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# HIGHER ORDER CONNECTIONS ON LIE GROUPOID: APPLICATION TO MATERIALS

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*Abstract.* We present a possibility of material representation by a higher order connection on a Lie groupoid instead of a representation by principal connections on a principal bundle. We also prove some interesting properties of higher order connections on a Lie groupoid induced by principal connections with respect to the material setting.

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## 1. INTRODUCTION

We present a method of deciding whether the constitutional equations of certain material are equivalent. We deal with a material whose representation is given by different isomorphic principal bundles with conjugate structure groups and by means of a Lie groupoid associated to these settings we compare the connections induced by constitutive equations on these principal bundles. We use the notion of semiholonomity of a higher order connection on a Lie groupoid, which was studied in [5]. We also compare the results with those obtained in [3] for connections on general bundles.

## 2. LIE GROUPOIDS

We start with a definition of a general groupoid, see [2].

**Definition 1.** A groupoid is defined as a pair $(\Phi, B)$ of a total set $\Phi$ and a base set $B$ endowed with two submersions

$$\alpha : \Phi \to B \text{ and } \beta : \Phi \to B$$

called the source and target maps, respectively, and a binary operation defined for those ordered pairs $(y, z) \in \Phi \times \Phi$ such that $\alpha(z) = \beta(y)$. This operation must satisfy the following properties:

(1) Associativity:

$$(xy)z = x(yz),$$

whenever the products are defined;

(2) Existence of identities: for each $b \in B$ there exists an element $\mathrm{id}_b \in \Phi$, called the identity at $b$, such that $z\,\mathrm{id}_b = z$ whenever $\alpha(z) = b$, and $\mathrm{id}_b z = z$ whenever $\beta(z) = b$;

(3) Existence of inverse: for each $z \in \Phi$ there exists a (unique) inverse $z^{-1}$ such that
$$zz^{-1} = \mathrm{id}_{\beta(z)} \text{ and } z^{-1}z = \mathrm{id}_{\alpha(z)}.$$

**Definition 2.** A groupoid $(\Phi, B)$ is said to be transitive if for each pair of points $a, b \in B$ there exists at least one element $z \in \Phi$ such that
$$\alpha(z) = a \text{ and } \beta(z) = b.$$

**Definition 3.** A groupoid $(\Phi, B)$ is a Lie groupoid if the total set $\Phi$ and the base set $B$ are differentiable manifolds, the projections $\alpha, \beta$ are smooth and so are the operations of composition and of inverse.

*Remark* 1. Note that any transitive Lie groupoid induces naturally a set of isomorphic principal bundles with mutually conjugate structure groups, more precisely for every Lie groupoid $\Phi$ and every $x \in B$
$$\Phi_x := \{\theta \in \Phi | \alpha(\theta) = x\}$$
is a principal bundle, whose structure group $G_x$ is the isotropy group of $\Phi$ over $x$. On the other hand, any principal bundle $\pi : P \to B$ with structure group $G$ induces a Lie groupoid $PP^{-1} := (P \times P)/G$, where the equivalence class is given by $pq^{-1} := (p, q) \sim (pg, qg)$ for $p, g \in P$ and $g \in G$, see e. g. [2, 4] for further details. Let us just note that the projections in this case are then defined by
$$\alpha(pq^{-1}) = \pi(p) \text{ and } \beta(pq^{-1}) = \pi(q).$$

## 3. Connections on Lie groupoids

First, following the paper [5], let us identify the jet prolongation $J^r(B \times \Phi)$ of a fibered manifold $B \times \Phi \to B$ with the set $J^r(B, \Phi)$ of $r$–jets of mappings $B \to \Phi$. On the nonholonomic jet prolongation $\widetilde{J}^r(B \times \Phi)$ for integers $r \geq q \geq 0$ we denote by $\pi^r_q$ the target surjection $\pi^r_q : \widetilde{J}^r(B, \Phi) \to \widetilde{J}^q(B, \Phi)$ with $\pi^r_r$ being the identity on $\widetilde{J}^r(B, \Phi)$. Together with $\pi^r_q$ we have also the surjections $J^k \pi^{r-k}_{q-k} : \widetilde{J}^r(B, \Phi) \to \widetilde{J}^q(B, \Phi)$. Then the following holds, [5].

**Lemma 1.** *The element $X \in \widetilde{J}^r(B, \Phi)$ is semiholonomic if and only if*
$$(J^k \pi^{r-k}_{q-k})(X) = \pi^r_q(X) \text{ for any integers } 1 \leq k \leq q \leq r. \tag{1}$$

The space of semiholonomic $r$–jets will be denoted by $\bar{J}^r(B, \Phi)$.

The notion of a higher order connection on a Lie groupoid was established by C. Ehresmann in [1]. Let $\sim : B \to \Phi$ denote an inclusion of the manifold units into the groupoid and let us consider the projections $\pi^r_k$ as above. We use the notation of [5].

**Definition 4.** A nonholonomic, semiholonomic or holonomic connection of order $r \geq 1$ on $\Phi$ is a smooth map

$$\Gamma : B \to \widetilde{J}^r(B, \Phi),\; B \to \bar{J}^r(B, \Phi) \text{ or } B \to J^r(B, \Phi),$$

respectively, satisfying

$$\pi_0^r \Gamma = \sim,\; (j^r \alpha)\Gamma(x) = j_x^r(u \to u),\; (j^r \beta)\Gamma(x) = j_x^r(u \to x)$$

for all $u, x \in B$.

*Remark* 2. According to [6], let us consider the set

$$\widetilde{Q}^r(\Phi) = \{X \in \widetilde{J}^r(B, \Phi) | \pi_0^r X = \sim (x),$$
$$(j^r \alpha)X = j_x^r(u \to u),\; (j^r \beta)X = j_x^r(u \to x),\; (\alpha(X) = x)\},$$

where $\alpha$ is the source map. Then $\alpha : \widetilde{Q}^r(\Phi) \to B$ is a fibered manifold and the $r$th order connections are the sections $B \to \widetilde{Q}^r(\Phi)$.

It is well known that for $r = 1$ this corresponds to the standard notion of a connection on any of the principal bundles determined by $\Phi$. Recall that in the language of jet prolongations, a principal connection on a principal bundle is defined as follows. Let us consider a principal bundle $(P, p, B, G)$, where $p : P \to B$ is a fibered manifold, $G$ is a Lie group and by $r$ we denote the principal right action $r : P \times G \to P$ and write $r^g = r(-, g) : P \to P$ for $g \in G$. We also denote by $r$ the canonical right action $r : J^1 P \times G \to J^1 P$ given by $r^g(j_x^1 s) = j_x^1(r^g \circ s)$ for all $g \in G$ and $j_x^1 s \in J^1 P$. A principal connection $\Gamma$ on a principal fiber bundle $P$ with a principal action $r$ is an $r$-equivariant section $\Gamma : P \to J^1 P$ of the first jet prolongation $J^1 P \to P$.

The above definition together with the constructions mentioned in Remark 1 proves the following claim.

**Proposition 1.** *A principal connection on a principal bundle $P \to B$ induces naturally a first order connection on the Lie groupoid $PP^{-1}$ and any first order connection on a Lie groupoid $\Phi$ induces a principal connection on the principal bundle $\Phi_x$ for any $x \in B$.*

The following concept of a construction of higher order connection can be found in [5]. Let now $\xi = \xi^{(1)} : x \to j_x^1 \xi_x$ be a first order connection on $\Phi$ and define for each integer $r \geq 1$ the map

$$\xi^{(r)} : B \to \widetilde{J}^r(B, \Phi),\; x \to j_x^1(u \to j_u^{r-1}[\xi_x(u)]).$$

Then for any $r$th order connection $\Gamma$ and first order connection $\xi$ on $\Phi$ the map

$$\Gamma * \xi : B \to \widetilde{J}^{r+1}(B, \Phi),\; x \to j_x^1 \Gamma \cdot \xi^{(r+1)}(x)$$

is well defined connection on $\Phi$ of order $r + 1$. Note that $\Gamma' = \Gamma * \pi_1^r \Gamma$ is called the prolongation of $\Gamma$. Furthermore, given $r$ first order connections $\xi_1, \xi_2, \ldots, \xi_r$ on $\Phi$, we can define recurrently the $r$th order connection on $\Phi$ as a composition $\xi_1 * \cdots * \xi_r$. On the other hand, given an $r$th order connection $\Gamma$ on $\Phi$, we can define $r$ first order connections

$$\xi_s = \xi_s(\Gamma) : B \to J^1(B, \Phi) : x \to (j_1 \pi_0^{(s-1)}) \pi_s^r \Gamma(x)$$

for $s = 1, \ldots, r$.

Using this notation, let us mention the classification property of higher order connections on a Lie groupoid $\Phi$, [5].

**Theorem 1.** *If $\Gamma$ is a semiholonomic connection on $\Phi$, then all $\xi_s(\Gamma)$ are equal, i.e $\Gamma = \xi * \cdots * \xi$. Moreover, a connection $\xi * \cdots * \xi$ is holonomic if and only if $\xi$ is curvature free.*

To recall some further properties of higher order connections on a Lie groupoid $\Phi$ we have to mention the following notions, [5]. Let us denote by $G = G(\Phi)$ the isotropy group bundle and by $L = L(\Phi)$ the isotropy Lie algebra bundle attached to $\Phi$, i. e.

$$G_x = \{\theta \in \Phi | \alpha(\theta) = \beta(\theta) = x\} \text{ and } L_x = T_{\widetilde{x}}(G_x),$$

where $\widetilde{x} \in \Phi$ is an image of $x \in B$ under the inclusion $\sim$ . Then the following holds, [5].

**Theorem 2.** *Every second order connection $\Gamma$ on $\Phi$ is uniquely determined by two first order connections $\xi_1(\Gamma)$, $\xi_2(\Gamma)$ and a linear map*

$$A(\Gamma) : TB \otimes TB \to L(\Phi).$$

Now if $\Gamma$ and $\bar{\Gamma}$ are two $r$th order connections on $\Phi$, we can consider the composition $\bar{\Gamma} \cdot \Gamma^{-1} : x \to \bar{\Gamma}(x) \cdot \Gamma^{-1}(x)$. To generalize the linear map $A(\Gamma)$ from the previous theorem we put

$$A(\Gamma) = \Gamma \cdot [\xi_1(\Gamma) * \cdots * \xi_r(\Gamma)]^{-1}.$$

Then the following holds, [5].

**Theorem 3.** *Let $\Gamma$ be an $r$th order connection on $\Phi$. Then $\Gamma$ is uniquely determined by $\xi_1(\Gamma), \ldots, \xi_r(\Gamma)$ and $A(\Gamma)$. Moreover, $\Gamma$ is semiholonomic if and only if all $\xi_s(\Gamma)$ are equal, and $A(\Gamma)$ is semiholonomic.*

Finally, let us recall that two $r$th order connections $\Gamma$, $\bar{\Gamma}$ on $\Phi$ are said to be equivalent in the $q$–th order $(1 \le q < r)$ if

$$\pi_q^{r,C} \Gamma = \pi_q^{r,C} \bar{\Gamma}$$

for all decreasing sequences $C = \{r \ge c_1 > \cdots > c_{r-q} \ge 1\}$, where

$$\pi_q^{r,c_i} \Gamma = (j^{c_i} \pi_{q-c_i}^{r-c_i}) \Gamma.$$

Especially they are equivalent in the $(r-1)$–st order if $\bar{\Gamma} \cdot \Gamma^{-1}$ is a section in

$$L(\Phi) \otimes (\overset{r}{\otimes} T^* B).$$

## 4. Applications to materials

In material sciences, the basic setting is often given in the form of a principal bundle, for the case of so called Cosserat media see [2], endowed with a principal connection obtained from a constitutive equation. Let us consider the following case.

**Proposition 2.** *Let a material admit different settings in the form of $r$ mutually isomorphic principal bundles $(P_i, B, G_i)$ with the same base manifold $B$ and with conjugate structure groups $G_i$. Let each of the settings be endowed with a material connection $\Gamma_i$. The set of settings $\{(P_i, B, G_i), \Gamma_i, i = 1,..,r\}$ is equivalent to a setting given by a Lie groupoid $\Phi = P_k P_k^{-1}$ for some $k \in \{1,\dots,r\}$ and an $r$th order connection $\Gamma$ on $\Phi$. Moreover, if $\Gamma$ is semiholonomic, then the connections $\Gamma_i$ are generated by equivalent constitutive equations.*

*Proof.* According to Remark 1, each principal bundle $(P_i, B, G_i)$ induces Lie groupoid $P_i P_i^{-1}$. Obviously, as the principal bundles are isomorphic with conjugate structure groups, Lie groupoids $P_i P_i^{-1}$ and $P_j P_j^{-1}$ are isomorphic for any $i, j \in \{1,\dots,r\}$. Furthermore, each of the connections $\Gamma_i$ corresponds to a connection on the Lie groupoid $P_i P_i^{-1}$ and thus, up to a Lie groupoid isomorphism, they correspond to $r$ first order connection $\Gamma_i$ on a Lie groupoid $P_k P_k^{-1}$ for certain $k \in \{1,\dots,r\}$. Then

$$\Gamma = \Gamma_1 * \cdots * \Gamma_r.$$

According to Theorem 1, if $\Gamma$ is semiholonomic, then all generating connections $\Gamma_i$ are equal and thus they were generated by equivalent constitutive equations. □

Let us consider the case when the connection $\Gamma$ is not semiholonomic, i. e. the generating connections are not all equal. Then we can consider $r!$ connections of order $r$ obtained from $r$ first order connections by changing the order in the expression $\Gamma_1 * \cdots * \Gamma_r$. Let us denote such connections by $\bar{\Gamma}^i, i \in I = \{1,\dots,r!\}$. Then the following holds.

**Proposition 3.** *If $\bar{\Gamma}^i$ and $\bar{\Gamma}^j$ are equivalent in $(r-1)$–st order for some couple $(i, j) \in I \times I, i \neq j$, then there exist at least 2 couples of equal generating connections, i. e. there are 2 couples of equivalent corresponding constitutive equations.*

*Proof.* If $\bar{\Gamma}^i$ and $\bar{\Gamma}^j$ are equivalent in $(r-1)$–st order, then

$$(j^k \pi^{r-k}_{r-k-1})\bar{\Gamma}^i = (j^k \pi^{r-k}_{r-k-1})\bar{\Gamma}^j$$

for $k = 0, 1, \dots, r-1$. Particularly,

$$\pi^r_{r-1}\bar{\Gamma}^i = \pi^r_{r-1}\bar{\Gamma}^j,$$

i. e. for the first order generating connections it holds that $\Gamma^i_l = \Gamma^j_l$ for $l = 1,\dots,r-1$ and $\Gamma^i_r \neq \Gamma^j_r$. But this means from the construction of $\bar{\Gamma}^i$ and $\bar{\Gamma}^j$

that there exist connections $\Gamma^i_m, \Gamma^j_n$ different from $\Gamma^i_r, \Gamma^j_r$ such that the relations

$$\Gamma^i_r = \Gamma^i_m \text{ and } \Gamma^j_r = \Gamma^j_n$$

hold. □

By analogous yet more combinatoric considerations we obtain

**Proposition 4.** *If $k$ connections $\bar{\Gamma}^{i_1}, \dots, \bar{\Gamma}^{i_k}$, $\{i_1, \dots, i_k\} \subset I$, are equivalent in $(r-1)$th order, then there exist at least $k$ couples of equal generating connections, i. e. there are $k$ couples of equivalent corresponding constitutive equations.*

Let us denote by $\langle \Gamma_1 * \cdots * \Gamma_k \rangle$ the set of all permutations of $k$ first order connections $\Gamma_1, \dots, \Gamma_k$. Then by excluding one of the equal connections and by iterating the process described in Proposition 4 we obtain the following assertion, which reduces the number of material connections to just those corresponding to non-equivalent constitutive equations.

**Corollary 1.** *Let us suppose that there is no pair of connections equivalent in $(l_m - 1)$–st order among the connections $\langle \Gamma_{l_1} * \cdots * \Gamma_{l_m} \rangle$, $l_1 < l_2 < \cdots < l_m$, $l_i \in \{1, \dots, r\}$, $i = 1, \dots, m$, $m < r$, respectively, and for any $n = m+1, \dots, r$ there exists a pair of connections equivalent in $(n-1)$th order. Then the appropriate material representation is given by a Lie groupoid and a nonholonomic connection of order $l_m$.*

Finally, we show an analogue and a generalization of a result proved in [3]. Indeed, in [3] we handled second order connections on fibered manifolds, while Theorem 3 gives us the possibility to prove similar result for $r$th order connections on a Lie groupoid $\Phi$. Note that for $r > 2$ there is no similar identification of connections on fibered manifolds. First, one can define the relation on the space of $r$th order nonholonomic connections on $\Phi$, for our purpose we identify such space with $(r+1)$–tuples $(\Gamma_1, \dots, \Gamma_r, A(\Gamma))$ as in Theorem 3. We say that the elements $(\Gamma_1, \dots, \Gamma_r, A(\Gamma)), (\bar{\Gamma}_1, \dots, \bar{\Gamma}_r, A(\bar{\Gamma}))$ are equivalent if and only if $\Gamma_1 = \bar{\Gamma}_1, \dots, \Gamma_r = \bar{\Gamma}_r$. It is easy to see that this is an equivalence relation and we denote by $[\theta] = [(\Gamma_1, \dots, \Gamma_r, A(\Gamma))]$ a class of this equivalence. Finally the class $[\theta]$ consists of semiholonomic connections if and only if $\Gamma_1 = \cdots = \Gamma_r$ for any $(\Gamma_1, \dots, \Gamma_r, A(\Gamma)) \in [\theta]$. Furthermore, $A(\Gamma)$ is semiholonomic.

**Proposition 5.** *Let $\Phi$ be a Lie groupoid and $[\theta]$ a class of $r$th order connections. Then the constitutive equations on a principal bundle $\Phi_x$ corresponding to first order connections $\Gamma_1, \dots, \Gamma_r$ on $\Phi$ are in the same projective class if and only if $[\theta]$ is semiholonomic.*

*Proof.* If the element $[(\Gamma_1, ...\Gamma_r, A(\Gamma))]$ belongs to $[\theta]$, then from Theorem 3 the semiholonomity is equivalent to the property $\Gamma_1 = \cdots = \Gamma_r$. In particular, $r$ constitutive equations determine $r$ projectively equivalent connections of the first order. □

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