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CONFIGURATION SPACE ANALYSIS OF RIGID MECHANISMS

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Abstract. In mechanical engineering and robotics, the main problem is usually to solve the equations of motion of a given rigid mechanism. The rigid mechanisms are sometimes restricted or controlled to move along a constrained path. In engineering sciences, the most common constraints are called *ideal joint constraints*. The ideal joint constraints are generally *holonomic* constraints involving only the coordinates and orientation of a mechanism with respect to a fixed coordinate frame. An important aspect is that ideal joint constraints can be formulated as a set of polynomial equations. This means that the configuration spaces of typical rigid mechanisms can be treated as *algebraic varieties* and the components of a constraint function as generators of a polynomial ideal. This geometry algebra equivalence and the advances of computational commutative algebra and algebraic geometry gives us means to actually *compute* the properties of configuration spaces.

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1. INTRODUCTION

The equations of motion arising from Lagrangian mechanics for multibody systems are typically differential algebraic equations (DAE) where algebraic equations in our case determine the holonomic constraints. The equations of motion in Lagrangian mechanics in the holonomic case are generally of the form

$$\begin{aligned} f(t,u,\dot{u},\ddot{u},\lambda) &= 0, \\ g(u) &= 0, \end{aligned} \tag{1.1}$$

where $u : I \mapsto g^{-1}(0) \subset \mathbb{R}^k$ is the trajectory of the system, λ denotes the Lagrangian multipliers defining the holonomic constraint forces and $g^{-1}(0)$ is usually the analytic or algebraic variety defining the configuration space. In this paper we will introduce modern methods of computational algebraic geometry to study the configuration space as an algebraic variety. We will present relevant theorems and give three easy examples of planar mechanisms and their configuration space analysis. In computations, we have used the well established computer program SINGULAR [9]. In most of the computations the key is to compute the Gröbner bases of a given ideal in a particular monomial ordering.

2. Preliminary definitions

Let $\mathbb{A} = \mathbb{K}[x_1, \ldots, x_k]$ be the ring of polynomials with coefficient field $\mathbb{K}$. We will always assume that $\mathbb{K}$ is algebraically closed. We can look at the vector components of the constraint map $g : \mathbb{R}^k \mapsto \mathbb{R}^n$ as functions or as generators of an ideal $\mathcal{I} = \langle g_1, \ldots, g_n \rangle \subset \mathbb{A}$. For standard facts about the ideal theory we refer to [3, 4, 8, 10]. Since the polynomial rings that we consider are Noetherian, we know that *every* ideal in $\mathbb{A}$ is finitely generated. The geometric object corresponding to an ideal $\mathcal{I} \subset \mathbb{A}$ is its *variety* $\mathrm{V}(\mathcal{I}) \subset \mathbb{K}^k$ which is the vanishing set of all polynomials in $\mathcal{I} \subset \mathbb{A}$. We will also frequently use the fact that any radical ideal can be written uniquely as a finite intersection of *prime ideals*. This prime decomposition implies directly the decomposition of the corresponding variety into its *irreducible* parts which is called the irreducible decomposition of a variety.

Let us then present the tools which allow us to do local analysis on varieties and distinguish between different types of singularities.

Definition 1. (Local ring) A local ring R is a ring which has exactly one maximal ideal.

Remark 1. In this paper we look at localizations of polynomial rings with respect to a point/maximal ideal. We consider equivalence classes of polynomials giving always the same value when evaluated at $\mathrm{V}(\mathcal{I})$. The equivalence classes are given by

$$[f] = \{g \in \mathbb{A} \mid f - g \in I(\mathrm{V}(\mathcal{I}))\}, \tag{2.1}$$

where $I(\mathrm{V}(\mathcal{I}))$ denotes the *ideal of a variety* $\mathrm{V}(\mathcal{I})$. As usual, the *coordinate ring* $\mathbb{K}(\mathrm{V}(\mathcal{I}))$ is

$$\mathbb{K}(\mathrm{V}(\mathcal{I})) = \{[f] \mid f \in \mathbb{A}\}. \tag{2.2}$$

Definition 2. (Localization of $\mathrm{V}(\mathcal{I})$ at a point) Let $\mathfrak{m}_p$ be the maximal ideal $\mathfrak{m}_p = \langle x_1 - p_1, \ldots, x_n - p_k \rangle \subset \mathbb{A}$ where $p = (p_1, \ldots, p_n) \in \mathrm{V}(\mathcal{I}) \subset \mathbb{K}^n$. Then we can write the localization of $\mathbb{A}$ at p as

$$\mathcal{O}_p = \{f/g \mid f \in \mathbb{A},\ g \notin \mathfrak{m}_p\} \tag{2.3}$$

The unique maximal ideal in this case is

$$m_p = \{f/g \mid f \in \mathfrak{m}_p,\ \ g \notin \mathfrak{m}_p\} \tag{2.4}$$

The *localization* of $\mathrm{V}(\mathcal{I})$ at p is the ideal

$$\mathcal{O}_{V,p} = \{f/g \mid f \in \mathbb{K}(\mathrm{V}(\mathcal{I})),\ g \notin \mathfrak{m}_p\} \subset \mathcal{O}_p, \tag{2.5}$$

Notice that $\mathcal{I} \subset \mathfrak{m}_p$ and $\mathcal{O}_{V,p}$ is a local ring which is a subring of a $\mathcal{O}_p$ with maximal ideal $\mathfrak{m}_p \mathcal{O}_{V,p} = M_p$.

3. SINGULARITIES AND DIMENSION

In this section we will briefly present the relevant definitions and theorems in order to compute our examples. Remember that the embedding dimension $\operatorname{edim}(\mathcal{O}_{V,p}) = \dim_K(M_p/M_P^2)$ of an algebraic variety is the minimal number of generators of M_p. Particulary important is that the Krull dimension of an ideal can be easily computed if the elements of the Gröbner basis of an ideal is known.

Definition 3 (Singular and regular points of a variety)**.** Suppose that $\mathcal{I}$ is a radical ideal. The local ring $\mathcal{O}_{V,p}$ is a *regular local ring* if

$$\dim_K(\mathcal{O}_{V,p}) = \operatorname{edim}(\mathcal{O}_{V,p}) = \dim(T_p\mathrm{V}(\mathcal{I})),$$

where $T_p\mathrm{V}(\mathcal{I})$ denotes the tangent space of $\mathrm{V}(\mathcal{I})$ at p. If the point p is not regular, it is *singular*.

The last equation in Definition 3 gives us the actual means to compute the singular points [5, 7, 8, 11].

Theorem 1 (Jacobian criterion)**.** *Let $\mathcal{I} = \langle g_1, \dots g_n\rangle \subset \mathbb{A}$ be a radical ideal and suppose that $\mathrm{V}(\mathcal{I}) \subset \mathbb{K}^k$ is equidimensional* and dim$(\mathrm{V}(\mathcal{I})) = k - \ell$. Then the singular variety of $\mathrm{V}(\mathcal{I})$ is*

$$S(\mathrm{V}(\mathcal{I})) = \mathsf{V}\big(\mathcal{I} + \mathsf{I}_l(dg)\big) = \mathsf{V}\big(\mathcal{I}\big) \cap \mathsf{V}\big(\mathsf{I}_l(dg)\big) \subset \mathbb{K}^k.$$

Here $\mathsf{I}_l(dg)$ denotes the lth Fitting ideal of the Jacobian of the constraint map g generated by $l \times l$ minors of dg. From this, it follows that, if $p \in S(\mathrm{V}(\mathcal{I}))$, then $\mathcal{O}_{V,p}$ is not a regular local ring. Moreover, if $1 \in \mathcal{I} + \mathsf{I}_l(dg)$, then the variety $\mathrm{V}(\mathcal{I})$ is naturally smooth since $\mathrm{V}(1) = \varnothing$.

Let us then introduce another important object in our analysis - the *tangent cone*[3, 4]. We can rearrange any polynomial $f \in \mathbb{A}$ by total degree d as a linear combination $f = f_{p,0} + f_{p,1} + \ldots + f_{p,d}$, where

$$f_{p,k} = \sum_{|\alpha|=k} c_\alpha (x-p)^\alpha.$$

Here, α is a multi index. Let us denote by $f_{p,\min}$ the nonzero homogeneous component of smallest degree.

Definition 4 (tangent cone)**.** Suppose that $\mathrm{V}(\mathcal{I}) \subset \mathbb{K}^k$ is an affine variety and let $p \in \mathrm{V}(\mathcal{I})$. Let $\mathcal{I}_{p,\min}$ be the ideal generated by the minimal parts of polynomials in $\mathcal{I}$. The *tangent cone* of $\mathrm{V}(\mathcal{I})$ at p, denoted by $C_p(\mathrm{V}(\mathcal{I}))$, is

$$C_p(\mathrm{V}(\mathcal{I})) = \mathrm{V}(\mathcal{I}_{p,\min}), \tag{3.1}$$

The following theorem allows us also to distinguish between singular and regular points [3, 4].

*A variety is called *equidimensional* if all of its irreducible components have the same dimension.

Theorem 2. *The following conditions are equivalent:*

(1) $p \in \mathrm{V}$ *is a regular point of* V*;*
(2) $\dim(C_p(\mathrm{V})) = \dim(T_p\mathrm{V})$*;*
(3) $C_p(\mathrm{V}) = T_p\mathrm{V}$.

Finally, let us present a theorem to distinguish certain types of singularities [12].

Theorem 3. *Suppose that* $\mathcal{I} \subset \mathbb{K}[x_1, \ldots, x_k]$ *is a ideal where* $\mathbb{K}$ *is algebraically closed. Let* $p \in \mathrm{V}(\mathcal{I})$ *be a singular point of* $\mathrm{V}(\mathcal{I})$ *and* $\mathcal{O}_p$ *be the local ring at* p*. If the prime decomposition of the radical of* $\mathcal{O}_{V,p}$ *in the local ring is*

$$\sqrt{\mathcal{O}_{V,p}} = \mathcal{I}_1 \cap \ldots \cap \mathcal{I}_r \subset \mathcal{O}_p,$$

then the corresponding irreducible varieties $\mathrm{V}(\mathcal{I}_i)$ *of prime ideals* $\mathcal{I}_i$ *represent varieties passing through the singular point* p *and the varieties intersect at this point. However, if the prime decomposition of* $\mathcal{O}_{V,p}$ *is* $\sqrt{\mathcal{O}_{V,p}} = \mathcal{I}$*, then* $\mathcal{I}$ *is an integral domain and the point* p *is a singularity of an irreducible variety* $\mathrm{V}(\mathcal{I})$.

4. Examples

Here we apply the previous theorems to three easy examples.

4.1. *Simple slider-crank mechanisms*

Let us consider a mechanism constructed from two and three bars attached to each other with a revolute joint and suppose that the last bar is constrained to move on x-axis.

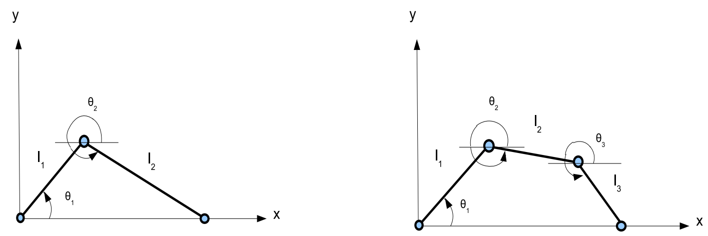

FIGURE 1. On the left the 2-bar slider-crank mechanism. On the right the 3-bar slider crank mechanism.

4.1.1. *2-bar slider-crank mechanism*

The constraint mapping $g : \mathbb{R}^2 \mapsto \mathbb{R}$ for the 2-bar slider-crank mechanism is then

$$g(\theta_1, \theta_2) = l_1 \sin(\theta_1) + l \sin(\theta_2).$$

The configuration space is then the analytic variety $g^{-1}(0)$. With substitutions $c_i = \cos(\theta_i)$ and $s_i = \sin(\theta_i)$, the constraint equations take the form

$$p_1 = l_1 s_1 + l_2 s_2 = 0,\ p_2 = c_1^2 + s_1^2 - 1 = 0,\ p_3 = c_2^2 + s_2^2 - 1 = 0.$$

The configuration space is then $\mathrm{V}(\langle p_1, p_2, p_3\rangle) = \mathrm{V}(\mathcal{I})$. Now it is easy to check that $\mathcal{I}$ is a radical ideal and, moreover, $\dim(\mathrm{V}(\mathcal{I})) = 1$. Let us then deduce the necessary condition for singularities from Theorem 1 in (l_1, l_2)-space. Since $\dim(\mathrm{V}(\mathcal{I})) = 1$, the singular variety of $\mathrm{V}(\mathcal{I})$ is

$$S(\mathrm{V}(\mathcal{I})) = \mathsf{V}\big(\mathcal{I} + \mathsf{I}_3(dp)\big)$$

Decomposing the Fitting ideal $\mathsf{I}_3(dp)$ gives

$$\sqrt{\mathsf{I}_3(dp)} = \mathcal{I}_1 \cap \ldots \cap \mathcal{I}_6,$$

Only the ideal $\mathcal{I}_5 = \langle c_1, c_2\rangle$ corresponds to a physically feasible solution so we only need to look at the intersection of $\mathrm{V}(\mathcal{I}_5)$ and $\mathrm{V}(\mathcal{I})$. To find the necessary condition, we compute the Gröbner basis of $S = \mathcal{I} + \mathcal{I}_5$ in the ring $\mathbb{Q}[(c_1, s_1, c_2, s_2), (l_1, l_2)]$ with the above *elimination ordering*. The computation shows

$$E = S \cap \mathbb{Q}[l_1, l_2] = \langle (l_1 + l_2)(l_1 - l_2)\rangle,$$

Hence $l_1 = l_2$ is a necessary condition for singularities in (l_1, l_2)-space. Next we choose for example $l_1 = l_2 = 1$ and compute the actual singular points. This gives

$$\sqrt{S} = \langle s_2 + 1, c_2, s_1 + s_2, c_1\rangle \cap \langle s_2 - 1, c_2, s_1 + s_2, c_1\rangle.$$

Thus the singular points are

$$\mathrm{V}(S) = \mathrm{V}(\sqrt{S}) = \{(0, 1, 0, -1), (0, -1, 0, 1)\} = q_1 \cup q_2.$$

Let us then carry out a local analysis for variety $\mathrm{V}(\mathcal{I})$ at the singular point $q_2 = (c_1, s_1, c_2, s_2) = (0, -1, 0, 1)$. Now we can compute the the tangent cone and get

$$C_{q_2}(\mathrm{V}(\mathcal{I})) = \mathrm{V}(\langle c_1, c_2, (s_1 - 1)^2 - (s_2 + 1)^2\rangle).$$

Near q_2, the variety $\mathrm{V}(\mathcal{I})$ looks like two lines $s_1 - 1 = \pm(s_2 + 1)$ intersecting in the plane $c_1 = c_2 = 0$. Next, we consider $\mathcal{I} = \langle p_1, p_2, p_3\rangle$ in the local ring $\mathcal{O}_{q_2}$ using *local ordering* and compute the prime decomposition of $\mathcal{O}_{V,q_2}$. As expected, we have

$$\mathcal{O}_{V,q_2} = H_1 \cap H_2.$$

By theorem 3, at least two irreducible varieties/motion modes pass through q_2. In fact, when we compute in global ordering after substitution $l_1 = l_2 = 1$, we simply have

$$\mathcal{I} = \mathcal{I}_1 \cap \mathcal{I}_2 = \langle c_2^2 + s_2^2 - 1, c_1 - c_2, s_1 + s_2\rangle \cap \langle c_2^2 + s_2^2 - 1, c_1 + c_2, s_1 + s_2\rangle,$$

$\mathrm{V}(\mathcal{I}) = \mathrm{V}(\mathcal{I}_1) \cup \mathrm{V}(\mathcal{I}_2)$ and $S(\mathrm{V}(\mathcal{I})) = \mathrm{V}(\mathcal{I}_1) \cap \mathrm{V}(\mathcal{I}_2)$.

4.1.2. *3-bar slider-crank mechanism*

Let us then do similar analysis for 3-bar slider crank mechanism. With substitutions $c_i = \cos(\theta_i)$ and $s_i = \sin(\theta_i)$, as before, the constraint equations take the form

$$p_1 = l_1 s_1 + l_2 s_2 + l_3 s_3 = 0, \quad p_{i+1} = c_i^2 + s_i^2 - 1 = 0, \quad 1 \le i \le 3.$$

Again, we check that $\mathcal{I}$ is radical and compute $\dim(\mathrm{V}(\mathcal{I})) = 2$. The singular variety is then

$$S(\mathrm{V}(\mathcal{I})) = \mathrm{V}\big(\mathcal{I} + \mathsf{I}_4(dp)\big).$$

Again, we first analyze the Fitting ideal $\mathsf{I}_4(dp)$ and find

$$\sqrt{\mathsf{I}_4(dp)} = \mathcal{I}_1 \cap \ldots \cap \mathcal{I}_{11}.$$

Only one of the prime components $\mathcal{I}_9 = \langle c_1, c_2, c_3\rangle$ is physically relevant so we compute the Gröbner basis of $S = \mathcal{I} + \mathcal{I}_9$ in the ring

$$\mathbb{Q}[(c_1, s_1, c_2, s_2, c_3, s_3), (l_1, l_2, l_3)]$$

with the above elimination ordering. Now we find

$$\begin{aligned} E &= S \cap \mathbb{Q}[l_1, l_2, l_3] = \langle (l_1 + l_2 + l_3)(l_1 - l_2 - l_3)(l_1 + l_2 - l_3)(l_1 - l_2 + l_3)\rangle \\ &= \langle h_1 h_2 h_3 h_4\rangle \end{aligned}$$

The necessary conditions for singularities are thus $h_2 = 0$, $h_3 = 0$ or $h_4 = 0$. Let us choose $l_1 = 2$ and $l_2 = l_3 = 1$ so that $h_2 = 0$ is fulfilled. This gives

$$\sqrt{S} = \langle c_1, s_1 - 1, c_2, s_2 + 1, c_3, s_3 + 1\rangle \cap \langle c_1, s_1 - 1, c_2, s_2 + 1, c_3, s_3 + 1\rangle.$$

The singular points are thus

$$\mathrm{V}(S) = \mathrm{V}(\sqrt{S}) = \{(0, 1, 0, -1, 0, -1), (0, -1, 0, 1, 0, 1)\} = q_1 \cup q_2.$$

Let us then investigate locally the variety at point q_2. Computing the tangent cone gives

$$C_{q_2}(\mathrm{V}(\mathcal{I})) = \mathrm{V}(\langle c_1, c_2, c_3, 2(s_1 + 1)^2 - (s_2 - 1)^2 - (s_3 - 1)^2\rangle).$$

Note that $2(s_1+1)^2-(s_2-1)^2-(s_3-1)^2 = 0$ actually gives a cone in the (s_1, s_2, s_3) space. Moreover, when we compute the prime decomposition of the radical of localization $\mathcal{O}_{V,q_2}$ of $\mathrm{V}(\mathcal{J})$ at $\mathcal{O}_{q_2}$, we find that

$$\sqrt{\mathcal{O}_{V,q_2}} = \mathcal{O}_{V,q_2}$$

Thus the singularity is not an intersection of different motion modes/irreducible varieties. It is still possible to visualize the configuration spaces of 2-bar and 3-bar slider crank mechanisms. In the first case, the configuration space ”breaks” naturally to two irreducible varieties. In the 3-bar case, such separation does not exist.

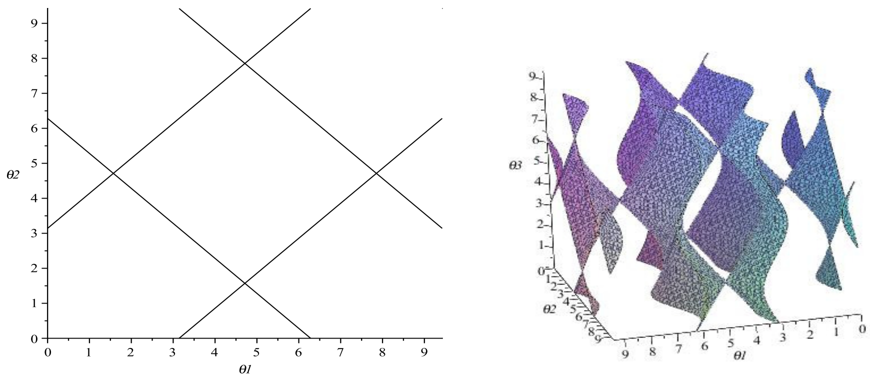

FIGURE 2. On the left, the configuration space of 2-bar slider-crank mechanism in the (θ_1, θ_2)-space. On the right, the configuration space of 3-bar slider crank mechanism in the $(\theta_1, \theta_2, \theta_3)$-space.

Remark 2. The plots agree with our computational results. Also, in the 2-bar slider crank case, there are no regular solutions $c : I \mapsto g^{-1}(0)$ for equations of motion (1.1) through the singular point $q_2 = c(t_0)$ from one motion mode to another since, for such solutions, automatically, $c'(t_0) = (0, 0)$. However, in the case of the 3-bar slider crank, such a problem does not exist.

4.2. *Closed four bar mechanism*

Let us finally consider the closed four bar mechanism. To simplify the analysis, we assume that one bar with a length of unity is fixed on the x-axis.
In this case, the constraint equations are given by

$$p_1 = l_1c_1 + l_2c_2 + l_3c_3 - 1 = 0,$$
$$p_2 = l_1s_1 + l_2s_2 + l_3s_3 = 0,$$
$$p_{2+i} = c_i^2 + s_i^2 - 1 = 0, \quad 1 \le i \le 3.$$

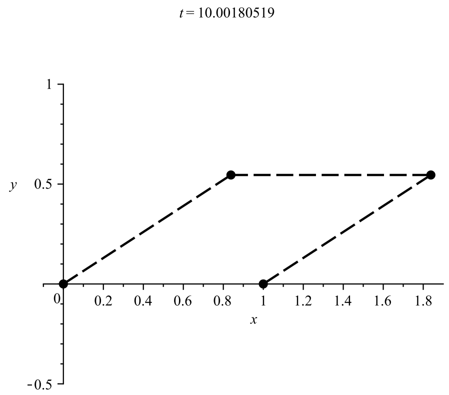

FIGURE 3. Simple closed four bar mechanism.

Setting $\mathcal{I} = \langle p_1, \ldots, p_5 \rangle$, we compute $\dim(\mathrm{V}(\mathcal{I})) = 1$. The singular variety is thus

$$S(\mathrm{V}(\mathcal{I})) = \mathrm{V}\big(\mathcal{I} + \mathsf{I}_5(dp)\big).$$

The prime decomposition of $\sqrt{\mathsf{I}_5(dp)}$ yields

$$\sqrt{\mathsf{I}_5(dp)} = \mathcal{I}_1 \cap \ldots \cap \mathcal{I}_{10}.$$

The only physically relevant component is given by

$$\mathcal{I}_8 = \langle -s_2c_3 + c_2s_3, -s_1c_3 + c_1s_3, -s_1c_2 + c_1s_2 \rangle$$

Computing the Gröbner basis of $S = \mathcal{I} + \mathcal{I}_8$ in the ring

$$\mathbb{Q}[(c_1, s_1, c_2, s_2, c_3, s_3), (l_1, l_2, l_3)]$$

we find

$$E = S \cap \mathbb{Q}[l_1, l_2, l_3] = \langle k_1 \cdot \ldots \cdot k_9 \rangle.$$

The necessary condition for $\mathrm{V}(S) \neq \varnothing$ in (l_1, l_2, l_3)-space is $k_i = 0$, where the nine polynomials k_i are of the form

$$k_i = l_1 \pm l_2 \pm l_3 \pm 1.$$

Let us choose $l_1 = l_2 = l_3 = 1$ so that $k_4 = l_1 - l_2 + l_3 - 1 = 0$. After substitutions we compute

$$\begin{aligned}\sqrt{S} &= \mathcal{J}_1 \cap \mathcal{J}_2 \cap \mathcal{J}_3 \\ \mathcal{J}_1 &= \langle c_1 + 1, s_1, c_2 - 1, s_2, c_3 - 1, s_3 \rangle \\ \mathcal{J}_2 &= \langle c_1 + 1, s_1, c_2 - 1, s_2, c_3 + 1, s_3 \rangle \\ \mathcal{J}_3 &= \langle c_1 + 1, s_1, c_2 + 1, s_2, c_3 - 1, s_3 \rangle.\end{aligned}$$

Also, after substitutions, the constraint ideal decomposes as $\sqrt{\mathcal{I}} = \mathcal{I} = \mathcal{I}_1 \cap \mathcal{I}_2 \cap \mathcal{I}_3$ and easy computation shows that the singularities q_i are the intersections

$$\begin{aligned}q_1 &= \mathrm{V}(\mathcal{J}_1) = \mathrm{V}(\mathcal{I}_1) \cap \mathrm{V}(\mathcal{I}_3) = (-1, 0, 1, 0, 1, 0) \\ q_2 &= \mathrm{V}(\mathcal{J}_2) = \mathrm{V}(\mathcal{I}_2) \cap \mathrm{V}(\mathcal{I}_3) = (1, 0, 1, 0, -1, 0) \\ q_3 &= \mathrm{V}(\mathcal{J}_3) = \mathrm{V}(\mathcal{I}_1) \cap \mathrm{V}(\mathcal{I}_2) = (1, 0, -1, 0, 1, 0).\end{aligned}$$

Let us still do the local analysis for $\mathrm{V}(\mathcal{I})$ for example at q_1. The tangent cone $C_{q_1}(\mathrm{V}(\mathcal{I}))$ is

$$C_{q_1}(\mathrm{V}(\mathcal{I})) = \mathrm{V}(\langle s_1, s_2, s_3, c_1 + c_2 + c_3 - 1, (c_2 - 1)(c_3 - 1) \rangle)$$

The singularity looks now again like the intersection of two lines in the hyper plane $s_1 = s_2 = s_3 = 0$. Then, the computation in the local ring shows that

$$\mathcal{O}_{V,q_1} = H_1 \cap H_2$$

This confirms that the two irreducible varieties $\mathrm{V}(\mathcal{I}_1)$ and $\mathrm{V}(\mathcal{I}_2)$ intersect at point q_1 as Theorem 3 suggests.

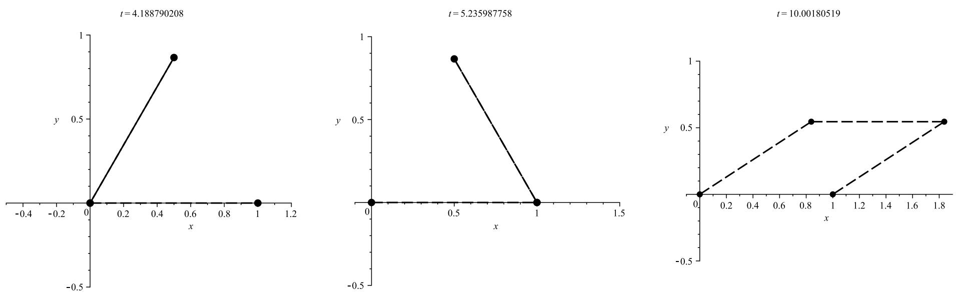

FIGURE 4. On the left the degenerate motion mode $\mathrm{V}(\mathcal{I}_1)$. In the middle the degenerate motion mode $\mathrm{V}(\mathcal{I}_2)$. On the right the "actual" four bar motion mode $\mathrm{V}(\mathcal{I}_3)$. In both degenerate motion modes, two bars are fixed on the horizontal axis and two bars rotate freely together. In the "actual" motion mode, the mechanism rotates as one parallellogram.

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Remark 3. Notice again that there are no regular solutions $c : I \mapsto g^{-1}(0)$ for (1.1) which would take the the mechanism through q_1 from $V(\mathcal{I}_3)$ to $V(\mathcal{I}_1)$ and again through q_2 from $V(\mathcal{I}_3)$ to $V(\mathcal{I}_1)$. At singular points $c(t_i) = q_i$, $1 \le i \le 3$, automatically, $c'(t_i) = (0,0,0)$ since the one dimensional varieties intersect transversally.

5. CONCLUSIONS

In this paper, we successfully analyzed configuration spaces of three simple mechanisms with computational algebraic geometry. The methods can be applied to more complicated mechanisms as well [1,2,15]. When we are able to represent the configuration space as an algebraic variety, we can use the tools of computational algebraic geometry to make both global and local statements about the configuration space. The local analysis for analytic varieties has been previously investigated in [13,14] and the Gröbner bases methods in [6]. The dimension or mobility in particular motion mode $V(\mathcal{I})$ can be regarded as the Krull dimension of the corresponding ideal $\mathcal{I}$. The Jacobian criterion together with elimination theory provides us with the means to determine whether and under which conditions, configuration space singularities exist. The tangent cone and the localization of motion mode $V(\mathcal{I})$ at a singular point gives us the means to investigate the nature of the singularity.

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