-open set $V \subseteq Y$, $F^+(V)$ ($F^-(V)$) is an open set in X.*
(c) *For every d_δ-closed set $K \subseteq Y$, $F^-(K)$ ($F^+(K)$) is a closed set in X.*

Lemma 1. For a multifunction $F : X \rightsquigarrow Y$, the following statements are equivalent:

(a) F is u. D_δ-c.
(b) $F(\overline{A}) \subset [F(A)]_{d_\delta}$ for all $A \subseteq X$
(c) $\overline{F^+(B)} \subseteq F^+([B]_{d_\delta})$ for all $B \subseteq X$
(d) For every d_δ-closed set $K \subseteq Y$, $F^+(K)$ is closed
(e) For every d_δ-open set $G \subseteq Y$, $F^+(G)$ is open

Proof. (a)⇒(b): Let $y \in F(\overline{A})$. Choose $x \in \overline{A}$ such that $y \in F(x)$. Let V be a regular F_σ-set containing $F(x)$ and thus y. Since F is u. D_δ-c., $F^+(V)$ is an open set containing x. This gives $F^+(V) \cap A \neq \emptyset$ which in turn implies that $V \cap F(A) \neq \emptyset$ and consequently $y \in [F(A)]_{d_\delta}$. Hence, $F(\overline{A}) \subset [F(A)]_{d_\delta}$.

(b)⇒(c): Let B be any subset of Y. Then $F(\overline{F^+(B)}) \subseteq [F(F^+(B))]_{d_\delta} \subseteq [B]_{d_\delta}$ and consequently $\overline{F^+(B)} \subseteq F^+([B]_{d_\delta})$.

(c)⇒(d): Since a set K is d_δ-closed if and only if $K = [K]_{d_\delta}$, therefore the implication (c)⇒(d) is obvious.

(d)⇒(e): Obvious.

(e)⇒(a): Since every cozero set is d_δ-open and since a multifunction is u. D_δ-c. if and only if the inverse image of every regular F_σ-set is open. Hence, (e)⇒(a). □

Theorem 10. *Let X, Y and Z be topological spaces and let the function $F : X \rightsquigarrow Y$ be u. D_δ-c. and $G : Y \rightsquigarrow Z$ be u. D_δ-super c. Then $G \circ F : X \rightsquigarrow Z$ is u. s. c.*

Proof. Since $(G \circ F)^+(V) = F^+(G^+(V))$, it is immediate in view of Lemma 1 and Theorem 1. □

Theorem 11. *Let $F : X \rightsquigarrow Y$ be a u. s. c. (l. s. c.) multifunction defined on a D_δ-completely regular space. Then F is u. D_δ-super c. (l. D_δ-super c.).*

Proof. In a D_δ-completely regular space, every open set is d_δ-open. □

Definition 10 ([5])**.** Let $f : X \to Y$ be a surjection from a topological space X onto a set$Y$. The topology on Y for which a subset $A \subset Y$ is open if and only if $f^{-1}(A)$ is d_δ-open in X is called the D_δ-quotient topology and the map f is called the D_δ-quotient map.

Theorem 12. *Let F be a multifunction from a topological space (X, τ_1) onto a topological the space (Y, τ_2), where τ_2 is D_δ-quotient topology on Y. Then F is l. D_δ-super c. Moreover, τ_2 is the finest topology on Y which makes the map $F : X \rightsquigarrow Y$ l. D_δ-super c.*

Proof. The l. D_δ-super continuity of F follows from the definition of the D_δ-quotient topology. □

Theorem 13. *Let $f : X \to Y$ be a D_δ-quotient map. Then a multifunction $F : Y \rightsquigarrow Z$ is l. s. c. if and only if $F \circ f$ is l. D_δ-super c.*

Proof. If U is an open set in Z and $F \circ f$ is l. D_δ-super c. then $(F \circ f)^+(U) = f^+(F^+(U)) = f^{-1}(F^+(U))$ which is d_δ-open in X. Since f is D_δ-quotient map, $F^+(U)$ is open in Y. Thus, F is l. s. c. Conversely, let $F : Y \rightsquigarrow Z$ be u. s. c. Let U be an open set in Z. By the l. D_δ-super continuity of $F \circ f$, $(F \circ f)^+(U) = f^{-1}(F^+(U))$ is d_δ-open in X. □

4. D_δ-complete regularization

In this Section we show that if the domain of upper-lower D_δ-supercontinuous multifunction F is retopologized in an appropriate way, then F is simply a upper-lower semi continuous multifunction.

Let (X, τ) be a topological space and let β denote the collection of all regular F_σ-subsets of (X, τ). Since the intersection of two regular F_σ-sets is a regular F_σ-set, the collection β is a base for a topology τ^* on X called the D_δ-complete regularization of τ. Clearly $\tau^* \subset \tau$. The space (X, τ) is D_δ-completely regular if and only if $\tau^* = \tau$ [5] .

Throughout the section, the symbol τ^* will have the same meaning as in the above paragraph.

Theorem 14. *A multifunction $F : (X, \tau) \rightsquigarrow (Y, \sigma)$ is u. D_δ-super c. if and only if $F : (X, \tau^*) \rightsquigarrow (Y, \sigma)$ is u. s. c.*

Theorem 15. *Let (X, τ) be topological space.Then the following are equivalent.*

(a) *The space (X, τ) is D_δ-completely regular.*
(b) *Every upper-lower semi continuous multifunction from (X, τ) into a space (Y, σ) is upper-lower D_δ-supercontinuous.*

Proof. (a)⇒(b): Obvious.

(b)⇒(a): Take $(Y, \sigma) = (X, \tau)$. Then the identity multifunction I_X on X is upper-lower semi continuous and hence upper-lower D_δ-supercontinuous. Thus, by Theorem 11, $1_X : (X, \tau^*) \to (X, \tau)$ is upper-lower semi continuous.Since $U \in \tau$ implies $1_X^{-1}(U) = U \in \tau^*$, $\tau \subset \tau^*$. Therefore, $\tau = \tau^*$ and so (X, τ) is D_δ-completely regular. □

Theorem 16. *Let $F : (X, \tau) \rightsquigarrow (Y, \sigma)$ be a multifunction. Then F is upper-lower D_δ-continuous multifunction from (X, τ) to a space (Y, σ) if and only if $F : (X, \tau) \rightsquigarrow (Y, \sigma^*)$ is upper-lower semi continuous.*

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