On the degree of geodesic mobility of Riemannian manifolds

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ON THE DEGREE OF GEODESIC MOBILITY OF RIEMANNIAN MANIFOLDS

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Abstract. We will consider the definition of the degree r of geodesic mobility of a Riemannian manifold and will prove that the well-known results on the geodesic mobility are essentially local. In the last section of our paper, we will prove that the geodesic mobility of an n-dimensional closed, simply-connected Riemannian manifold with positive constant sectional curvature satisfies the identity $r = (n + 1)(n + 2)/2, n \geq 2$.

1. INTRODUCTION

The solving of the problem of geodesic mapping of Riemannian manifolds was started by Levi-Civita (see [5]) who studied the trajectory equivalence of dynamical systems. The definition of the degree of geodesic mobility of a Riemannian manifold was introduced by Sinyukov (see [12, p. 137], see [2, 8], [11, p. 168]) as a numerical characteristic of the cardinality of the geodesic class. The method for finding the degree of geodesic mobility is based on an analysis of the integrability conditions of fundamental equations of geodesic mappings (see [12, pp. 137–147]). Interesting results of global geodesic mappings for special Riemannian manifolds are formulated in papers [1, 6–11].

In this paper, we will prove that the well-known results on the geodesic mobility are essentially local and that the geodesic mobility r of an n-dimensional closed, simply-connected Riemannian manifold with positive constant sectional curvature satisfies the identity

$$r = (n + 1)(n + 2)/2.$$

2. THE DEGREE OF GEODESIC MOBILITY

Suppose that (M, g) is an n-dimensional Riemannian manifold and ∇ is the Levi-Civita connection corresponding to the metric tensor $g = (g_{ij}), n \geq 2$.

A mapping of the Riemannian manifold (M, g) to a Riemannian manifold (M', g') is called *geodesic* if it takes every geodesic of (M, g) to a geodesic of (M', g').

It is well known (see, for example [12, p. 137], [8], [11, p. 168]) that the Riemannian manifold (M, g) admits a geodesic mapping if and only if there exists a covariant regular symmetric tensor field $a = (a_{ij})$ on this manifold such that

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik} \tag{2.1}$$

where $\lambda_i = \frac{1}{2} \nabla_i (\text{trace}_g a)$. If $\lambda_i = 0$, then the geodesic mapping is called *trivial*.

The equations (2.1) have been completed by Sinyukov to a system of differential equations of Cauchy type in covariant derivatives (see [12, p. 134]; [8]; [11, p. 168]). In particular, Sinyukov has proved that if the system of differential equations (2.1) is a totally-integrable system of differential equations in a neighborhood of an arbitrary point of Riemannian manifold (M, g), then (M, g) is a manifold of constant curvature (see [12, p. 138]).

The *degree of geodesic mobility r* of an n-dimensional Riemannian manifold (M, g) is the dimension of linear space of solutions of system of differential equations (2.1). The notion of geodesic mobility of a Riemannian manifold was introduced by Sinyukov (see [12, p. 137], [8]). From Sinyukov's theorem, we conclude that the n-dimensional Riemannian manifolds of constant curvature and only these manifolds have the maximal degree $r = (n + 1)(n + 2)/2$ for $n \geq 2$.

On the other hand, an arbitrary Riemannian manifold has mobility no smaller than 1, because system (2.1) has at least one solution of the form $a = \alpha g$ for an arbitrary real constant α on this manifold. Therefore, we can conclude that $1 \leq r \leq (n + 1)(n + 2)/2$ for any n-dimensional Riemannian manifold (M, g).

The result formulated in Sinyukov's theorem and other results on the geodesic mobility (see [11, pp. 147-149]) are essentially local. For example, we consider an n-dimensional hyperbolic space, which is a Riemannian manifold (M, g) of constant negative curvature. In this case, we have $r = (n + 1)(n + 2)/2$. If we factorize hyperbolic space by a suitable discrete group of motions (see [13]), we obtain a compact manifold of constant negative curvature.

It is well known (see [7]) that on a compact Riemannian manifold with non positive constant curvature, any tensor field $a = (a_{ij})$ satisfying the equations (2.1) has the trivial form $a = \text{const} \cdot g$. Therefore, this manifold does not admit a "globally defined" non-trivial geodesic mapping, i.e. $r = 1$. Finally, we can conclude that the number $r = (n + 1)(n + 2)/2$ is a *local* numerical characteristic of the cardinality of the geodesic class of a hyperbolic space. On the other hand, if we consider the number r as a *global* numerical characteristic of the cardinality of the geodesic class of a compact hyperbolic space, then $r = 1$.

3. The main theorem

In this section, we shall prove the main proposition of our paper.

Theorem. *Let* (M, g) *be an n-dimensional closed and simply-connected Riemannian manifold with positive constant sectional curvature and r be the degree of geodesic mobility as a global numerical characteristic of the cardinality of the geodesic class of* (M, g). *Then*

$$r = (n + 1)(n + 2)/2.$$

Proof. Let $\mathbb{R}^{n+1}$ be an $(n+1)$-dimensional Euclidean space endowed with the inner product $\langle \cdot, \cdot \rangle$. If $\{x^{\alpha}\}$ for $\alpha, \beta, \gamma = 1, \ldots, n, n + 1$ are Cartesian orthogonal coordinates in $\mathbb{R}^{n+1}$ then

$$\mathbb{S}^n : \quad (x^1)^2 + \cdots + (x^n)^2 + (x^{n+1})^2 = 1$$

is a unit sphere in $\mathbb{R}^{n+1}$. If the sphere $\mathbb{S}^n$ is represented locally by $x^{\alpha} = f^{\alpha}(u^k)$ in terms of its local coordinates $\{u^k\}$, then putting $f_k^{\alpha} = \partial_k x^{\alpha}$ for $\partial_k = \partial/\partial u^k$, we can define the second fundamental form of the sphere $\mathbb{S}^n$ by the Gauss equations (see [14])

$$f_{ij}^{\alpha} = \nabla_i f_j^{\alpha} = \partial_i f_j^{\alpha} - f_k^{\alpha} \Gamma_{ji}^k + \Gamma_{\beta\gamma}^{\alpha} f_i^{\beta} f_j^{\gamma}$$

where ∇_i is the van der Waerden–Bortolotti covariant differentiation along $\mathbb{S}^n$, Γ_{ji}^k are the Christoffel symbols of the metric tensor $g_{ij} = f_i^{\alpha} f_j^{\beta} g_{\alpha\beta}$ on $\mathbb{S}^n$, $\Gamma_{\beta\gamma}^{\alpha}$ are the Christoffel symbols of the metric tensor $g_{\alpha\beta}$ on a $(n+1)$-dimensional Euclidean space. In a Cartesian orthogonal system, $g_{\alpha\beta} = \delta_{\alpha\beta}$ and $\Gamma_{\beta\gamma}^{\alpha} = 0$, where $\delta_{\alpha\beta}$ is Kronecker's delta.

It is well known that $\mathbb{S}^n$ is a totally umbilical submanifold and hence $f_{ij}^{\alpha} = g_{ij} N^{\alpha}$, where $N^{\alpha} = -x^{\alpha}$ are the components of the unit normal vector $N \in C^{\infty}(T\mathbb{S}^n)^{\perp}$.

If $A = (A_{\alpha\beta})$ is a constant symmetric tensor, then we can define a tensor field $a = (a_{ij})$ on $\mathbb{S}^n$ by the following identities $a_{ij} = A_{\alpha\beta} f_i^{\alpha} f_j^{\beta}$. Applying ∇_k to these identities, we obtain

$$\nabla_k a_{ij} = f_k^{\gamma} (\nabla_{\gamma} A_{\alpha\beta}) f_i^{\alpha} f_j^{\beta} + A_{\alpha\beta}(f_{ki}^{\alpha} f_j^{\beta} + f_i^{\alpha} f_{kj}^{\beta})$$

$$= A_{\alpha\beta}(N^{\alpha} f_j^{\beta} g_{ik} + N^{\beta} f_i^{\alpha} g_{jk}).$$

If we suppose that $\lambda_j = A_{\alpha\beta} N^{\alpha} f_j^{\beta}$, then we can rewrite the last equations in the following form

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik},$$

which shows that $\mathbb{S}^n$ admits a globally defined geodesic mapping because $a = (a_{ij})$ is a tensor field globally defined on $\mathbb{S}^n$.

The number of independent components of the constant symmetric tensor $A = (A_{\alpha\beta})$ is equal to $(n + 1)(n + 2)/2$. Therefore, we can determine a sharp upper bound on the degree of geodesic mobility r as a *global* numerical characteristic of the cardinality of the geodesic class for a unit sphere $\mathbb{S}^n$. In addition, we recall that Hopf has shown (see [3, 4]) that a closed, simply-connected Riemannian manifold

with constant sectional curvature 1 is necessarily isometric to the unit sphere $\mathbb{S}^n$, equipped with its standard metric. This finishes the proof.   □

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