



Miskolc Mathematical Notes
Vol. 14 (2013), No 2, pp. 591-599

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2013.920

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ON DECOMPOSITION PROBLEMS ON MANIFOLDS WITH A SPECIAL DIFFERENTIAL OPERATOR

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Abstract. This paper deals with the properties of a special differential operator with respect to the general decomposition of tensor fields on manifolds with affine connection. It is shown that properties of a special differential operator are transferred to the components of a given decomposition.

2000 *Mathematics Subject Classification:* Primary 53A55, 15A69; Secondary 53B05.

Keywords: decomposition of tensors, recurrency, differential operator, covariant derivative

1. INTRODUCTION

The trace decomposition of tensors on an n -dimensional Riemannian manifold M_n was described by Weyl [8]. This decomposition problem may be naturally generalized for certain cases (see for example [2, 4–7, 9]).

In this paper, we generalize some results from [3] concerning decompositions of tensors and tensor fields. Crasmareanu [1] has studied decompositions of tensor fields for the case of covariant derivative of such tensors which satisfy certain conditions. We consider a special differential operator and generalize those results.

2. GENERAL DECOMPOSITION OF TENSORS

It is well known that on every manifold M_n , a positive definite metric g determining the structure of a Riemannian manifold may be introduced. For practical reasons (e. g., in theoretical physics), the pseudo-Riemannian manifolds with an indefinite pseudometric g are considered, too. In what follows, we will use in both cases the terms of *Riemannian manifold* and *metric*.

Let M_n be a manifold with a metric g . We introduce the following notation: For every ordered system of indices i_1, i_2, \dots, i_q , $1 \leq i_\rho \leq n$, and every pair (i_ρ, i_σ) , $\rho < \sigma$, of them, we denote by

$$M_{i_1, \dots, i_{\rho-1} \overset{\rho}{\vee} i_{\rho+1} \dots i_{\sigma-1} \overset{\sigma}{\vee} i_{\sigma+1} \dots i_q}^{(\rho, \sigma)}$$

The paper was supported by Grant P201/11/0356 of the Czech Science Foundation.

the tensor of the type $(0, q - 2)$, where the ρ -th and the σ -th indices are omitted.

Then we have the following theorem:

Theorem 1. *Let $T_{i_1 \dots i_q}$ be a $(0, q)$ -tensor and*

$$(i_{k_1}, i_{l_1}), (i_{k_2}, i_{l_2}), \dots, (i_{k_s}, i_{l_s}), \quad s \leq \frac{1}{2} q(q - 1), \quad (2.1)$$

be pairs of indices satisfying $k_\sigma < l_\sigma$. Then there exists the decomposition of the tensor $T_{i_1 \dots i_q}$ of the form

$$T_{i_1 \dots i_q} = \tilde{T}_{i_1 \dots i_q} + \sum_{\sigma=1}^s g_{i_{k_\sigma} i_{l_\sigma}} \cdot M_{i_1, \dots, i_{k_\sigma-1} \overset{k_\sigma}{Y} i_{k_\sigma+1} \dots i_{l_\sigma-1} \overset{l_\sigma}{Y} i_{l_\sigma+1} \dots i_q}^{(k_\sigma, l_\sigma)}, \quad (2.2)$$

where the tensor \tilde{T} satisfies

$$\tilde{T}_{i_1 \dots i_{k_\sigma} \dots i_{l_\sigma} \dots i_q} \cdot g^{i_{k_\sigma} i_{l_\sigma}} = 0, \quad (2.3)$$

for all indices (2.1). The tensor \tilde{T} is uniquely determined and $M^{(k_\sigma, l_\sigma)}$ are certain tensors of the type $(0, q - 2)$.

Proof. Let M_n be a Riemannian manifold with metric g . Now, let us consider an a priori chosen point x_0 at M_n . Values of metric g as well as all the given tensors will be considered in x_0 .

Taking a coordinate system in which the matrix of g fulfills

$$g_{ij} = g_0 \cdot D, \quad \text{where } g_0 \in R, \quad \text{and } D = \text{diag}(\pm 1, \pm 1, \dots, \pm 1), \quad (2.4)$$

we obtain the inverse matrix g^{-1} in the form of

$$g^{ij} = g_0^{-1} \cdot D. \quad (2.5)$$

The existence of the coordinated system above is guaranteed only for the concrete point x_0 . A metric (2.4) exists globally in Euclidean and pseudo-Euclidean spaces.

Therefore, the relation (2.3) may be written as

$$\sum_{k_\sigma, l_\sigma=1}^n \tilde{T}_{i_1 \dots i_{k_\sigma} \dots i_{l_\sigma} \dots i_q} \cdot g_{i_{k_\sigma} i_{l_\sigma}} = 0. \quad (2.6)$$

Let us define the inner product \circ on T_q^0 by

$$\overset{1}{T} \circ \overset{2}{T} \stackrel{\text{def}}{=} \sum_{i_1, \dots, i_q=1}^n \overset{1}{T}_{i_1 \dots i_q} \cdot \overset{2}{T}_{i_1 \dots i_q}. \quad (2.7)$$

Now, let us construct a linear subspace $\overset{*}{T} \subseteq T_p^0$ which is generated by all the tensors of the form

$$\sum_{\sigma=1}^s g_{i_{k_\sigma} i_{l_\sigma}} \cdot M_{i_1, \dots, i_{k_\sigma-1} \overset{k_\sigma}{Y} i_{k_\sigma+1} \dots i_{l_\sigma-1} \overset{l_\sigma}{Y} i_{l_\sigma+1} \dots i_q}^{(k_\sigma, l_\sigma)}. \quad (2.8)$$

Considering a linear subspace in T_p^0 which contains all the tensors fulfilling (2.3), we obtain $\tilde{T} \circ T^* = 0$ for any tensors \tilde{T} of this subspace in the obvious way. Therefore, this subspace is contained in the orthogonal complement of the subspace T^* , which implies the unicity of the decomposition (2.2) of the tensor $T_{i_1 \dots i_p}$. \square

3. SPECIAL DECOMPOSITIONS OF TENSORS

Comparing the decomposition in the following theorem with decompositions in Theorem 1 (and Theorem 4) as well as in the fundamental theorem proved by Weyl (see [8]), we may remark that in the following one, not only tensor \tilde{T} is uniquely determined but also all tensors $M^{(*)}$.

Theorem 2. *Let $T_{i_1 i_2 \dots i_q}$ be a $(0, q)$ -tensor and*

$$(i_1, i_2), (i_1, i_3), \dots, (i_1, i_q) \tag{3.1}$$

be some pairs of indices. Then for $n > q - 1$ there exists the unique decomposition of $T_{i_1 \dots i_q}$ of the form

$$T_{i_1 \dots i_q} = \tilde{T}_{i_1 \dots i_q} + \sum_{\sigma=2}^n g_{i_1 i_\sigma} \cdot M_{\gamma_{i_2, \dots, i_{\sigma-1}}^\sigma \gamma_{i_{\sigma+1} \dots i_q}^\sigma}^{(1, \sigma)}, \tag{3.2}$$

where $\tilde{T}_{i_1 \dots i_q}$ satisfies

$$\tilde{T}_{i_1 \dots i_\sigma \dots i_q} \cdot g^{i_1 i_\sigma} = 0, \tag{3.3}$$

for any pairs of indices (3.1). The tensor \tilde{T} and all $(0, q - 2)$ -tensors $M^{(i_1 i_\sigma)}$ are uniquely determined.

Proof. With respect to Theorem 1 we are only to prove the uniqueness of the tensors $M^{(i_1 i_\sigma)}$. Contracting (3.2) with $g^{i_1 i_s}$ for $s = 2, \dots, n$ and using (3.3) for the tensor \tilde{T} , we obtain the following system of linear equations

$$T_{i_1 \dots i_q} g^{i_1 i_s} = 0 + n \cdot M_{\gamma_{i_2, \dots, i_{\sigma-1}}^\sigma \gamma_{i_{\sigma+1} \dots i_q}^\sigma}^{(1, \sigma)} + \sum_{\sigma=2, \sigma \neq s}^q M_{\gamma_{i_2, \dots, i_{\sigma-1}}^\sigma \gamma_{i_{\sigma+1} \dots i_{s-1}}^\sigma \gamma_{i_{s+1} \dots i_q}^s}^{(1, \sigma)}. \tag{3.4}$$

It follows from the fixed point theorem that such system of equations with variables $M^{(i_1 i_2)}, \dots, M^{(i_1 i_n)}$ has for $n > q - 1$ exactly one solution. \square

Let us recall the raising indices of tensors. If given a tensor T of the type $(0, p + q)$, we may construct a tensor of the type (p, q) by the following:

$$T_{j_1 \dots j_q}^{i_1 \dots i_p} \stackrel{\text{def}}{=} g^{i_1 \alpha_1} g^{i_2 \alpha_2} \dots g^{i_p \alpha_p} T_{\alpha_1 \dots \alpha_p j_1 \dots j_q}. \tag{3.5}$$

Raising indices, we obtain from Theorem 2 the known unique decomposition of the tensor $T_{i_2 \dots i_p}^{i_1}$:

$$T_{i_2 \dots i_p}^{i_1} = \tilde{T}_{i_2 \dots i_p}^{i_1} + \delta_{i_2}^{i_1} M_{i_3 \dots i_p}^{12} + \delta_{i_3}^{i_1} M_{i_2 i_4 \dots i_p}^{13} + \dots + \delta_{i_p}^{i_1} M_{i_2 \dots i_{p-1}}^{1p}, \tag{3.6}$$

where the tensor \tilde{T} is traceless, i. e.,

$$\tilde{T}_{\dots \alpha \dots}^{\alpha} = 0 \tag{3.7}$$

and the tensor \tilde{T} as well as the $(0, p - 1)$ -tensors M are uniquely determined.

As we have mentioned above, the decomposition (3.6) is presented by Weyl in [8], but the uniqueness of tensors M is not contained there.

In view of (3.5), Theorem 1 implies immediately

Theorem 3. *Let $T_{i_1 \dots i_p}^{j_1 \dots j_q}$ be a (p, q) -tensor and*

$$(i_{k_1}, j_{l_1}), (i_{k_2}, j_{l_2}), \dots, (i_{k_s}, j_{l_s}), \quad s \leq pq, \tag{3.8}$$

be pairs of indices satisfying $1 \leq k_\sigma \leq p, 1 \leq l_\sigma \leq q$.

Then there exists a decomposition of $T_{i_1 \dots i_p}$ of the form

$$T_{j_1 \dots j_q}^{i_1 \dots i_p} = \tilde{T}_{j_1 \dots j_q}^{i_1 \dots i_p} + \sum_{\sigma=1}^s \delta_{j_{l_\sigma}}^{i_{k_\sigma}} \cdot M_{j_1, \dots, j_{l_\sigma-1}, j_{l_\sigma+1}, \dots, j_q}^{(k_\sigma) i_1, \dots, i_{k_\sigma-1}, i_{k_\sigma+1}, \dots, i_p}^{(l_\sigma)}, \tag{3.9}$$

where the tensor \tilde{T} satisfies

$$\tilde{T}_{j_1 \dots j_{l_\sigma} \dots j_q}^{i_1 \dots i_{k_\sigma} \dots i_p} \cdot \delta_{i_{k_\sigma}}^{j_{l_\sigma}} = 0, \tag{3.10}$$

for any indices (3.8). The tensor \tilde{T} is uniquely determined and M are certain tensors of the type $(p - 1, q - 1)$.

The condition (3.10) means that the tensor \tilde{T} is traceless over any pair of indices (3.8). This may be expressed by

$$\tilde{T}_{j_1 \dots j_{l_\sigma-1} \alpha j_{l_\sigma+1} \dots j_q}^{i_1 \dots i_{k_\sigma-1} \alpha i_{k_\sigma+1} \dots i_p} = 0. \tag{3.11}$$

Let us remark that Theorem 3 holds on every manifold since the metric tensor (which was used in the proof of Theorem 1) may be constructed in any case.

Immediately, we obtain the following Theorem, which is presented in [3, 6].

Theorem 4. Let $T_{i_1 \dots i_p}^{j_1 \dots j_p}$ be a tensor of the type (p, q) . If $n + 1 \geq p + q$, then there exists the unique decomposition of $T_{i_1 \dots i_p}^{j_1 \dots j_p}$ in the following form.

$$T_{i_1 \dots i_p} = \tilde{T}_{i_1 \dots i_p} + \sum_{t=1}^{\min\{p,q\}} \sum_{\oplus} \delta_{j_{\sigma_1}}^{i_{\rho_1}} \delta_{j_{\sigma_2}}^{i_{\rho_2}} \dots \delta_{j_{\sigma_t}}^{i_{\rho_t}} M^{\star \dots}, \tag{3.12}$$

where

$$\oplus = \left\{ \begin{array}{l} \rho_1, \rho_2, \dots, \rho_t = 1, 2, \dots, p \quad (\rho_1 < \rho_2 < \dots < \rho_t) \\ \sigma_1, \sigma_2, \dots, \sigma_t = 1, 2, \dots, q \quad (\sigma_i \text{ are mutually different}), \end{array} \right. \tag{3.13}$$

$$\star = \left\{ \begin{array}{cccc} \rho_1 & \rho_2 & \dots & \rho_t \\ \sigma_1 & \sigma_2 & \dots & \sigma_t \end{array} \right\}$$

and tensors $\tilde{T}_{i_1 \dots i_p}$ and M are traceless.

4. SPECIAL DIFFERENTIAL OPERATOR L

Now, let us introduce the following concept on tensors of the type (p, q) .

Definition 1. A map $L : T_q^p \mapsto T_q^p$ is said to be a *special differential operator* if for any tensors $\overset{1}{T}, \overset{2}{T}$ the following properties are satisfied:

$$\begin{aligned} L(\overset{1}{T} \pm \overset{2}{T}) &= L\overset{1}{T} \pm L\overset{2}{T}, \\ L(\overset{1}{T} \cdot \overset{2}{T}) &= L\overset{1}{T} \cdot \overset{2}{T} + \overset{1}{T} \cdot L\overset{2}{T}, \\ Lg &= 0; \quad L\delta = 0. \end{aligned} \tag{4.1}$$

By δ we denote the Kronecker tensor.

Let us give some examples of the operator L :

- (a) ∇ — the covariant derivative, i. e., ∇ is an affine connection on M_n
- (b) $\circ R$ — “semisymmetric” operator; $T \circ R$ is in the coordinate description defined by

$$\nabla_l \nabla_m T_{i_1 \dots i_q}^{h_1 \dots h_p} - \nabla_m \nabla_l T_{i_1 \dots i_q}^{h_1 \dots h_p}.$$

Using the Bianci identity, we may write $T \circ R$ in the form

$$-T_{i_1 \dots i_q}^{\alpha \dots h_p} R_{\alpha lm}^{h_1} - \dots - T_{i_1 \dots i_q}^{h_1 \dots \alpha} R_{\alpha lm}^{h_p} + T_{\alpha \dots i_q}^{h_1 \dots h_p} R_{i_1 lm}^{\alpha} + \dots + T_{i_1 \dots \alpha}^{h_1 \dots h_p} R_{i_q lm}^{\alpha}, \tag{4.2}$$

where R_{ijk}^h are components of a Riemannian tensor R on M_n .

- (c) $\circ Z$ — “pseudosymmetric” operator; this operator is defined analogously as the previous one — in the formula (4.2), we replace the tensor R by the tensor Z defined as follows

$$Z_{ijk}^h = R_{ijk}^h - B(\delta_k^h g_{ij} - \delta_j^h g_{ik}),$$

where B is a function on M_n .

5. ON MANIFOLDS WITH $LT = 0$

It is well known that a symmetric manifold in the sense of Cartan is characterized by the condition

$$\nabla R = 0.$$

Projective symmetric and conformal symmetric manifolds are characterized by the conditions

$$\nabla W = 0 \text{ and } \nabla C = 0, \text{ respectively;}$$

by W and C we denote the Weyl tensors of projective and conformal curvature, respectively.

Semisymmetric and pseudosymmetric manifolds are characterized by the conditions

$$R \circ R = 0 \text{ and } R \circ Z = 0, \text{ respectively.}$$

Further, projective and conformal semisymmetric manifolds are characterized by

$$W \circ R = 0 \text{ and } C \circ R = 0, \text{ respectively.} \quad (5.1)$$

Replacing in (5.1) the Riemannian tensor by the Weyl tensor, we obtain the characterization of projective pseudosymmetric and conformal pseudosymmetric manifolds.

It follows from these remarks that it is useful to investigate manifolds with a general condition $LT = 0$.

Applying Definition 1 of the operator L and Theorem 1, we may prove the following lemmas.

Lemma 1. *Let T be a tensor of the type $(0, p)$ with the decomposition (2.2) for the selected pairs of indices (2.1). If $L\tilde{T} = 0$ and $LM^* = 0$, where $* = (k_\sigma, l_\sigma)$, then $LT = 0$.*

Lemma 2. *Let T be a tensor of the type $(0, p)$ with the decomposition (2.2) for the selected pairs of indices (2.1). If $LT = 0$ then $L\tilde{T} = 0$.*

Consider a $C^\infty(M)$ -module of k -differential forms $\Omega^k(M)$ on a manifold M_n . Then for any k -differential form $\alpha \in \Omega^k(M)$, the special differential operator L may be determined by the relation $LT = \alpha T$.

Let a 1-differential form $\omega \in \Omega^1(M)$ with

$$\nabla_X T = \omega(X) \cdot T \quad (5.2)$$

be given (see [1]). Then we speak about *T-recurrency*. The following special cases are well-known:

- ◇ $\nabla R = \omega \cdot R$, the *Walker recurrency* (or simply *recurrency*),
- ◇ $\nabla W = \omega \cdot W$, the *projective recurrency*,
- ◇ $\nabla C = \omega \cdot C$, the *conformal recurrency*,
- ◇ $\nabla \text{Ric} = \omega \cdot \text{Ric}$, the *Ricci recurrency*.

Therefore, it will be useful to deal with the case (5.2).

Considering a 1-differential form ω with (5.2) and choosing indices according to the Theorem 2, we may prove the following theorem.

Theorem 5. *Let T be a $(0, p)$ -tensor decomposed in the sense of (3.2). Then*

$$LT = \omega T, \quad (5.3)$$

if and only if

$$L\tilde{T} = \omega\tilde{T} \quad \text{and} \quad L\overset{1\sigma}{M} = \omega\overset{1\sigma}{M} \quad (5.4)$$

where ω is a k -differential form in $\Omega^k(M)$.

Proof. Consider a decomposition (3.2) of T for pairs of indices (3.1). First, let us suppose

$$L\tilde{T} = \omega\tilde{T} \quad \text{and} \quad L\overset{1\sigma}{M} = \omega\overset{1\sigma}{M}. \quad (5.5)$$

Substituting (5.5) into (3.2), we obtain immediately $LT = \omega T$.

Now, let us suppose $LT = \omega T$. Applying L to the expression (3.2) of T , we have

$$LT_{i_1 \dots i_p} = L\tilde{T}_{i_1 \dots i_p} + g_{i_1 i_2} \overset{12}{L\tilde{M}}_{i_3 \dots i_p} + g_{i_1 i_3} \overset{13}{L\tilde{M}}_{i_2 i_4 \dots i_p} + \dots + g_{i_1 i_p} \overset{1p}{L\tilde{M}}_{i_2 \dots i_{p-1}} \quad (5.6)$$

i. e.

$$\omega T = L\tilde{T}_{i_1 \dots i_p} + g_{i_1 i_2} \overset{12}{\omega\tilde{M}}_{i_3 \dots i_p} + g_{i_1 i_3} \overset{13}{\omega\tilde{M}}_{i_2 i_4 \dots i_p} + \dots + g_{i_1 i_p} \overset{1p}{\omega\tilde{M}}_{i_2 \dots i_{p-1}}. \quad (5.7)$$

Applying the form ω to the expression of T in the form (3.2), we obtain

$$\omega T = L\tilde{T}_{i_1 \dots i_p} + g_{i_1 i_2} \overset{12}{\omega\tilde{M}}_{i_3 \dots i_p} + g_{i_1 i_3} \overset{13}{\omega\tilde{M}}_{i_2 i_4 \dots i_p} + \dots + g_{i_1 i_p} \overset{1p}{\omega\tilde{M}}_{i_2 \dots i_{p-1}}. \quad (5.8)$$

Comparing (5.7) and (5.8), we have

$$\begin{aligned} (L\tilde{T} - \omega\tilde{T}) &+ (g_{i_1 i_2} L\overset{12}{M}_{i_3 \dots i_p} - g_{i_1 i_2} \omega\overset{12}{M}_{i_3 \dots i_p}) \\ &+ (g_{i_1 i_3} L\overset{13}{M}_{i_2 i_4 \dots i_p} - g_{i_1 i_3} \omega\overset{13}{M}_{i_2 i_4 \dots i_p}) + \\ &+ \dots + (g_{i_1 i_p} L\overset{1p}{M}_{i_2 \dots i_{p-1}} - g_{i_1 i_p} \omega\overset{1p}{M}_{i_2 \dots i_{p-1}}) = 0. \end{aligned} \quad (5.9)$$

Contracting (5.9) with $g^{i_1 i_\sigma}$ for $\sigma = 2, \dots, n$, we deduce a system of linear equations as follows

$$n \cdot (L\overset{1\sigma}{M} - \omega\overset{1\sigma}{M}) + \sum_{\tau=2, \tau \neq \sigma}^p (L\overset{1\tau}{M} - \omega\overset{1\tau}{M}) = 0.$$

For $n > p - 1$, such a system has the unique solution $L\overset{1\sigma}{M} - \omega\overset{1\sigma}{M} = 0$ for $\sigma = 2, \dots, n$, which implies that $L\tilde{T} - \omega\tilde{T} = 0$. \square

The following theorem is the consequence of Theorem 5.

Theorem 6. *Let T be a $(0, p)$ -tensor decomposed in the sense of (3.2). Then*

$$LT = 0$$

if and only if $L\tilde{T} = 0$ and $L\overset{1\sigma}{M} = 0$.

REFERENCES

- [1] M. Crasmareanu, "Trace Decomposition and Recurrency." *Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica*, vol. 40, pp. 43–46, 2001.
- [2] M. Jukl, L. Juklová, and J. Mikeš, "The decomposition of tensor spaces with quaternionic structure," in *APLIMAT 2007*, Bratislava, 2007, pp. 217–222.
- [3] D. Krupka, "The trace decomposition of tensor spaces," *Linear and Multilinear Algebra*, vol. 54, no. 4, pp. 235–263, 2006.
- [4] L. Lakomá and M. Jukl, "The decomposition of tensor spaces with almost complex structure," *Rendiconti del circolo matematico di Palermo, Ser. II, Suppl.*, vol. 72, pp. 145–150, 2004.
- [5] L. Lakomá and J. Mikeš, "On the special trace decomposition problem on quaternionic structure," in *Proc. of the Third Internat. Workshop on Diff. Geom. and its Appl.; The First German-Romanian Seminar on Geom., Sibiu, Romania*, 1997, pp. 225–229.
- [6] J. Mikeš, "On general trace decomposition problem," in *Diff. Geom. and its Appl., Proc. of 6th Int. Conf., Brno, Czech Republic*, 1995, pp. 45–50.
- [7] V. V. Vinshnevskij, A. P. Shirokov, and V. V. Shurygin, *Spaces over algebras. (Prostranstva nad algebrami)*. Kazan: Izd. Kazan. Univ., 1985.
- [8] H. Weyl, *The classical groups*. Princeton: Princeton Univ. Press., 1946, fifteenth printing (1997).
- [9] K. Yano, *Differential geometry on complex and almost complex spaces*. Pergamon press, 1965.

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