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GEOMETRY OF ALMOST CLIFFORDIAN MANIFOLDS: NIJENHUIS TENSOR

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Abstract. We generalize some classical results on Nijenhuis tensor for an almost Cliffordian manifold based on arbitrary Clifford algebra and suggest its relations with the integrability of the corresponding G -structure. We prove the set of properties for Nijenhuis tensors with respect to arbitrary Clifford algebra.

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1. ALMOST CLIFFORDIAN MANIFOLDS

Let $\mathcal{O} = \mathcal{C}\ell(s, t)$ be a Clifford algebra. If M is an km -dimensional manifold, where $k = 2^{s+t}$ and $m \in \mathbb{N}$, then an almost Clifford manifold is given by a reduction of the structure group $\mathrm{GL}(km, \mathbb{R})$ of the principal frame bundle of M to

$$\mathrm{GL}(m, \mathcal{O}) = \{A \in \mathrm{GL}(km, \mathbb{R}) \mid AI = IA, I \in \mathcal{O}\},$$

where \mathcal{O} is arbitrary Clifford algebra. In other words, an almost Clifford manifold is a smooth manifold equipped by the set of anti commuting and commuting affinors $I_i, i = 1, \dots, t, I_i^2 = -E$ and $J_j, j = 1, \dots, s, J_j^2 = E$ such that the free associative unitary algebra generated by $\langle I_i, J_j, E \rangle$ is isomorphically equivalent to \mathcal{O} . In particular, on the elements of this reduced bundle, one can define affinors in the form F_1, \dots, F_k globally.

Definition 1. Let M be a smooth manifold such that $\dim(M) = m$. Let A be a smooth ℓ -dimensional ($\ell < m$) vector subbundle in $T^*M \otimes TM$ such that the identity affiner $E = \mathrm{id}_{TM}$ restricted to $T_x M$ belongs to $A_x M \subset T_x^* M \otimes T_x M$ at each point $x \in M$. We say that M is equipped with an ℓ -dimensional A -structure.

It is easy to see that an almost Clifford structure is not an A -structure because the affinors in the form $F_0, \dots, F_\ell \in A$ have to be defined only locally.

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Definition 2. The A -structure, where A is a Clifford algebra \mathcal{O} , is called an almost Cliffordian manifold.

In particular, the almost Clifford and almost Cliffordian structures are G -structures based on Clifford algebras. Two most important examples are an almost hypercomplex geometry and an almost quaternionic geometry, which are based on Clifford algebra $\mathcal{C}\ell(0, 2)$. Note that the geometric property of an almost hypercomplex structure reads that there is no nontrivial G -invariant subspace \mathcal{D} in $\mathbb{V} \otimes \wedge^2 \mathbb{V}^*$ because the first prolongation $\mathfrak{g}^{(1)}$ of the Lie algebra \mathfrak{g} vanishes. For an almost quaternionic structure, the situation is more complicated, because $\mathfrak{g}^{(1)} = \mathbb{V}^*$ and there is a class of these structures indexed by \mathbb{V} , see [1]. Note that for Cliffordian structures based on $\mathcal{C}\ell(0, 3)$ is $\mathfrak{g}^{(1)} = \mathbb{V}^*$ and there is a class of these structures indexed by \mathbb{V} too, see [2].

2. NIJENHUIS TENSOR

The Nijenhuis tensor plays an important role in the theory of integrability. As a classical concept, Nijenhuis introduced $N_J \in \wedge^2 T^*M \otimes TM$ of an almost complex structure $J \in T^*M \otimes TM$. This tensor is an obstruction for an almost complex structure which distinguished it from the complex structure, i. e. their integrability. Recall that Nijenhuis tensor $N(P, Q) \in \wedge^2 T^*M \otimes TM$ for a pair of tensors $P, Q \in T^*M \otimes TM$ is given by the expression

$$N(P, Q)(X, Y) = [PX, QY] - P[QX, Y] - Q[X, PY] + [QX, PY] \\ - Q[PX, Y] - P[X, QY] + (PQ + QP)[X, Y].$$

An almost quaternionic manifold M is integrable if and only if the Nijenhuis tensors $N(I, I)$ and $N(J, J)$ vanish, where $I, J, IJ \in T^*M \otimes TM$ is a quaternionic structure. Let us finally note that in [3], the author proved a similar fact for Clifford algebra $\mathcal{C}\ell(0, 3)$.

If $P = Q$, then, by straightforward computing,

$$N(P, P)(X, Y) = 2([PX, PY] - P[PX, Y] - P[X, PY] + P^2[X, Y])$$

and if $P = E$, where E is an identity, then

$$N(E, Q)(X, Y) = 0.$$

Following [6, 7], we reformulate Nijenhuis tensor calculus with respect to arbitrary Clifford algebra. In those papers, a set of classical results for $\mathcal{C}\ell(0, 2)$ and $\mathcal{C}\ell(2, 0)$ was proved. In fact, most of the arguments work generally, but we have to formulate and prove them for any Clifford algebra. First of all, following papers [6, 7], we recall the operation $\bar{\wedge}$

$$(S \bar{\wedge} P)(X, Y) := S(PX, Y) + S(X, PY) \quad (1)$$

and

$$(P \bar{\wedge} S)(X, Y) := PS(X, Y) \tag{2}$$

where $S \in \otimes^2 \mathbb{V} \otimes \mathbb{V}^*$ and $N \in \mathbb{V} \otimes \mathbb{V}^*$. Now, one can easily check the following identities

$$N(L, QP) + N(Q, LP) = N(L, QP) \bar{\wedge} P + L \bar{\wedge} N(Q, P) + Q \bar{\wedge} N(L, P) \tag{3}$$

$$(S \bar{\wedge} Q) \bar{\wedge} P - (S \bar{\wedge} P) \bar{\wedge} Q = S \bar{\wedge} QP - S \bar{\wedge} PQ \tag{4}$$

$$(L \bar{\wedge} S) \bar{\wedge} P = L \bar{\wedge} (S \bar{\wedge} P) \tag{5}$$

Lemma 1. *Let $\mathcal{O} = \mathcal{C}\ell(s, t)$ be a Clifford algebra. If $F, G \in \mathcal{O}$ such that $F \neq G$, then the following identities hold:*

$$N(FG, F) = -\frac{1}{2}N(F, G) \bar{\wedge} F - \frac{1}{2}G \bar{\wedge} N(F, F) + \frac{1}{4}N(F, F) \bar{\wedge} G \tag{6}$$

$$0 = N(F, G) \bar{\wedge} F + 2F \bar{\wedge} N(F, G) + G \bar{\wedge} N(F, F) + \frac{1}{2}N(F, F) \bar{\wedge} G \tag{7}$$

$$N(G, H) = N(F, G) \bar{\wedge} G + G \bar{\wedge} N(F, G) + F \bar{\wedge} N(G, G). \tag{8}$$

Proof. Putting $L = Q = F, P = G$ in (3), we find

$$N(F, FG) + N(F, FG) = N(F, F) \bar{\wedge} G + F \bar{\wedge} N(F, G) + F \bar{\wedge} N(F, G),$$

that is,

$$N(FG, F) = \frac{1}{2}N(F, F) \bar{\wedge} G + F \bar{\wedge} N(F, G). \tag{9}$$

Putting $L = G, P = Q = F$ in (3), we find

$$N(G, F^2) + N(F, GF) = N(G, F) \bar{\wedge} F + G \bar{\wedge} N(F, F) + F \bar{\wedge} N(G, F),$$

that is,

$$N(F, GF) = N(G, F) \bar{\wedge} F + G \bar{\wedge} N(F, F) + F \bar{\wedge} N(G, F). \tag{10}$$

Adding (9) and (10) and dividing the sum by 2, we find

$$N(FG, F) = -\frac{1}{2}N(F, G) \bar{\wedge} F - \frac{1}{2}G \bar{\wedge} N(F, F) + \frac{1}{4}N(F, F) \bar{\wedge} G$$

and subtracting (10) from (9), we find

$$N(F, G) \bar{\wedge} F + 2F \bar{\wedge} N(F, G) + G \bar{\wedge} N(F, F) + \frac{1}{2}N(F, F) \bar{\wedge} G = 0 \tag{11}$$

Finally, putting $L = P = G, Q = F$ in (3), we find

$$N(F, G^2) + N(G, FG) = N(F, G) \bar{\wedge} G + F \bar{\wedge} N(G, G) + G \bar{\wedge} N(F, G),$$

that is,

$$N(G, H) = N(F, G) \bar{\wedge} G + G \bar{\wedge} N(F, G) + F \bar{\wedge} N(G, G).$$

□

Lemma 2. Let $\mathcal{O} = \mathcal{C}\ell(s, t)$ be a Clifford algebra. If $F, G \in \mathcal{O}$ such that $F \neq G$, then the following identity holds:

$$\begin{aligned} \epsilon_1 N(F, F) - \epsilon_2 N(G, G) + N(H, F)\bar{\wedge}G + G\bar{\wedge}N(H, F) \\ - N(G, H)\bar{\wedge}F + F\bar{\wedge}N(G, H) + 2H\bar{\wedge}N(F, G) = 0, \end{aligned} \quad (12)$$

where $H = FG$ and $\epsilon_i = 1$ for $K^2 = -1$ and $\epsilon_i = -1$ for $K^2 = 1$.

Proof. Putting $L = FG$, $Q = F$, $P = G$ in (3), we find

$$\begin{aligned} N(FG, FG) - N(F, FGG) = N(FG, F)\bar{\wedge}G + FG\bar{\wedge}N(F, G) \\ + F\bar{\wedge}N(FG, G), \end{aligned}$$

that is,

$$\begin{aligned} N(H, H) = \epsilon N(F, F) + N(FG, F)\bar{\wedge}G + FG\bar{\wedge}N(F, G) \\ + F\bar{\wedge}N(FG, G), \end{aligned} \quad (13)$$

where $\epsilon = 1$ for $G^2 = -1$ and $\epsilon = -1$ for $G^2 = 1$.

Putting $L = FG$, $Q = G$, $P = F$ in (3), we find

$$\begin{aligned} N(FG, GF) - N(G, FGF) = N(FG, G)\bar{\wedge}F + FG\bar{\wedge}N(G, F) \\ + G\bar{\wedge}N(FG, F), \end{aligned}$$

that is,

$$N(H, H) = \epsilon N(G, G) - N(G, H)\bar{\wedge}F - H\bar{\wedge}N(F, G) - G\bar{\wedge}N(H, F), \quad (14)$$

where $\epsilon = 1$ for $G^2 = -1$ and $\epsilon = -1$ for $G^2 = 1$. Thus, from (13) and (14), we find

$$\begin{aligned} N(H, H) = \frac{1}{2} \{ \epsilon N(F, F) + \epsilon N(G, G) + N(H, F)\bar{\wedge}G \\ + G\bar{\wedge}N(H, F) - N(G, H)\bar{\wedge}F + F\bar{\wedge}N(G, H) + 2H\bar{\wedge}N(F, G) \} = 0 \end{aligned}$$

and

$$\begin{aligned} \epsilon N(F, F) - \epsilon N(G, G) + N(H, F)\bar{\wedge}G \\ + G\bar{\wedge}N(H, F) - N(G, H)\bar{\wedge}F + F\bar{\wedge}N(G, H) + 2H\bar{\wedge}N(F, G) = 0, \end{aligned}$$

which completes the proof. □

Lemma 3. Let $\mathcal{O} = \mathcal{C}\ell(s, t)$ be a Clifford algebra. If $F \in \mathcal{O}$, then the following identity holds:

$$N(F, F)\bar{\wedge}F = -2F\bar{\wedge}N(F, F).$$

Proof. One can easily check that putting $L = P = Q = F$ gives $N(F, F^2) + N(F, F^2) = N(F, F)\bar{\wedge}F + F\bar{\wedge}N(F, F) + F\bar{\wedge}N(F, F)$ □

Theorem 1. *Let \mathcal{O} be a Clifford algebra $\mathcal{C}\ell(s, t)$ and let $F, G \in \mathcal{O}$ such that $F \neq G$. If the Nijenhuis tensors $N(F, F)$ and $N(G, G)$ vanish, then $N(FG, FG)$ vanishes.*

Proof. Since $N(F, F) = 0$, we have from (9) $N(H, F) = F \bar{\wedge} N(F, G)$, and from (11),

$$N(F, G) \bar{\wedge} F = -2F \bar{\wedge} N(F, G). \quad (15)$$

Since $N(G, G) = 0$, we have from (9), where we changed F and G ,

$$N(F, G) \bar{\wedge} G = -2G \bar{\wedge} N(F, G) \quad (16)$$

and from (11), where we changed F and G ,

$$N(F, G) \bar{\wedge} G = -2G \bar{\wedge} N(F, G). \quad (17)$$

Now, if we substitute $N(F, F) = 0$, $N(G, G) = 0$, and (16) into (12), then the part containing ϵ_i vanishes and the proof is correct for any Clifford algebra, i. e., we find

$$\begin{aligned} & (F \bar{\wedge} N(F, G)) \bar{\wedge} G + G \bar{\wedge} (F \bar{\wedge} N(F, G)) \\ & - (G \bar{\wedge} N(F, G)) \bar{\wedge} F - F \bar{\wedge} (G \bar{\wedge} N(F, G)) + 2H \bar{\wedge} N(F, G) = 0, \end{aligned}$$

from which

$$(F \bar{\wedge} N(F, G)) \bar{\wedge} G - (G \bar{\wedge} N(F, G)) \bar{\wedge} F = 0, \quad (18)$$

since

$$G \bar{\wedge} (F \bar{\wedge} N(F, G)) = -F \bar{\wedge} (G \bar{\wedge} N(F, G)) = -H \bar{\wedge} N(F, G)$$

by virtue of $GF = -FG = -H$. Now, using (5), (15) and (17), we find, from (18)

$$\begin{aligned} F \bar{\wedge} (N(F, G) \bar{\wedge} G) - G \bar{\wedge} (N(F, G) \bar{\wedge} F) &= 0, \\ -2F \bar{\wedge} (G \bar{\wedge} N(F, G)) + 2G \bar{\wedge} (F \bar{\wedge} N(F, G)) &= 0, \\ -4FG \bar{\wedge} N(F, G) &= 0, \end{aligned}$$

that is,

$$H \bar{\wedge} N(F, G) = 0. \quad (19)$$

Since $H^2 = -1$, we have from (19) $N(F, G) = 0$. \square

Corollary 1. *Let \mathcal{O} be a Clifford algebra $\mathcal{C}\ell(s, t)$. If the Nijenhuis tensors $N(I_i, I_i)$ vanish, where I_i are the algebra generators of \mathcal{O} , then*

$$N(F_i, F_j) = 0,$$

where F_i are vector space generators.

3. CLASSES OF SUBORDINATED CONNECTIONS

Recall the concept of A -planar curves on A -structures equipped with the linear connection ∇ . For any tangent vector $X \in T_x M$, we shall write $A_x(X)$ for the vector subspace

$$A_x(X) = \{F_i(X) | F_i \in A_x M\} \subset T_x M$$

and call it the A -hull of the vector X . Similarly, A -hull of a vector field is a subbundle in TM obtained pointwise. Let M be a smooth manifold equipped with an A -structure and a linear connection ∇ . A smooth curve $c : \mathbb{R} \rightarrow M$ is said to be A -planar if

$$\nabla_{\dot{c}} \dot{c} \in A(\dot{c}).$$

In [5], the authors proved a set of facts about the class of \mathcal{D} -connections. The theorems below, about Cliffordian structures, are proved in paper [5] and some examples of this concept can be found in papers [2,4]. The theorems about \mathcal{D} -connections can be found in [1].

Following [4,5], we have a set of results on Clifford and Cliffordian manifolds.

Corollary 2. *Let M be a smooth manifold equipped with a G -structure, where $G = \text{GL}(n, \mathcal{O})$, $\mathcal{O} = \mathcal{C}\ell(s, t)$, $s + t > 1$, i. e. an almost Clifford manifold. Then the G -structure is of type 1 and there exists a unique \mathcal{D} -connection.*

One can see that an almost Cliffordian manifold M is given as a G -structure provided that there is a reduction of the structure group of the principal frame bundle of M to $G := \text{GL}(m, \mathcal{O}) \text{GL}(1, \mathcal{O}) = \text{GL}(m, \mathcal{O}) \times \text{GL}(1, \mathcal{O})$, the action of G on $T_x M$ looks like QXq , where $Q \in \text{GL}(m, \mathcal{O})$, $q \in \text{GL}(1, \mathcal{O})$, where the right action of $\text{GL}(1, \mathcal{O})$ is blockwise. In this case the tensor fields in the form F_1, \dots, F_k can be defined only locally. It is easy to see that the Lie algebra $\mathfrak{gl}(m, \mathcal{O})$ of a Lie group $\text{GL}(m, \mathcal{O})$ is of the form

$$\mathfrak{gl}(m, \mathcal{O}) = \{A \in \mathfrak{gl}(km, \mathbb{R}) | AI_i = I_i A, AJ_j = J_j A\}$$

and the Lie algebra \mathfrak{g} of a Lie group $\text{GL}(m, \mathcal{O}) \text{GL}(1, \mathcal{O})$ is of the form $\mathfrak{g} = \mathfrak{gl}(m, \mathcal{O}) \oplus \mathfrak{gl}(1, \mathcal{O})$.

Let us note that the cases of $\mathcal{C}\ell(s, t)$, where $s + t = 2$, were studied in [6,7] and the case of $\mathcal{C}\ell(0, 3)$ was studied in a detailed way in [2].

Corollary 3. *Let M be an almost Cliffordian manifold based on Clifford algebra $\mathcal{O} = \mathcal{C}\ell(s, t)$, where $\dim(M) \geq 2(s + t)$, i. e., a smooth manifold equipped with a G -structure, where $G = \text{GL}(n, \mathcal{O}) \text{GL}(1, \mathcal{O})$ or equivalently an A -structure, where $A = \mathcal{O}$. Then the class of \mathcal{D} -connections preserves A and shares the same A -planar curves, which are isomorphic to $(\mathbb{R}^{km})^*$.*

4. CONCLUSION

From the classical theory, the Nijenhuis tensor is a part of a torsion of any almost complex connection ∇ and is J -antilinear in each argument ,

$$N_J(X, Y) = T_{\nabla}(X, Y) + JT_{\nabla}(JX, Y) + JT_{\nabla}(X, JY) - T_{\nabla}(JX, JY)$$

and the connection ∇ called minimal such that $N_J = 4T_{\nabla}$, i.e. the structure is integrable if and only if N_J vanishes. Let M be an almost quaternionic manifold or an almost quaternionic manifold of the second kind (paraquaternionic). In [6] and [7], the authors proved that the structure is integrable if and only if the structure tensor $T_Q = N(I, I) + N(J, J) + N(K, K)$ vanishes. In this case, there is the class of \mathcal{D} -connections without torsion.

A similar fact was proved for an almost Cliffordian manifold, where $\mathcal{C}\ell(0, 3)$, see [3]. In this article, the authors proved that the structure tensor which is locally generated by F_i is given by

$$T_Q = \sum_{i=1}^6 N(F_i, F_i) + \sum_{i=1}^6 N(F_i, F_i)\partial(\alpha_a \otimes F_a),$$

where ∂ denotes Spencer’s operator of alternation. The following step is to find a description of the structure preserving connection based on the Nijenhuis tensor for any Cliffordian manifold.

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