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ON METRIZABILITY OF A CLASS OF 2-MANIFOLDS WITH LINEAR CONNECTION

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Abstract. In [17], we proved a theorem which shows how to find, under particular assumptions that guarantee metrizable (among others, recurrency of the curvature is necessary), all (at least local) pseudo-Riemannian metrics compatible with a given torsion-free linear connection without flat points in a domain of two-dimensional manifold. The result has the form of an implication only. In general, if there are flat points, or if curvature is not recurrent, we cannot give any good answer as it can also be demonstrated by examples. Note that in higher dimension, the problem of metrizable is not easy to solve, [6, 7]. Here, we try to apply this apparatus to the class of (torsion-free, locally homogeneous) connections with constant Christoffels in open domains of 2-manifolds (called *connections of Type A* in [1, 8]).

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1. INTRODUCTION

In [17], we developed a simple method which helps to decide whether a given linear connection on a 2-dimensional manifold arises as a Riemannian connection of some pseudo-Riemannian metric. Note that in an arbitrary dimension, the problem is difficult to solve. In the real analytic Riemannian case, probably the most effective method was offered by O. Kowalski in [6, 7], see also [14–16]. In the simplest case, for nowhere flat two-manifolds, we formulated the necessary and sufficient conditions for local metrizable in [17], and in favourable case, we described all compatible metrics in terms of the Ricci tensor. The purpose of the paper is to show a particular application, related to the class of locally homogeneous connections in 2-manifolds, [1, 8, 11]. First, recall some notation. Let (M_n, ∇) be a manifold with a linear connection and let R denote the corresponding curvature tensor, $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X, Y]}Z$ for X, Y, Z from $\mathcal{X}(M)$. The induced map $Z \mapsto R(X, Y)Z$, $R(X, Y): T_x M \rightarrow T_x M$ for X, Y, Z from $T_x M$ is linear and skew-symmetric. The Ricci tensor Ric in type $(0, 2)$ is a

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trace of the endomorphism: $\text{Ric}(Y, Z) = \text{Tr}\{X \mapsto R(X, Y)Z\}$, $X, Y, Z \in \mathfrak{X}(M)$, hence it carries less information than R in general. The formula

$$\text{Ric}(Z, Y) - \text{Ric}(Y, Z) = \text{Tr } R(Y, Z)$$

holds [10, p. 14].

1.1. The Ricci tensor of a pseudo-Riemannian manifold

On a pseudo-Riemannian manifold (M_n, g) with the metric tensor g , besides the curvature R in type (1, 3), we consider the type (0, 4) tensor introduced, up to a sign, as $\tilde{R}(X, Y, Z, W) = g(R(X, Y)Z, W)$, usually also called the curvature tensor; the relations

$$\tilde{R}(X, Y, Z, W) = \tilde{R}(Z, W, X, Y) = -\tilde{R}(Y, X, Z, W) = -\tilde{R}(X, Y, W, Z)$$

hold. In a coordinate system $(U, (x^i))$ based at a point $x \in M$, components R_{hijk} of \tilde{R} and R_{ijk}^ℓ of R satisfy $R_{hijk} = R_{jkhi} = -R_{ihjk} = -R_{hikj}$, $R_{hijk} = g_{hs} R_{ijk}^s$, and $g^{\ell h} R_{hijk} = R_{ijk}^\ell$. On (M_n, g) , the Ricci tensor of type (1, 1) has components $R_j^i = g_s^i R_{sj}$, and the scalar curvature ϱ is its trace,

$$\varrho = \text{Tr Ric} = R_s^s = g^{ij} R_{ij}.$$

Recall that a Riemannian manifold (M_n, g) is an *Einstein space* if there is a real constant λ such that $\text{Ric}(X, Y) = \lambda g(X, Y)$. (M_n, g) is *isotropic in a point* $x \in M$ if the curvature is the same constant, $K(x)$, on every (two-plane) section, and *isotropic* if it is isotropic in every point, [3]. If x is an isotropic point of (M_n, g) , then the following formula holds in x in any local coordinates around x : $R_{hijk} = K(g_{hj}g_{ik} - g_{hi}g_{jk})$.

From now on, suppose $n = 2$. Let (x^1, x^2) denote local coordinates in a coordinate neighborhood U of a manifold (M_2, g) . In dimension two, the non-zero components of \tilde{R} are equal R_{1212} up to a sign, and the sectional curvature

$$K(x) = \frac{R_{1212}}{g_{11}g_{22} - g_{12}g_{21}}$$

is sometimes called the *Gaussian curvature* ([12, p. 62.], [4, p. 137]). Any two-dimensional manifold is isotropic and the formula $R_{hijk} = K(g_{hj}g_{ik} - g_{hi}g_{jk})$ holds [4, p. 137]. Components of the Riemann curvature R of type (1, 3) are related to the Ricci tensor by $R_{hjk}^i = \delta_j^i R_{kh} - \delta_k^i R_{jh}$, [9, 12]. We easily check by direct evaluation (cf. [12, 17]):

Lemma 1. *The curvature tensor of a pseudo-Riemannian two-manifold (M_2, g) satisfies*

$$R_{hjk}^i = K(x)(\delta_k^i g_{hj} - \delta_j^i g_{hk})$$

where $K(x)$ is the Gaussian curvature and the Ricci tensor is a multiple of the metric tensor:

$$\text{Ric} = Kg = \frac{1}{2} \varrho g, \qquad g = \frac{2}{\varrho} \text{Ric}$$

where $\varrho(x)$ is the scalar curvature.

Consequently, the Ricci tensor of the Levi–Civita connection of a 2-dimensional pseudo-Riemannian manifold (M_2, g) is always symmetric and is proportional to the metric tensor. As a corollary, the Ricci tensor is *non-degenerate* for a nowhere flat (M_2, g) .

Obviously, (M_2, g) is an Einstein space if and only if it has constant (scalar or sectional) curvature (cf. [2, p. 44]). If this is the case, then (due to $\nabla g = 0$) the Ricci tensor is covariantly constant, $\nabla \text{Ric} = 0$, and *vice versa*.

Lemma 2. ([17]) *The Ricci tensor of a nowhere flat 2-dimensional pseudo-Riemannian manifold (M_2, g) with non-zero sectional curvature is recurrent, i. e.*

$$\nabla \text{Ric} = \omega \otimes \text{Ric}, \tag{1.1}$$

and the corresponding (non-zero) 1-form is exact, $\omega = df$ for some function f .

Cf. [5, I, p. 280].

1.2. Two-manifolds with a linear connection

Let us now pay attention to a torsion-free linear connection ∇ on a 2-manifold M_2 . As is well known, in this case, the curvature tensor R of type (1, 3) can be completely recovered from the Ricci tensor,

$$R(X, Y)Z = \text{Ric}(Y, Z)X - \text{Ric}(X, Z)Y$$

for $X, Y, Z \in \mathfrak{X}(M)$. In components $R^i_{hjk} = \partial_j^i R_{hk} - \partial_k^i R_{jh}$; $R^i_{hij} = R_{jh}$ for $j \neq i$ and $R^i_{hjj} = 0$. We get immediately

Lemma 3. *In (M_2, ∇) , $R = 0$ if and only if $\text{Ric} = 0$.*

Lemma 4. *For (M_2, ∇) , the Ricci tensor Ric is recurrent if and only if R is recurrent.*

Proof. If Ric is recurrent, then there is a one-form $\omega = \omega_j dx^j$ such that in local coordinates,

$$\nabla_\ell R^i_{hjk} = \delta_j^i \nabla_\ell R_{kh} - \delta_k^i \nabla_\ell R_{jh} = \delta_j^i \omega_\ell R_{kh} - \delta_k^i \omega_\ell R_{jh} = \omega_\ell R^i_{hjk}$$

and, hence, $\nabla R = \omega \otimes R$. The converse is similar: if R is recurrent, then $\nabla_\ell R_{jk} = \omega_\ell R^i_{kij} = \omega_\ell R_{jk}$, and $\nabla \text{Ric} = \omega \otimes \text{Ric}$. □

Since, in the case $R = 0$, the tensor Ric is in fact also *recurrent* ($\omega = 0$), *recurrency is a necessary condition* for metrization of two-manifolds.

2. METRIZABILITY OF 2-MANIFOLDS WITH LINEAR CONNECTION

As far as $R_{hjj}^i = 0$ and $R_{hij}^i = R_{jh}$ hold for $j \neq i$, the curvature tensor of a linear connection ∇ on M_2 is completely determined by its Ricci tensor; explicitly,

$$\begin{aligned} R_{11} &= -R_{112}^2 = R_{121}^2, \\ R_{21} &= -R_{121}^1 = R_{112}^1, \\ R_{12} &= -R_{212}^2 = R_{221}^2, \\ R_{22} &= -R_{221}^1 = R_{212}^1. \end{aligned}$$

It is now clear that there are two necessary conditions for local metrizable of a symmetric connection on a pseudo-Riemannian 2-manifold: the Ricci tensor must be symmetric and must also be recurrent for some closed 1-form (Lemma 2). Moreover, Ric may be degenerate only if $R = 0$ holds, and if this is the case, then $\text{Ric} = 0$. For the sake of global metrizable, the corresponding 1-form must also be exact. As is well known, a flat connection is always globally metrizable; we can even prescribe the signature.

When the Ricci tensor (or equivalently, the curvature) is non-zero in one fixed point $x_0 \in M$, then due to continuity, it is non-zero in some neighborhood of x_0 . Note that the subset of non-flat points is open. The following can be proved:

Theorem 1. *Let (M_2, ∇) be a 2-manifold with a torsion-free linear connection such that the Ricci tensor is symmetric, $R_{ij} = R_{ji}$, recurrent, i. e. (1.1) holds for some 1-form ω , and the regularity condition $|R_{ij}| \neq 0$ is satisfied. Then, at least locally, there is a metric compatible with the connection.*

Theorem 2. *Let (M_2, ∇) be a two-dimensional manifold with a complete torsion-free linear connection and with the curvature R non-zero everywhere. If the Ricci tensor of ∇ is regular, symmetric, and (1.1) holds where the 1-form is gradient, i. e., $\omega = df$ for some function f , then $g = e^{-f} \cdot \text{Ric}$ is a global metric tensor such that $\nabla g = 0$.*

Hence, a now-where flat complete torsion-free linear connection on M_2 is metrizable if and only if its Ricci tensor is symmetric, regular and recurrent with the corresponding 1-form being exact. If this is the case and $\nabla \text{Ric} = df \otimes \text{Ric}$ holds for some smooth function f , then there is a family of (mutually homothetic) global metrics corresponding to ∇ , $g_t = \exp(-f + t) \cdot \text{Ric}$, $t \in \mathbb{R}$.

The cases

- (i) $\text{Ric} = 0$ identically

and

- (ii) Ric is symmetric, recurrent and non-degenerate

are always favourable for local metrizability of a 2-manifold. Isolated flat points do not matter. But, as examples show, the Ricci tensor might vanish in some “greater” subset and then we can hardly give any reasonable prediction.

3. TORSION-FREE CONNECTIONS WITH CONSTANT CHRISTOFFELS IN DIMENSION 2

Consider $n = 2$. Let us apply the above methods, cf. [17], in the case of torsion-free connections with constant Christoffels on 2-manifolds (motivation for examining this class comes from classification of locally homogeneous linear connections, [1, 8, 11]). A torsion-free connection ∇ in a domain $U \subset M_2$ is given uniquely by a family of functions (components) $A(u, v), \dots, F(u, v)$ in u, v :

$$\nabla_{\partial u} \partial u = A \partial u + B \partial v, \quad \nabla_{\partial u} \partial v = C \partial u + D \partial v, \quad \nabla_{\partial v} \partial v = E \partial u + F \partial v,$$

where $\partial u = \frac{\partial}{\partial u}$, similarly for v , and (u, v) are local coordinates in U (i. e. $\Gamma_{11}^1 = A, \Gamma_{11}^2 = B$ etc.). In the case of constant Christoffels, the curvature and the Ricci tensor are constant,

$$\begin{aligned} R_{11} &= B(F - C) + D(A - D), \\ R_{12} &= R_{21} = CD - BE, \\ R_{22} &= E(A - D) + C(F - C), \end{aligned} \tag{3.1}$$

and constant components of the covariant derivative ∇Ric are

$$\begin{aligned} -R_{11;1} &= 2(AR_{11} + BR_{12}), \\ -R_{11;2} &= 2(CR_{11} + DR_{12}), \\ -R_{12;1} &= CR_{11} + (A + D)R_{12} + BR_{22}, \\ -R_{12;2} &= ER_{11} + (C + F)R_{12} + DR_{22}, \\ -R_{22;1} &= 2(CR_{12} + DR_{22}), \\ -R_{22;2} &= 2(ER_{12} + FR_{22}). \end{aligned}$$

Lemma 5. *A torsion-free linear connection on $M_2 = \mathbb{R}^2[u, v]$ with constant Christoffels is locally flat in the following cases:*

- (a) $B = D = 0, A, C, E, F \in \mathbb{R}$ are related by $AE - C^2 + CF = 0$;
- (b) $D \neq 0, C = BE/D, A, B, E, F \in \mathbb{R}$ are related by $D^2A - EB^2 + DBF - D^3 = 0$.

Proof. Suppose $\text{Ric} = 0$ holds. From technical reasons, let us distinguish the cases $D = 0, D \neq 0$ in our considerations (although D is not a preferable coefficient, we can start with something else, e.g. with C ; the main purpose is only to prepare the situation for application of Maple).

Let $D = 0$. Then, the condition $R_{12} = 0$ implies $BE = 0$. If $E = 0$ as well, then we get $B(F - C) = 0 = C(F - C)$ from (3.1) and further either $B = C = 0$ holds, we get the subcase

$$(a1) \quad B = C = D = E = 0, \quad A, F \in \mathbb{R},$$

or $F = C$, we obtain the subcase

$$(a2) \quad D = E = 0, \quad F = C, \quad A, B, C \in \mathbb{R};$$

the condition from (a) obviously holds. If $D = B = 0$, then the equation $C(F - C) + AE = 0$ must hold, case (a).

Now let $D \neq 0$. We get $C = BE/D$ from $R_{12} = 0$. The other two components of the Ricci tensor vanish if and only if the following holds:

$$\begin{aligned} B(F - BE/D) + D(A - D) &= 0, \\ (BE/D)(F - BE/D) + E(A - D) &= 0. \end{aligned}$$

This system is equivalent to the unique third-order algebraic equation $D^2A - EB^2 + DBF - D^3 = 0$, hence (b) is checked. \square

Note that if we choose $D \neq 0$ and either A, B, E or B, E, F , we calculate the remaining coefficients uniquely. It means:

Corollary 1. *The class of torsion-free linear connections with constant Christoffels contains a three-parameter family of metrizable locally flat connections.*

Step by step, the following can be proved, [13].

Lemma 6. *If the Ricci tensor of a torsion-free linear connection is recurrent on $M_2 = \mathbb{R}^2[u, v]$, has constant Christoffels and satisfies $R_{12} = 0$ then either Ric vanishes or is degenerate.*

Lemma 7. *If the Ricci tensor a torsion-free linear connection with constant Christoffels on $M_2 = \mathbb{R}^2[u, v]$ is symmetric, recurrent, non-vanishing and non-degenerate then the components must satisfy $R_{11} \cdot R_{22} \neq 0$.*

Hence, if a torsion-free connection with constant Christoffels on M_2 and non-vanishing Ricci tensor should be symmetric and recurrent, then it satisfies necessarily $R_{11} \cdot R_{12} \cdot R_{22} \neq 0$.

Lemma 8. *If (M_2, ∇) is a 2-manifold with a torsion-free linear connection which has constant Christoffels, the corresponding Ricci tensor is recurrent and all its components are non-vanishing on M_2 , then Ric is degenerate.*

Corollary 2. *In the above notation, exactly the following choices of constants give rise to a metrizable connection ∇ on M_2 :*

- (1) $B = C = D = E = 0, \quad A, F \in \mathbb{R};$
- (2) $D = E = 0, \quad F = C, \quad A, B, C \in \mathbb{R};$
- (3) $B = D = 0, \quad A, C, E, F \in \mathbb{R}$ satisfying the equality $AE - C^2 + CF = 0;$
- (4) $D \neq 0, \quad C = BE/D, \quad A, B, E, F \in \mathbb{R}$ satisfying the equality $D^2A - EB^2 + DBF - D^3 = 0.$

Theorem 3. *The only (locally) metrizable connections in the class of torsion-free linear connections with constant Christoffels defined on 2-manifolds are exactly locally flat connections.*

Note that Theorem 3 follows also from the results published in [1].

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