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ON COMPLETE TOTALLY UMBILICAL AND MAXIMAL SPACE-LIKE SURFACES IN PSEUDO-RIEMANNIAN MANIFOLDS

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Abstract. Space-like surfaces with special second fundamental forms have been an important tool in the study of pseudo-Riemannian manifolds. This paper focuses on adapting some theorems of Riemannian geometry in the large to the study of geometry of space-like surfaces of pseudo-Riemannian manifolds, specifically, various comparison theorems that use sectional and Ricci curvatures. The purpose of this paper is to establish a global geometry of complete space-like totally umbilical and maximal surfaces in pseudo-Riemannian manifolds.

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1. DEFINITIONS AND NOTATIONS

First, we set up our terminology and notations (see [2, 4]). Let (M, g) be an n-dimensional connected pseudo-Riemannian manifold (also called a semi-Riemannian manifold) of class C^∞ whose metric g has the signature (p, q) for $p + q = n$ and M' an n'-dimensional differentiable manifold of class C^∞ imbedded in (M, g) with an imbedding map $f : M' \to M$. We call the image $f(M')$ a surface (also called a submanifold) in (M, g) and identify it with the manifold M'.

The differential df of the imbedding map $f : M' \to M$ will denote by f_*, so that a vector field X' in TM' corresponds to a vector field $f_* X'$ in TM.

We denote by $g' = g(f_*, f_*)$ the metric tensor $g' = f^* g$ induced in M' from g by f, where f^* is the mapping conjugate of f_*. Then, the surface M' is called *space-like* if the metric tensor g' is positive definite and hence $n' \leq q$. In this case, (M', g') is a Riemannian manifold and for each of its points x, there exists a normal subspace of a tangent space $T_x M$ such that the following orthogonal decomposition $T_x M' = (T_x M')^\perp \oplus T_x M'$ is true.

We denote by ∇' the covariant differential operator corresponding to the Riemannian metric g'. Then, the *second fundamental form Q of the surface* (M', g') is defined by the formula

$$Q(X', Y') := (\nabla'_{X'} f_*) Y' = \nabla_{f_* X'} f_* Y' - f_*(\nabla'_{X'} Y')$$

for any $X',Y' \in C^\infty TM'$. The Gauss curvature equation for (M', g') has the form

$$R'(X', Y', V', W') = R(f_*X', f_*Y', f_*V', f_*W') + g(Q(X', W'), Q(Y', V')) - g(Q(Y', W'), Q(X', V')) \quad (1.1)$$

for all $X', Y', V', W' \in C^\infty TM'$, where R' is the curvature tensor of (M', g'). In particular, if (M, g) is a pseudo-Riemannian manifold of constant curvature C, then (1.1) can be rewritten in the form

$$R'(X', Y', V', W') = C \cdot (g'(X', W')\, g'(Y', V') - g'(Y', W')\, g'(X', V')) + g(Q(X', W'), Q(Y', V')) - g(Q(Y', W'), Q(X', V')). \quad (1.2)$$

A surface (M', g') is said to be totally umbilical in (M, g) if its second fundamental form satisfies $Q = g' \cdot H$ where H is the *mean curvature vector* defined by the formula $n \cdot H = \mathrm{trace}_{g'} Q$. In this case, (1.1) can be rewritten as the following equations

$$R'(X', Y', V', W') = R(f_*X', f_*Y', f_*V', f_*W') + g(H, H) \cdot (g'(X', W')g'(Y', V') - g'(Y', W')g'(X', V')) \quad (1.3)$$

for all $X', Y', V', W' \in C^\infty TM'$. In addition, if (M, g) is a manifold of constant curvature C, then from (1.2) and (1.3), we obtain

$$R'(X', Y', V', W') = (C + g(H, H)) \cdot (g'(X', W')g'(Y', V') - g'(Y', W')g'(X', V')) \quad (1.4)$$

and hence, (M', g') is a manifold of constant curvature $C' = C + g(H, H)$.

On the other hand, from (1.3), we obtain

$$\sec'(x, \pi) = \sec(x, \pi) + g(H_x, H_x) \quad (1.5)$$

where

$$\sec'(x, \pi) = \sec'(X'_x \wedge Y'_x) := -\frac{R'(X'_x, Y'_x, X'_x, Y'_x)}{g'(X'_x, X'_x)g'(Y'_x, Y'_x) - g'(X'_x, Y'_x)^2}$$

and

$$\sec(x, \pi) = \sec(X'_x \wedge Y'_x) := -\frac{R(f_*X'_x, f_*Y'_x, f_*X'_x, f_*Y'_x)}{g(f_*X'_x, f_*X'_x)g(f_*Y'_x, f_*Y'_x) - g(f_*X'_x, f_*Y'_x)^2}$$

are space-like sectional curvatures of (M', g') and (M, g) at $x \in M'$ with respect to the space-like nondegenerated plane $\pi = \mathrm{span}\{X'_x, Y'_x\} \subset T_xM'$.

In particular, let (M', g') be an n-dimensional space-like totally umbilical hypersurface (also called a hypersubmanifold) in a *Lorentzian manifold* (M, g). It means that (M, g) is a $(n+1)$-dimensional pseudo-Riemannian manifold (M, g) with metric g of Lorentzian signature $(-, +, \ldots, +)$. In this case, we can rewrite the equality (1.5) in the following form

$$\sec'(x, \pi) = \sec(x, \pi) - |H^2| \tag{1.6}$$

where $|H^2| = |g(H, H)|$ is a *mean curvature* of (M', g').

Next, a surface (M', g') is said to be *maximal* (in contrast to the Riemannian case where it is called the *minimal* surface) if its mean curvature vector $H = 0$. In particular, if (M', g') is an n-dimensional space-like maximal hypersurface in a Lorentzian manifold (M, g), the following equation can be obtained from (1.1)

$$\mathrm{Ric}'(Y', V') = \mathrm{Ric}(f_* Y', f_* V') + R(\mathcal{N}, f_* Y', f_* V', \mathcal{N}) + g'(\mathcal{A} Y', \mathcal{A} V') \tag{1.7}$$

where Ric' and Ric are Ricci tensors of (M', g') and (M, g), respectively. Moreover, $\mathcal{N}$ is the (globally defined) unitary time-like normal vector field on M' and $\mathcal{A}$ is the shape operator of (M', g') in (M, g) with respect to $\mathcal{N}$ such that $Q(Y', V') = -g'(\mathcal{A} Y', V')\mathcal{N}$. In particular, from (1.7), we obtain

$$\mathrm{Ric}'(V'_x) = \mathrm{Ric}(f_* V'_x) - \sec(x, \pi) + \|\mathcal{A} V'\|^2 \cdot \|V'_x\|^{-2} \tag{1.8}$$

where $\mathrm{Ric}'(V'_x) = \mathrm{Ric}'(V'_x, V'_x) \cdot \|V_x\|^{-2}$ and $\mathrm{Ric}(f_* V'_x) = \mathrm{Ric}'(f_* V'_x, f_* V'_x) \cdot \|V_x\|^{-2}$ are Ricci curvatures at $x \in M'$ with respect to the nonzero vector $V'_x \in T_x M'$,

$$\|\mathcal{A} V'\|^2 = g'(\mathcal{A} V'_x, \mathcal{A} V'_x)$$

and

$$\|V_x\|^2 = g'(V'_x, V'_x) = g(f_* V'_x, f_* V'_x).$$

Moreover, here, $\sec(x, \pi) := \sec(\mathcal{N}_x \wedge f_* V'_x)$ is a *time-like sectional curvature* of the *time-like plane* $\pi = \mathrm{span}\{\mathcal{N}_x, f_* V'_x\}$ and $V'_x \in T_x M'$ is a nonzero vector. Next, along with (1.8) from (1.7), we obtain the equation

$$s = s' + 2\,\mathrm{Ric}(\mathcal{N}, \mathcal{N}) + \mathrm{trace}(\mathcal{A}^2) \tag{1.9}$$

where s' and s are scalar curvatures of (M', g') and (M, g), respectively.

Finally, (M', g') is said to be *totally geodesic* if its second fundamental form Q vanishes identically. Obviously, such a surface is totally umbilical and maximal at the same time. In this paper, we exclude such surfaces from consideration.

2. Totally umbilical surfaces in pseudo-Riemannian manifolds

Let (M', g') be a n-dimensional complete space-like totally umbilical surface in a pseudo-Riemannian manifold (M, g). In this case, we have

Theorem 2.1. *Let* (M', g') *be a complete space-like totally umbilical surface in a pseudo-Riemannian manifold* (M, g).

(1) *If the mean curvature vector* H *of* (M', g') *and the sectional curvature* sec *of* (M, g) *satisfy the inequality*

$$\sec(\pi) \leq -g(H, H)$$

for all plane sections π *of* TM', *then* (M', g') *is diffeomorphic to* $\mathbb{R}^n$.

(2) *If there exists a constant* $\delta > 0$ *such that the mean curvature vector* H *of* (M', g') *and the sectional curvature* sec *of* (M, g) *satisfy the inequality*

$$\sec(\pi) \geq -g(H, H) + \delta$$

for all plane sections π *of* TM', *then* (M', g') *is compact, its diameter satisfies the inequality* $\operatorname{diam} M' \leq \pi/\sqrt{\delta}$ *and the first fundamental group is finite.*

Proof. Firstly, we recall that Mayer's theorem states (see [5, pp. 212–213]) that a complete Riemannian manifold (M', g') is compact if there exists a positive constant δ such that $\sec'(x, \pi) \geq \delta > 0$ for an arbitrary plane $\pi = \operatorname{span}\{X'_x, Y'_x\}$ at each point $x \in M'$. Moreover, its diameter satisfies the inequality $\operatorname{diam} M' \leq \pi/\sqrt{\delta}$ and the first fundamental group is finite.

Secondly, by (1.3), the inequality $\sec'(x, \pi) \geq \delta > 0$ can be rewritten in the form

$$\sec(x, \pi) \geq -g(H_x, H_x) + \delta.$$

Thirdly, if a complete Riemannian manifold (M', g') has nonpositive sectional curvature at all plane sections, then (M', g') is diffeomorphic to $\mathbb{R}^n$ (see [5, p. 201]). After these remarks, the theorem statements become obvious. □

Next, from Theorem 2.1 and (1.4), we conclude that if (M, g) is a manifold of constant curvature C such that $C + g(H, H) > 0$, then its complete space-like totally umbilical surface (M', g') is compact. In addition, we recall that any compact, simply-connected Riemannian manifold (M', g') with constant sectional curvature $C' > 0$ is necessarily isometric to a Euclidian sphere $\mathbb{S}^n$ of radius $(C')^{-1/2}$, equipped with its standard metric (see [5, p. 217–218]). Thus, using Theorem 2.1 and the result above, we can formulate the following corollary.

Corollary 2.1. *Let* (M', g') *be a complete space-like totally umbilical surface in a pseudo-Riemannian manifold* (M, g) *of constant curvature* C. *If the mean curvature vector* H *of* (M', g') *satisfies the inequality*

$$g(H, H) + C > 0,$$

then (M', g') *is compact and its first fundamental group is finite. Further, suppose that* (M', g') *is simply-connected, then* (M', g') *is isometric to a Euclidian sphere* $\mathbb{S}^n$ *of radius* $(g(H, H) + C)^{-1/2}$.

Now, we suppose that (M', g') is an n-dimensional space-like totally umbilical hypersurface in a Lorentzian manifold (M, g). In this case, from the inequality $\sec(x, \pi) \leq |{H_x}^2|$, we obtain $\sec'(x, \pi) \leq 0$ for an arbitrary plane $\pi \subset T_x M'$ at each point $x \in M'$. It follows that (M', g') is diffeomorphic to $\mathbb{R}^n$ (see [5, p. 201]).

On the other hand, if the inequality $\sec(x, \pi) > |{H_x}^2| + \delta$ holds for some positive constant δ, then from (1.5), we obtain $\sec'(x, \pi) \geq \delta > 0$. Therefore, as a consequence of Mayer's theorem (see [5, pp. 212–213]), we have the following result.

Corollary 2.2. *Let (M', g') be a complete space-like totally umbilical hypersurface in a Lorentzian manifold (M, g).*

(1) *If the mean curvature $|H^2|$ of (M', g') and the sectional curvature* sec *of (M, g) satisfy the inequality*

$$\sec(\pi) \leq |H^2|$$

for all plane sections π of TM', then (M', g') is diffeomorphic to $\mathbb{R}^n$.

(2) *If there exists a positive constant δ such that the mean curvature $|H^2|$ of (M', g') and the sectional curvature* sec *of (M, g) satisfy the inequality*

$$\sec(\pi) \geq |H^2| + \delta$$

for all plane sections π of TM', then (M', g') is compact, its diameter satisfies the inequality

$$\operatorname{diam} M' \leq \frac{\pi}{\sqrt{\delta}}$$

and the first fundamental group is finite.

We define a de Sitter space $\mathbb{S}_1^{n+1}(C)$ as a Lorentzian manifold (M, g) with positive constant sectional curvature C and recall that any space-like compact hypersurface (M', g') is a de Sitter space $\mathbb{S}_1^{n+1}(C)$ diffeomorphic to a Euclidian sphere $\mathbb{S}^n$. In particular, compact totally umbilical space-like hypersurfaces in $\mathbb{S}_1^{n+1}(C)$ are round n-spheres (see [1, 8]). Now, we can formulate the following corollary.

Corollary 2.3. *Let (M', g') be a complete space-like totally umbilical hypersurface in $\mathbb{S}_1^{n+1}(C)$. If the mean curvature $|H^2|$ of (M', g') satisfies the inequality*

$$|H^2| < C,$$

then (M', g') is a round n-sphere. Furthermore, if (M', g') is simply-connected, then (M', g') is isometric to a Euclidian sphere $\mathbb{S}^n$ of radius $(C - |H^2|)^{-1/2}$.

Montiel (see [8]) proved that every compact space-like surface in $\mathbb{S}_1^{n+1}(C)$ of constant mean curvature is totally umbilical. Based on these results and Corollary 2.4, we can formulate the following obvious corollary.

Corollary 2.4. *Let (M', g') be a compact simply-connected space-like hypersurface in a de Sitter space $\mathbb{S}_1^{n+1}(C)$. If the mean curvature $|H^2|$ of (M', g') is constant and*

$$|H^2| < C,$$

then (M', g') is isometric to a Euclidian sphere $\mathbb{S}^n$ of radius $(C - |H^2|)^{-1/2}$.

3. Maximal space-like hypersurfaces in Lorentzian manifolds

Let (M', g') be an n-dimensional complete space-like maximal hypersurface in a Lorentzian manifold (M, g).

Theorem 3.1. *Let (M', g') be a space-like complete non-totally geodesic, maximal hypersurface in a Lorentzian manifold (M, g). If there exists a positive constant δ such that the space-like Ricci curvature and time-like sectional curvature of (M, g) satisfy the inequality*

$$\mathrm{Ric}(f_* V_x') \geq \sec(\mathcal{N}_x \wedge f_* V_x') + \delta$$

for any vector $V_x \in T_x M'$ at each point $x \in M'$, then (M', g') is compact, its diameter satisfies the inequality

$$\operatorname{diam} M' \leq \frac{\pi}{\sqrt{\delta}}$$

and its first fundamental group is finite.

Proof. Firstly, from (1.8), we obtain

$$\begin{aligned}\mathrm{Ric}'(V_x') &= \mathrm{Ric}(f_* V_x') - \sec(\mathcal{N}_x \wedge f_* V_x') + \|\mathcal{A} V'\|^2 \cdot \|V_x'\|^{-2} \\ &\geq \mathrm{Ric}(V_x') - \sec(\mathcal{N}_x \wedge f_* V_x')\end{aligned}$$

for the nonzero vector $V_x \in T_x M'$. Secondly, if we suppose that

$$\mathrm{Ric}(f_* V_x') - \sec(\mathcal{N}_x \wedge f_* V_x') \geq \delta > 0$$

for some positive constant δ and an arbitrary nonzero vector $V_x \in T_x M'$ at each point $x \in M'$, then $\mathrm{Ric}'(V_x') \geq \delta > 0$. This means that (M', g') is compact, its diameter satisfies the inequality

$$\operatorname{diam} M' \leq \frac{\pi}{\sqrt{\delta}}$$

and the first fundamental group is finite (see [5, p. 216]). In particular, if we suppose for (M, g) that its all time-like sectional curvatures $\sec(x, \pi) \leq -\delta < 0$ and space-like Ricci curvatures $\mathrm{Ric}(V) \geq 0$ for all space-like nonzero vectors V, then (1.8) implies that

$$\mathrm{Ric}'(V_x') = \mathrm{Ric}(f_* V_x') - \sec(\mathcal{N}_x \wedge f_* V_x') + \|\mathcal{A} V'\|^2 \cdot \|V_x'\|^{-2} \geq \delta > 0$$

and hence, (M', g') is compact. This completes the proof. □

Next, from (1.9), we obtain

$$s' = s + 2\,\mathrm{Ric}(\mathcal{N}, \mathcal{N}) + \sum_{i=1,\ldots,n} (\lambda_i)^2,$$

where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of $\mathcal{A}$ which are called the *principal curvatures* of (M', g'). In addition, we recall that (M, g) is said to satisfy the *strong energy condition* or the *time-like convergence condition* (see [6, p. 95]) if $\mathrm{Ric}(V_x, V_x) \geq 0$ for every time-like vector $V_x \in T_x M$ at each point $x \in M$. Then, the following corollary holds.

Corollary 3.1. *Let (M', g') be a space-like and maximal hypersurface Lorentzian manifold (M, g) which satisfies the strong energy condition, then $s' \geq s$ for the scalar curvatures s' and s of (M', g') and (M, g), respectively. The equality holds on a totally geodesic hypersurface.*

Cheng and Yau [3] and T. Ishihara [7] proved that a complete maximal space-like submanifold (M', g') of $\mathbb{S}_1^{n+1}(C)$ is totally geodesic. Then, the Ricci curvature Ric' of a space-like hypersurface (M', g') in $\mathbb{S}_1^{n+1}(C)$ has the form

$$\mathrm{Ric}'(X') = (n-1)\,C > 0$$

for any nonzero $X' \in TM'$. This inequality is a necessary condition for a complete Riemannian manifold (M', g') to be compact (see [5, p. 216]). Thus, using this fact and Mayer's theorem (see [5, pp. 212–213]), we can formulate the following corollary.

Corollary 3.2. *If (M', g') is a space-like complete maximal hypersurface in a de Sitter space-time $\mathbb{S}_1^{n+1}(C)$, then (M', g') is compact, totally geodesic and its first fundamental group is finite. Further suppose that (M', g') is simply-connected then (M', g') is isometric to a Euclidian sphere $\mathbb{S}^n$ of radius $C^{-1/2}$.*

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