Conformal mappings of nearly quasi-Einstein manifolds

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CONFORMAL MAPPINGS OF NEARLY QUASI-EINSTEIN MANIFOLDS

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Abstract. In this paper, we consider a conformal mapping between two nearly quasi-Einstein manifolds $V_n$ and $\tilde{V}_n$. We find some properties of this transformation from $V_n$ to $\tilde{V}_n$ and some theorems are proved.

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1. INTRODUCTION

A non-flat $n$-dimensional Riemannian or a semi-Riemannian manifold $(M, g)$, $(n > 2)$ is said to be an Einstein manifold if the condition

$$S(X, Y) = \frac{r}{n} g(X, Y)$$

holds on $M$, where $S$ and $r$ denote the Ricci tensor and the scalar curvature of $(M, g)$, respectively. Einstein manifolds play an important role in Riemannian Geometry, as well as in general theory of relativity. For this reason, these manifolds have been studied by many authors.

A non-flat $n$-dimensional Riemannian manifold $(M, g)$, $(n > 2)$ is defined to be a quasi-Einstein manifold if its Ricci tensor $S$ of type $(0, 2)$ is not identically zero and satisfies the following condition

$$S(X, Y) = a g(X, Y) + b A(X) A(Y)$$

where $a, b \in \mathbb{R}$ and $A$ is a non-zero 1-form such that $g(X, U) = A(X)$ for all vector fields $X$ on $M$. [5]. Then $A$ is called the associated 1-form and $U$ is called the generator of the manifold. Also M. C. Chaki and R. K. Maity [2] studied the quasi-Einstein manifolds by considering $a$ and $b$ as scalars such that $b \neq 0$ and $U$ as a unit vector field.

A non-flat $n$-dimensional Riemannian manifold $(M, g)$, $(n > 2)$ is called a nearly quasi-Einstein manifold if its Ricci tensor $S$ of type $(0, 2)$ is not identically zero and...
satisfies the following condition
\[
S(X, Y) = a g(X, Y) + b E(X, Y) \tag{1.3}
\]
where \(a\) and \(b\) are non-zero scalars and \(E\) is a non-zero symmetric tensor of type 
\((0,2),\ [4]\). Then \(E\) is called the associated tensor and \(a\) and \(b\) are called the 
associated scalars of \(M\). An \(n\)-dimensional nearly quasi-Einstein manifold is denoted by 
\(N(QE)_n\). An example of \(N(QE)_4\) has been given in [4].

Putting \(X = Y = e_i\) in (1.3), we get
\[
r = na + b \tilde{E}. \tag{1.4}
\]
Here \(r\) is the scalar curvature of \(N(QE)_n\) and \(\tilde{E} = E(e_i, e_i)\) where \(\{e_i\}, \ i = 1, 2, \ldots, n\) is an orthonormal basis of the tangent space at each point of the manifold.

In this paper, we investigate a conformal mapping between two nearly quasi-
Einstein manifolds.

2. CONFORMAL MAPPINGS OF NEARLY QUASI-EINSTEIN MANIFOLDS

In this section, we suppose that \(V_n\) and \(\tilde{V}_n\), \((n > 2)\) are two nearly quasi-Einstein manifolds with metrics \(g\) and \(\tilde{g}\), respectively.

**Definition 1.** A conformal mapping is a diffeomorphism of \(V_n\) onto \(\tilde{V}_n\) such that
\[
\tilde{g} = e^{2\sigma} g \tag{2.1}
\]
where \(\sigma\) is a function on \(V_n\). If \(\sigma\) is constant, then it is called a homothetic mapping. In local coordinates, (2.1) is written as
\[
\tilde{g}_{ij}(x) = e^{2\sigma(x)} g_{ij}(x), \quad \tilde{g}^{ij} = e^{-2\sigma} g^{ij}. \tag{2.2}
\]

Besides those equations, we have the Christoffel symbols, the components of the curvature tensor, the Ricci tensor, and the scalar curvature, respectively
\[
\hat{\Gamma}^h_{ij} = \Gamma^h_{ij} + \delta^h_i \sigma_j + \delta^h_j \sigma_i - \sigma^h g_{ij}, \tag{2.3}
\]
\[
\hat{R}^h_{i jk} = R^h_{i jk} + \delta^h_k \sigma_{ij} - \delta^h_i \sigma_{jk} + g^{h\alpha}(\sigma_{\alpha k} g_{ij} - \sigma_{\alpha j} g_{ik})
+ \Delta_1 \sigma(\hat{\delta}_k^h \hat{g}_{ij} - \hat{\delta}_j^h \hat{g}_{ik}), \tag{2.4}
\]
\[
\hat{S}_{ij} = S_{ij} + (n - 2) \sigma_{ij} + (\Delta_2 \sigma + (n - 2) \Delta_1 \sigma) g_{ij}, \tag{2.5}
\]
\[
\hat{r} = e^{-2\sigma}(r + 2(n - 1) \Delta_2 \sigma + (n - 1)(n - 2) \Delta_1 \sigma), \tag{2.6}
\]
where \(S_{ij} = R^h_{i jh} \), \(r = S_{\alpha \beta} g^{\alpha \beta} \), \(\sigma_{ij} = \frac{\partial \sigma}{\partial x^i} = \nabla_i \sigma \), \(\sigma^h = \sigma_{\rho} g^{\rho h} \) and
\[
\sigma_{ij} = \nabla_j \nabla_i \sigma - \nabla_i \nabla_j \sigma. \tag{2.7}
\]
\(\Delta_1 \sigma\) and \(\Delta_2 \sigma\) are the first and the second Beltrami’s symbols which are determined by
\[
\Delta_1 \sigma = g^{\alpha \beta} \nabla_\alpha \sigma \nabla_\beta \sigma, \quad \Delta_2 \sigma = g^{\alpha \beta} \nabla_\beta \nabla_\alpha \sigma \tag{2.8}
\]
where $\nabla$ is the covariant derivative according to the Riemannian connection in $V_n$. We denote the objects of space conformally corresponding to $V_n$ by a bar, i.e., $\bar{V}_n$. If $\bar{V}_n$ is a $N(QE)_n$, then we have, from (1.3), (2.2), and (2.5),

$$
\tilde{b}E_{ij} = bE_{ij} + (n - 2)\sigma_{ij} + (\Delta_2 \sigma + (n - 2)\Delta_1 \sigma + a - \bar{\alpha}e^{2p})g_{ij}.
$$

(2.9)

**Definition 2.** A vector field $\xi$ in a Riemannian manifold $M$ is called torse-forming if it satisfies the condition $\nabla_X \xi = \rho X + \phi(X)\xi$ where $X \in TM$, $\phi(X)$ is a linear form and $\rho$ is a function, [12]. In the local transcription, this reads

$$
\nabla_i \xi^h = \rho \delta_i^h + \xi^h \phi_i.
$$

(2.10)

where $\xi^h$ and $\phi_i$ are the components of $\xi$ and $\phi$, and $\delta_i^h$ is the Kronecker symbol. A torse-forming vector field $\xi$ is called recurrent if $\rho = 0$; concircular if the form $\phi_i$ is a gradient covector, i.e., there is a function $\psi(x)$ such that $\phi = d\psi(x)$; convergent, if it is concircular and $\rho = \text{const} \cdot \exp(\psi)$.

Therefore, recurrent vector fields are characterized by the following equation

$$
\nabla_X \xi = \phi(X)\xi.
$$

(2.11)

Also, from the Definition 2., for a concircular vector field $\xi$, we get

$$
(\nabla_Y \xi)X = \rho g(X, Y)
$$

(2.12)

for all $X, Y \in TM$. A Riemannian space with a concircular vector field is called equidistant, [10, 11].

Conformal mappings of Riemannian spaces (or semi-Riemannian spaces) have been studied by many authors, [1,3,6,9]. In this section, we investigate the conformal mappings of nearly quasi-Einstein manifolds preserving the associated tensor $E$.

**Theorem 1.** If $V_n$ admits a conformal mapping preserving the associated tensor $E$ and the associated scalar $b$, then $V_n$ is an equidistant manifold.

*Proof.* Suppose that $V_n$ admits a conformal mapping preserving the associated tensor $E$ and the associated scalar $b$. Using (2.9), we obtain

$$
(n - 2)\sigma_{ij} + (\beta + a - \bar{\alpha}e^{2p})g_{ij} = 0
$$

(2.13)

where $\beta = \Delta_2 \sigma + (n - 2)\Delta_1 \sigma$. In this case, we get

$$
\sigma_{ij} = \alpha g_{ij}
$$

(2.14)

where $\alpha = \frac{1}{n-2}(\bar{\alpha}e^{2p} - a - \beta)$ is a function. Putting $\xi = -\exp(-\sigma)$ and using (2.7), (2.12) and (2.14), we get that $V_n$ is an equidistant manifold. Hence, the proof is complete. □
Theorem 2. An equidistant manifold $V_n$ admits a conformal mapping preserving the associated tensor $E$ if the associated scalars $\tilde{a}$ and $\tilde{b}$ satisfy both of the conditions
\[
\tilde{b} = b, \\
\tilde{a} = e^{-2\sigma} (a + \gamma),
\]
where $\gamma = \left(\frac{n-1}{n}\right)[2 \Delta_2 \sigma + (n-2)\Delta_1 \sigma]$.

Proof. Suppose that $V_n$ is an equidistant manifold. Then, there exists a concircular vector field $\xi$ satisfying the condition (2.12), that is, we have
\[
\nabla_j \xi_i = \rho g_{ij} \tag{2.15}
\]
where $\xi_i \equiv \nabla_i \xi$. Putting $\sigma = -\ln(-\xi(x))$ and using the condition (2.5), we obtain
\[
\tilde{S}_{ij} = S_{ij} + \gamma g_{ij} \tag{2.16}
\]
where $\gamma = \left(\frac{n-1}{n}\right)[2 \Delta_2 \sigma + (n-2)\Delta_1 \sigma]$. Considering (1.3) in (2.16) and using (2.2), we get
\[
\tilde{a}e^{2\sigma} g_{ij} + \tilde{b} \tilde{E}_{ij} = (a + \gamma) g_{ij} + b E_{ij}. \tag{2.17}
\]
Taking $\tilde{a} = e^{-2\sigma} (a+\gamma)$ and $\tilde{b} = b$, (2.17) implies that $\tilde{E}_{ij} = E_{ij}$. This completes the proof. \[\square\]

The conharmonic transformation is a conformal transformation preserving the harmonicity of a certain function. If the conformal mapping is also conharmonic, then we have, [8]
\[
\nabla_i \sigma^i + \frac{1}{2} (n-2) \sigma^2 \sigma_i = 0. \tag{2.18}
\]

Theorem 3. Let $V_n$ be a conformal mapping with preservation of the associated tensor $E$ and the associated scalar $b$. A necessary and sufficient condition for this conformal mapping to be conharmonic is that the associated scalar $\tilde{a}$ be transformed by $\tilde{a} = e^{-2\sigma} a$.

Proof. Suppose that $V_n$ admits a conformal mapping preserving the associated tensor $E$ and the associated scalar $b$. Using (2.7), (2.8) and (2.9), we obtain
\[
(n-2) \nabla_j \nabla_i \sigma - (n-2) \sigma_i \sigma_j + [\nabla_h \sigma^h + (n-2) \sigma^h \sigma_h + a - \tilde{a}e^{2\sigma}] g_{ij} = 0. \tag{2.19}
\]
Multiplying (2.19) by $g^{ij}$, we get
\[
\nabla_h \sigma^h + \frac{1}{2} (n-2) \sigma^2 \sigma_h + \frac{n}{2(n-1)} (a - \tilde{a}e^{2\sigma}) = 0. \tag{2.20}
\]
If this mapping is conharmonic, using (2.18) in (2.20), we obtain $\tilde{a} = e^{-2\sigma} a$. The converse is also true. This completes the proof. \[\square\]
Definition 3. A $\varphi (\text{Ric})$-vector field is a vector field on an $n$-dimensional Riemannian manifold $(M, g)$ and Levi-Civita connection $\nabla$, which satisfies the condition

$$\nabla \varphi = \mu \text{Ric} \tag{2.21}$$

where $\mu$ is a constant and Ric is the Ricci tensor, [7]. When $(M, g)$ is an Einstein space, the vector field $\varphi$ is concircular. Moreover, when $\mu = 0$, the vector field $\varphi$ is covariantly constant. In local coordinates, (2.21) can be written as

$$\nabla_{\partial_i} \varphi_j = \mu S_{ij} \tag{2.22}$$

where $S_{ij}$ denote the components of the Ricci tensor and $\varphi_j = \varphi^\alpha_{,\alpha} g_{ij}$.

Suppose that $V_n$ admits a $\sigma (\text{Ric})$-vector field. Then, we have

$$\nabla_{\partial_j} \sigma_i = \mu S_{ij} \tag{2.23}$$

where $\mu$ is a constant. Now, we can state the following theorem.

**Theorem 4.** Let us consider the conformal mapping (2.1) of a nearly quasi-Einstein manifold $V_n$ with constant associated scalars being also conharmonic with the $\sigma (\text{Ric})$-vector field. A necessary and sufficient condition for the length of $\sigma$ to be constant is that the trace of the associated tensor $E$ of $V_n$ be constant.

**Proof.** We consider that the conformal mapping (2.1) of a nearly quasi-Einstein manifold $V_n$ admitting a $\sigma (\text{Ric})$-vector field is also conharmonic. In this case, comparing (2.18) and (2.23), we get

$$r = \frac{(2 - n)}{2\mu} \sigma^j \sigma_j \tag{2.24}$$

where $r$ is the scalar curvature of $V_n$. If $V_n$ is of the constant associated scalars, from (1.4) and (2.24), we find

$$\bar{E} = \frac{1}{b} \left( \frac{(2 - n)}{2\mu} \sigma^j \sigma_j - na \right).$$

If the length of $\sigma$ is constant, then $\sigma^j \sigma_j = c$, where $c$ is a constant. Thus, we can see that $\bar{E}$ is constant. The converse is also true. Hence, the proof is complete. □

3. **AN EXAMPLE OF A NEARLY QUASI-EINSTEIN MANIFOLD**

In this section, we consider a Riemannian metric $g$ on $\mathbb{R}^4$ by the formula

$$ds^2 = g_{ij} dx^i dx^j = (x^4)^{\frac{n}{2}} \left[ (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right] + (dx^4)^2 \tag{3.1}$$

where $i, j = 1, 2, 3, 4$ and $x^1, x^2, x^3, x^4$ are the standard coordinates of $\mathbb{R}^4$. Then the only non-vanishing components of the Christoffel symbols, the curvature tensor,
the Ricci tensor and the scalar curvature are

\[
\begin{align*}
\Gamma^1_{14} &= \Gamma^2_{24} = \Gamma^3_{34} = \frac{2}{3x^4}, & \Gamma^4_{11} &= \Gamma^4_{22} = \Gamma^4_{33} = -\frac{2}{3}(x^4)^{1/3}, \\
R_{1441} &= R_{2442} = R_{3443} = -\frac{2}{9(x^4)^{2/3}}, & R_{1221} &= R_{1331} = R_{2332} = \frac{4}{9}(x^4)^{2/3}, \\
S_{11} &= S_{22} = S_{33} = \frac{2}{3(x^4)^{2/3}}, & S_{44} &= -\frac{2}{3}(x^4)^2, & r &= \frac{4}{3(x^4)^{2/3}}.
\end{align*}
\]

Therefore \( \mathbb{R}^d \) with the considered metric is a Riemannian manifold \((M_4, g)\) of non-vanishing scalar curvature. Let us now consider the associated scalars \( a, b \), and the associated tensor \( E \) as follows:

\[
a = -\frac{2}{3(x^4)^2}, \quad b = -\frac{1}{9x^4}
\]

and

\[
E_{ij}(x) = \begin{cases} 
-12(x^4)^{1/3} & \text{for } i = j = 1, 2, 3 \\
0 & \text{for } i = j = 4 \text{ and } i \neq j
\end{cases}
\]

at any point \( x \in M \). To verify the relation (1.3), it is sufficient to check the relations \( S_{ii} = ag_{ii} + bE_{ii}, i = 1, 2, 3, 4 \) since for the other cases, (1.3) holds trivially. From (3.2), (3.3), and (3.4), we obtain

\[
R \text{ H.S of } S_{11} = ag_{11} + bE_{11} = \frac{2}{3(x^4)^{2/3}} \neq S_{11}.
\]

Similarly, \( S_{22}, S_{33}, \) and \( S_{44} \) are also satisfied. Hence, \((M_4, g)\) endowed with the metric (3.1) is a \( N(QE)_4 \) with the conditions (3.3) and (3.4).

Let \((M_4, g)\) endowed with the metric (3.1) be a conformal mapping with preservation of the associated tensor \( E \) and the associated scalar \( b \). Also, we choose \( \sigma \) and \( \tilde{a} \) as follows:

\[
\sigma = \ln(x^1x^2x^3), \quad \tilde{a} = \frac{2}{3(x^1x^2x^3x^4)^2}
\]

where \( x^1, x^2, x^3 > 0. \) Now, we show that these choices satisfy Theorem 3.

From (3.5), we get \( \nabla \sigma = \frac{\partial \sigma}{\partial x^i} = \sigma_i = \frac{1}{x^i} \) for \( i = 1, 2, 3, 4 \) and \( \sigma_4 = 0 \). Moreover, the only non-vanishing covariant derivatives of \( \sigma_i \) \( (i = 1, 2, 3, 4) \) are

\[
\begin{align*}
\nabla_1 \sigma_4 &= \nabla_4 \sigma_1 = -\frac{2}{3x^1x^4}, \\
\nabla_2 \sigma_4 &= \nabla_4 \sigma_2 = -\frac{2}{3x^2x^4}, \\
\nabla_3 \sigma_4 &= \nabla_4 \sigma_3 = -\frac{2}{3x^3x^4}.
\end{align*}
\]
and
\[ \nabla_1 \sigma_1 = -\frac{1}{(x^1)^2}, \quad \nabla_2 \sigma_2 = -\frac{1}{(x^2)^2}, \quad \nabla_3 \sigma_3 = -\frac{1}{(x^3)^2}. \tag{3.9} \]

Using (3.6)–(3.9), we find
\[ g^{11} \nabla_1 \sigma_1 + g^{11} \sigma_1 \sigma_1 = 0 \tag{3.10} \]
and similarly the other cases hold. Therefore, the condition (2.18) is satisfied.

Moreover, from (3.3) and (3.5), we obtain
\[ \tilde{a} e^{2\sigma} = -\frac{2}{3(x^1 x^2 x^3 x^4)^2} \times e^{2 \ln(x^1 x^2 x^3 x^4)} = -\frac{2}{3(x^4)^2} = a. \tag{3.11} \]

From (3.10) and (3.11), we see that the equation (2.20) is satisfied. Hence, Theorem 3 holds for \((M_4, g)\) endowed with the metric (3.1) and the conditions (3.3) and (3.5).

Now, we also show that \((M_4, g)\) endowed with the metric (3.1) is not a quasi-Einstein manifold.

If possible, we have \(S_{ij} = a g_{ij} + b A_i A_j\), where \(i, j = 1, 2, 3, 4\). For \(i = j\), we get from this relation
\[ S_{ii} = a g_{ii} + b A_i A_i, \tag{3.12} \]
for all \(i = 1, 2, 3, 4\). Since \(S_{ii} \neq 0\) and \(g_{ii} \neq 0\), we can choose \(a \neq 0\), \(b \neq 0\) and \(A_i \neq 0\) for all \(i = 1, 2, 3, 4\) such that (3.12) holds. However, for these values of \(a, b\) and \(A_i\) and for \(i \neq j\), the equation \(S_{ij} = a g_{ij} + b A_i A_j\) cannot be satisfied because for \(i \neq j\), \(S_{ij} = g_{ij} = 0\) but \(A_i \neq 0\).

Therefore, \((M_4, g)\) is not a quasi-Einstein manifold. Thus, a \(N(QE)\) is not necessarily a quasi-Einstein manifold.

References


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