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Füsun Özen Zengin and Bahar Kirik

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CONFORMAL MAPPINGS OF NEARLY QUASI-EINSTEIN MANIFOLDS

FÜSUN ÖZEN ZENGİN AND BAHAR KIRIK

Abstract. In this paper, we consider a conformal mapping between two nearly quasi-Einstein manifolds V_n and $\bar{V}_n$. We find some properties of this transformation from V_n to $\bar{V}_n$ and some theorems are proved.

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1. Introduction

A non-flat n-dimensional Riemannian or a semi-Riemannian manifold (M,g), $(n>2)$ is said to be an Einstein manifold if the condition

$$S(X,Y)=\frac{r}{n}g(X,Y) \tag{1.1}$$

holds on M, where S and r denote the Ricci tensor and the scalar curvature of (M,g), respectively. Einstein manifolds play an important role in Riemannian Geometry, as well as in general theory of relativity. For this reason, these manifolds have been studied by many authors.

A non-flat n-dimensional Riemannian manifold (M,g), $(n>2)$ is defined to be a quasi-Einstein manifold if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the following condition

$$S(X,Y)=ag(X,Y)+bA(X)A(Y) \tag{1.2}$$

where $a,b\in\mathbb{R}$ and A is a non-zero 1-form such that $g(X,U)=A(X)$ for all vector fields X on M, [5]. Then A is called the associated 1-form and U is called the generator of the manifold. Also M. C. Chaki and R. K. Maity [2] studied the quasi-Einstein manifolds by considering a and b as scalars such that $b\neq 0$ and U as a unit vector field.

A non-flat n-dimensional Riemannian manifold (M,g), $(n>2)$ is called a nearly quasi-Einstein manifold if its Ricci tensor S of type (0,2) is not identically zero and satisfies the following condition

$$S(X,Y) = ag(X,Y) + bE(X,Y) \tag{1.3}$$

where a and b are non-zero scalars and E is a non-zero symmetric tensor of type (0,2), [4]. Then E is called the associated tensor and a and b are called the associated scalars of M. An n-dimensional nearly quasi-Einstein manifold is denoted by $N(QE)_n$. An example of $N(QE)_4$ has been given in [4].

Putting $X = Y = e_i$ in (1.3), we get

$$r = na + b\tilde{E}. \tag{1.4}$$

Here r is the scalar curvature of $N(QE)_n$ and $\tilde{E} = E(e_i, e_i)$ where $\{e_i\}$, $i = 1, 2, .., n$ is an orthonormal basis of the tangent space at each point of the manifold.

In this paper, we investigate a conformal mapping between two nearly quasi-Einstein manifolds.

2. Conformal mappings of nearly quasi-Einstein manifolds

In this section, we suppose that V_n and $\bar{V}_n$, $(n > 2)$ are two nearly quasi-Einstein manifolds with metrics g and $\bar{g}$, respectively.

Definition 1. A conformal mapping is a diffeomorphism of V_n onto $\bar{V}_n$ such that

$$\bar{g} = e^{2\sigma} g \tag{2.1}$$

where σ is a function on V_n. If σ is constant, then it is called a homothetic mapping. In local coordinates, (2.1) is written as

$$\bar{g}_{ij}(x) = e^{2\sigma(x)} g_{ij}(x), \quad \bar{g}^{ij} = e^{-2\sigma} g^{ij}. \tag{2.2}$$

Besides those equations, we have the Christoffel symbols, the components of the curvature tensor, the Ricci tensor, and the scalar curvature, respectively

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \delta^h_i \sigma_j + \delta^h_j \sigma_i - \sigma^h g_{ij}, \tag{2.3}$$

$$\begin{aligned} \bar{R}^h_{ijk} = {} & R^h_{ijk} + \delta^h_k \sigma_{ij} - \delta^h_j \sigma_{ik} + g^{h\alpha}(\sigma_{\alpha k} g_{ij} - \sigma_{\alpha j} g_{ik}) \\ & + \Delta_1 \sigma (\delta^h_k g_{ij} - \delta^h_j g_{ik}), \end{aligned} \tag{2.4}$$

$$\bar{S}_{ij} = S_{ij} + (n-2)\sigma_{ij} + (\Delta_2 \sigma + (n-2)\Delta_1 \sigma) g_{ij}, \tag{2.5}$$

$$\bar{r} = e^{-2\sigma}(r + 2(n-1)\Delta_2 \sigma + (n-1)(n-2)\Delta_1 \sigma), \tag{2.6}$$

where $S_{ij} = R^\alpha_{ij\alpha}$, $r = S_{\alpha\beta} g^{\alpha\beta}$, $\sigma_i = \frac{\partial \sigma}{\partial x^i} = \nabla_i \sigma$, $\sigma^h = \sigma_\alpha g^{\alpha h}$ and

$$\sigma_{ij} = \nabla_j \nabla_i \sigma - \nabla_i \sigma \nabla_j \sigma, \tag{2.7}$$

$\Delta_1 \sigma$ and $\Delta_2 \sigma$ are the first and the second Beltrami's symbols which are determined by

$$\Delta_1 \sigma = g^{\alpha\beta} \nabla_\alpha \sigma \nabla_\beta \sigma, \quad \Delta_2 \sigma = g^{\alpha\beta} \nabla_\beta \nabla_\alpha \sigma \tag{2.8}$$

where ∇ is the covariant derivative according to the Riemannian connection in V_n. We denote the objects of space conformally corresponding to V_n by a bar, i. e., $\bar{V}_n$. If $\bar{V}_n$ is a $N(QE)_n$, then we have, from (1.3), (2.2), and (2.5),

$$\bar{b}\bar{E}_{ij} = bE_{ij} + (n-2)\sigma_{ij} + (\Delta_2\sigma + (n-2)\Delta_1\sigma + a - \bar{a}e^{2\sigma})g_{ij}. \tag{2.9}$$

Definition 2. A vector field ξ in a Riemannian manifold M is called torse-forming if it satisfies the condition $\nabla_X\xi = \rho X + \phi(X)\xi$ where $X \in TM$, $\phi(X)$ is a linear form and ρ is a function, [12]. In the local transcription, this reads

$$\nabla_i\xi^h = \rho\,\delta_i^h + \xi^h\phi_i \tag{2.10}$$

where ξ^h and ϕ_i are the components of ξ and ϕ, and δ_i^h is the Kronecker symbol. A torse-forming vector field ξ is called recurrent if $\rho = 0$; concircular if the form ϕ_i is a gradient covector, i. e., there is a function $\psi(x)$ such that $\phi = d\psi(x)$; convergent, if it is concircular and $\rho = \text{const}\cdot\exp(\psi)$.

Therefore, recurrent vector fields are characterized by the following equation

$$\nabla_X\xi = \phi(X)\xi. \tag{2.11}$$

Also, from the Definition 2., for a concircular vector field ξ, we get

$$(\nabla_Y\xi)X = \rho g(X,Y) \tag{2.12}$$

for all $X, Y \in TM$. A Riemannian space with a concircular vector field is called equidistant, [10, 11].

Conformal mappings of Riemannian spaces (or semi-Riemannian spaces) have been studied by many authors, [1,3,6,9]. In this section, we investigate the conformal mappings of nearly quasi-Einstein manifolds preserving the associated tensor E.

Theorem 1. *If V_n admits a conformal mapping preserving the associated tensor E and the associated scalar b, then V_n is an equidistant manifold.*

Proof. Suppose that V_n admits a conformal mapping preserving the associated tensor E and the associated scalar b. Using (2.9), we obtain

$$(n-2)\sigma_{ij} + (\beta + a - \bar{a}e^{2\sigma})g_{ij} = 0 \tag{2.13}$$

where $\beta = \Delta_2\sigma + (n-2)\Delta_1\sigma$. In this case, we get

$$\sigma_{ij} = \alpha g_{ij} \tag{2.14}$$

where $\alpha = \frac{1}{n-2}(\bar{a}e^{2\sigma} - a - \beta)$ is a function. Putting $\xi = -\exp(-\sigma)$ and using (2.7), (2.12) and (2.14), we get that V_n is an equidistant manifold. Hence, the proof is complete. □

Theorem 2. *An equidistant manifold V_n admits a conformal mapping preserving the associated tensor E if the associated scalars $\bar{a}$ and $\bar{b}$ satisfy both of the conditions*

$$\bar{b} = b,$$
$$\bar{a} = e^{-2\sigma}(a + \gamma),$$

where $\gamma = \left(\frac{n-1}{n}\right)[2\ \Delta_2\sigma + (n-2)\Delta_1\sigma]$.

Proof. Suppose that V_n is an equidistant manifold. Then, there exists a concircular vector field ξ satisfying the condition (2.12), that is, we have

$$\nabla_j \xi_i = \rho g_{ij} \tag{2.15}$$

where $\xi_i \equiv \nabla_i \xi$. Putting $\sigma = -\ln(-\xi(x))$ and using the condition (2.5), we obtain

$$\bar{S}_{ij} = S_{ij} + \gamma g_{ij} \tag{2.16}$$

where $\gamma = \left(\frac{n-1}{n}\right)[2\ \Delta_2\sigma + (n-2)\Delta_1\sigma]$. Considering (1.3) in (2.16) and using (2.2), we get

$$\bar{a}e^{2\sigma} g_{ij} + \bar{b}\bar{E}_{ij} = (a + \gamma)g_{ij} + bE_{ij}. \tag{2.17}$$

Taking $\bar{a} = e^{-2\sigma}(a+\gamma)$ and $\bar{b} = b$, (2.17) implies that $\bar{E}_{ij} = E_{ij}$. This completes the proof. □

The conharmonic transformation is a conformal transformation preserving the harmonicity of a certain function. If the conformal mapping is also conharmonic, then we have, [8]

$$\nabla_i \sigma^i + \frac{1}{2}(n-2)\sigma^i \sigma_i = 0. \tag{2.18}$$

Theorem 3. *Let V_n be a conformal mapping with preservation of the associated tensor E and the associated scalar b. A necessary and sufficient condition for this conformal mapping to be conharmonic is that the associated scalar $\bar{a}$ be transformed by $\bar{a} = e^{-2\sigma}a$.*

Proof. Suppose that V_n admits a conformal mapping preserving the associated tensor E and the associated scalar b. Using (2.7), (2.8) and (2.9), we obtain

$$(n-2)\nabla_j\nabla_i\sigma - (n-2)\sigma_i\sigma_j + [\nabla_h\sigma^h + (n-2)\sigma^h\sigma_h + a - \bar{a}e^{2\sigma}]g_{ij} = 0. \tag{2.19}$$

Multiplying (2.19) by g^{ij}, we get

$$\nabla_h\sigma^h + \frac{1}{2}(n-2)\sigma^h\sigma_h + \frac{n}{2(n-1)}(a - \bar{a}e^{2\sigma}) = 0. \tag{2.20}$$

If this mapping is conharmonic, using (2.18) in (2.20), we obtain $\bar{a} = e^{-2\sigma}a$. The converse is also true. This completes the proof. □

Definition 3. A φ(Ric)-vector field is a vector field on an n-dimensional Riemannian manifold (M, g) and Levi-Civita connection ∇, which satisfies the condition

$$\nabla\varphi = \mu\,\mathrm{Ric} \tag{2.21}$$

where μ is a constant and Ric is the Ricci tensor, [7]. When (M, g) is an Einstein space, the vector field φ is concircular. Moreover, when $\mu = 0$, the vector field φ is covariantly constant. In local coordinates, (2.21) can be written as

$$\nabla_j \varphi_i = \mu S_{ij} \tag{2.22}$$

where S_{ij} denote the components of the Ricci tensor and $\varphi_i = \varphi^\alpha g_{i\alpha}$.

Suppose that V_n admits a σ(Ric)-vector field. Then, we have

$$\nabla_j \sigma_i = \mu S_{ij} \tag{2.23}$$

where μ is a constant. Now, we can state the following theorem.

Theorem 4. *Let us consider the conformal mapping* (2.1) *of a nearly quasi-Einstein manifold V_n with constant associated scalars being also conharmonic with the σ*(Ric)*-vector field. A necessary and sufficient condition for the length of σ to be constant is that the trace of the associated tensor E of V_n be constant.*

Proof. We consider that the conformal mapping (2.1) of a nearly quasi-Einstein manifold V_n admitting a σ(Ric)-vector field is also conharmonic. In this case, comparing (2.18) and (2.23), we get

$$r = \frac{(2-n)}{2\mu}\sigma^i \sigma_i \tag{2.24}$$

where r is the scalar curvature of V_n. If V_n is of the constant associated scalars, from (1.4) and (2.24), we find

$$\tilde{E} = \frac{1}{b}\left(\frac{(2-n)}{2\mu}\sigma^i \sigma_i - na\right).$$

If the length of σ is constant, then $\sigma^i \sigma_i = c$, where c is a constant. Thus, we can see that $\tilde{E}$ is constant. The converse is also true. Hence, the proof is complete. □

3. AN EXAMPLE OF A NEARLY QUASI-EINSTEIN MANIFOLD

In this section, we consider a Riemannian metric g on $\mathbb{R}^4$ by the formula

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2 \tag{3.1}$$

where $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3, x^4 are the standard coordinates of $\mathbb{R}^4$. Then the only non-vanishing components of the Christoffel symbols, the curvature tensor, the Ricci tensor and the scalar curvature are

$$
\begin{gathered}
\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{3x^4}, \quad \Gamma^4_{11} = \Gamma^4_{22} = \Gamma^4_{33} = -\frac{2}{3}(x^4)^{\frac{1}{3}}, \\
R_{1441} = R_{2442} = R_{3443} = -\frac{2}{9(x^4)^{2/3}}, \\
R_{1221} = R_{1331} = R_{2332} = \frac{4}{9}(x^4)^{2/3}, \\
S_{11} = S_{22} = S_{33} = \frac{2}{3(x^4)^{2/3}}, \quad S_{44} = -\frac{2}{3(x^4)^2}, \ r = \frac{4}{3(x^4)^2}.
\end{gathered} \tag{3.2}
$$

Therefore $\mathbb{R}^4$ with the considered metric is a Riemannian manifold (M_4, g) of non-vanishing scalar curvature. Let us now consider the associated scalars a, b, and the associated tensor E as follows:

$$
a = -\frac{2}{3(x^4)^2}, \qquad b = -\frac{1}{9x^4} \tag{3.3}
$$

and

$$
E_{ij}(x) = \begin{cases} -12(x^4)^{1/3} & \text{for } i = j = 1, 2, 3 \\ 0 & \text{for } i = j = 4 \text{ and } i \neq j \end{cases} \tag{3.4}
$$

at any point $x \in M$. To verify the relation (1.3), it is sufficient to check the relations $S_{ii} = ag_{ii} + bE_{ii}$, $i = 1, 2, 3, 4$ since for the other cases, (1.3) holds trivially. From (3.2), (3.3), and (3.4), we obtain

$$
R.H.S \ \ of \ \ S_{11} = ag_{11} + bE_{11} = \frac{2}{3(x^4)^{2/3}} = S_{11}.
$$

Similarly, S_{22}, S_{33}, and S_{44} are also satisfied. Hence, (M_4, g) endowed with the metric (3.1) is a $N(QE)_4$ with the conditions (3.3) and (3.4).

Let (M_4, g) endowed with the metric (3.1) be a conformal mapping with preservation of the associated tensor E and the associated scalar b. Also, we choose σ and $\bar{a}$ as follows:

$$
\sigma = \ln(x^1 x^2 x^3), \qquad \bar{a} = -\frac{2}{3(x^1 x^2 x^3 x^4)^2} \tag{3.5}
$$

where $x^1, x^2, x^3 > 0$. Now, we show that these choices satisfy Theorem 3.

From (3.5), we get $\nabla_i \sigma = \frac{\partial \sigma}{\partial x^i} = \sigma_i = \frac{1}{x^i}$ for $i = 1, 2, 3$ and $\sigma_4 = 0$. Moreover, the only non-vanishing covariant derivatives of σ_i $(i = 1, 2, 3, 4)$ are

$$
\nabla_1 \sigma_4 = \nabla_4 \sigma_1 = -\frac{2}{3x^1 x^4}, \tag{3.6}
$$

$$
\nabla_2 \sigma_4 = \nabla_4 \sigma_2 = -\frac{2}{3x^2 x^4}, \tag{3.7}
$$

$$
\nabla_3 \sigma_4 = \nabla_4 \sigma_3 = -\frac{2}{3x^3 x^4}, \tag{3.8}
$$

and

$$\nabla_1\sigma_1 = -\frac{1}{(x^1)^2}, \qquad \nabla_2\sigma_2 = -\frac{1}{(x^2)^2}, \qquad \nabla_3\sigma_3 = -\frac{1}{(x^3)^2}. \tag{3.9}$$

Using (3.6)–(3.9), we find

$$g^{11}\nabla_1\sigma_1 + g^{11}\sigma_1\sigma_1 = 0 \tag{3.10}$$

and similarly the other cases hold. Therefore, the condition (2.18) is satisfied.

Moreover, from (3.3) and (3.5), we obtain

$$\bar{a}e^{2\sigma} = -\frac{2}{3(x^1x^2x^3x^4)^2} \times e^{2\ln(x^1x^2x^3)} = -\frac{2}{3(x^4)^2} = a. \tag{3.11}$$

From (3.10) and (3.11), we see that the equation (2.20) is satisfied. Hence, Theorem 3 holds for (M_4, g) endowed with the metric (3.1) and the conditions (3.3) and (3.5).

Now, we also show that (M_4, g) endowed with the metric (3.1) is not a quasi-Einstein manifold.

If possible, we have $S_{ij} = ag_{ij} + bA_iA_j$, where $i, j = 1, 2, 3, 4$. For $i = j$, we get from this relation

$$S_{ii} = ag_{ii} + bA_iA_i, \tag{3.12}$$

for all $i = 1, 2, 3, 4$. Since $S_{ii} \neq 0$ and $g_{ii} \neq 0$, we can choose $a \neq 0$, $b \neq 0$ and $A_i \neq 0$ for all $i = 1, 2, 3, 4$ such that (3.12) holds. However, for these values of a, b and A_i and for $i \neq j$, the equation $S_{ij} = ag_{ij} + bA_iA_j$ cannot be satisfied because for $i \neq j$, $S_{ij} = g_{ij} = 0$ but $A_i \neq 0$.

Therefore, (M_4, g) is not a quasi-Einstein manifold. Thus, a $N(QE)_n$ is not necessarily a quasi-Einstein manifold.

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Authors' addresses

Füsun Özen Zengin
Istanbul Technical University, Faculty of Science and Letters, Department of Mathematics, Maslak, 34469 Istanbul, Turkey
E-mail address: fozen@itu.edu.tr

Bahar Kirik
Istanbul Technical University, Faculty of Science and Letters, Department of Mathematics, Maslak, 34469 Istanbul, Turkey
E-mail address: bkirik@itu.edu.tr