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THE THIRD TYPE BUNCH OF CONNECTIONS INDUCED BY AN ANALOG OF NORDEN'S NORMALIZATION FOR THE GRASSMANN-LIKE MANIFOLD OF CENTERED PLANES

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Abstract. The Grassmann-like manifold of centered m-planes (i.e. passing through a fixed point) and the principal bundle over it are considered in n-dimensional projective space. An analogy of Norden's construction of strong normalization induces bunches of connections of three types in the fibering associated to the Grassmann-like manifold which are related to each other. The third type bunch of group connections is constructed and a unique connection from this bunch is distinguished.

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In the n-dimensional projective space P_n, we consider the moving frame $\{A, A_I\}$ $(I, \ldots = \overline{1,n})$ with the derivation formulas

$$\begin{aligned} dA &= \theta A + \omega^I A_I, \\ dA_I &= \theta A_I + \omega_I^J A_J + \omega_I A, \end{aligned}$$

where the form θ plays the role of a proportionality factor and the structure forms ω^I, ω_J^I, ω_I of the projective group GP(n), which acts effectively on P_n, satisfy the Cartan equations

$$\begin{aligned} D\omega^I &= \omega^J \wedge \omega_J^I, \\ D\omega_J^I &= \omega_J^K \wedge \omega_K^I + \delta_J^I \omega_K \wedge \omega^K + \omega_J \wedge \omega^I, \\ D\omega_I &= \omega_I^J \wedge \omega_J. \end{aligned}$$

We consider the Grassmann-like manifold $\mathrm{Gr}^*(m,n)$ [3] of centered m-dimensional planes. This manifold is given by the equations

$$\omega^a = \Lambda_\alpha^a \omega^\alpha + \Lambda_\alpha^{ab} \omega_b^\alpha \quad (a, \ldots = \overline{1,m}; \quad \alpha, \ldots = \overline{m+1,n}),$$

where the components of the first order fundamental object $\Lambda = \{\Lambda^a_\alpha, \Lambda^{ab}_\alpha\}$ satisfy the differential equations

$$\Delta\Lambda^a_\alpha + \Lambda^{ab}_\alpha\omega_b + \omega^a_\alpha = \Lambda^a_{\alpha\beta}\omega^\beta + \Lambda^{ab}_{\alpha\beta}\omega^\beta_b,$$
$$\Delta\Lambda^{ab}_\alpha = \bar{\Lambda}^{ab}_{\alpha\beta}\omega^\beta + \Lambda^{abc}_{\alpha\beta}\omega^\beta_c.$$

The principal fiber bundle $G^*(\mathrm{Gr}^*(m,n))$ with the structure equations

$$D\omega^\alpha = \omega^\beta\wedge\omega^\alpha_\beta + \Lambda^a_\beta\omega^\beta\wedge\omega^\alpha_a + \Lambda^{ab}_\beta\omega^\beta_b\wedge\omega^\alpha_a,$$
$$D\omega^\alpha_a = \omega^b_a\wedge\omega^\alpha_b + \omega^\beta_a\wedge\omega^\alpha_\beta + \omega_a\wedge\omega^\alpha;$$
$$D\omega^a_b = \omega^c_b\wedge\omega^a_c + (\delta^a_b\Lambda^c_\alpha\omega_c + \delta^a_b\omega_\alpha + \Lambda^a_\alpha\omega_b)\wedge\omega^\alpha$$
$$+ (\delta^a_b\Lambda^{ec}_\alpha\omega_e - \delta^c_b\omega^a_\alpha + \Lambda^{ac}_\alpha\omega_b)\wedge\omega^\alpha_c,$$
$$D\omega^\alpha_\beta = \omega^\gamma_\beta\wedge\omega^\alpha_\gamma + (\delta^\alpha_\beta\Lambda^a_\gamma\omega_a + \delta^\alpha_\beta\omega_\gamma + \delta^\alpha_\gamma\omega_\beta)\wedge\omega^\gamma + (\delta^\alpha_\beta\Lambda^{ba}_\gamma\omega_b + \delta^\alpha_\gamma\omega^a_\beta)\wedge\omega^\gamma_a,$$
$$D\omega^a_\alpha = \omega^b_\alpha\wedge\omega^a_b + \omega^\beta_\alpha\wedge\omega^a_\beta + (\Lambda^a_\beta\omega_\alpha)\wedge\omega^\beta + (\Lambda^{ab}_\beta\omega_\alpha)\wedge\omega^\beta_b,$$
$$D\omega_a = \omega^b_a\wedge\omega_b + \omega^\alpha_a\wedge\omega_\alpha,$$
$$D\omega_\alpha = \omega^a_\alpha\wedge\omega_a + \omega^\beta_\alpha\wedge\omega_\beta$$

is constructed over the manifold $\mathrm{Gr}^*(m,n)$, the stationarity subgroup G^* of the centered plane L^*_m is the typical fiber. In the principal fiber bundle the connection is given in G. F. Laptev's [3] method by the field of the connection object Γ. This object contains the subobject

$$\Gamma_1 = \{\Gamma^a_{b\alpha}, L^{ac}_{b\alpha}, \Gamma^\alpha_{\beta\gamma}, L^{\alpha a}_{\beta\gamma}\}.$$

An analog of strong Norden's normalization for this manifold is carried out. It consists of the fields of the planes C_{n-m-1} and N_{m-1}:

$$L^*_m \cap C_{n-m-1} = \varnothing, \quad A \notin N_{m-1} \subset L^*_m.$$

The planes C_{n-m-1} and N_{m-1} will be set by the points $B_\alpha = A_\alpha + \lambda^a_\alpha A_a + \lambda_\alpha A$ and $B_a = A_a + \lambda_a A$, correspondingly.

The differential equations for the components of the clothing quasitensor $\lambda = \{\lambda^a_\alpha, \lambda_\alpha, \lambda_a\}$ have the form

$$\Delta\lambda^a_\alpha + \omega^a_\alpha = \lambda^a_{\alpha\beta}\omega^\beta + \lambda^{ab}_{\alpha\beta}\omega^\beta_b,$$
$$\Delta\lambda_\alpha + \lambda^a_\alpha\omega_a + \omega_\alpha = \lambda_{\alpha\beta}\omega^\beta + \chi^a_{\alpha\beta}\omega^\beta_a,$$
$$\Delta\lambda_a + \omega_a = \lambda_{a\alpha}\omega^\alpha + \lambda^b_{a\alpha}\omega^\alpha_b.$$

This normalization induces the connections of three types [1, 3] $\overset{01}{\Gamma}$, $\overset{02}{\Gamma}$, $\overset{03}{\Gamma}$ in the fibering associated with the manifold $\mathrm{Gr}^*(m,n)$. Moreover,

$$\overset{01}{\Gamma} = \frac{1}{2}\Big(\overset{02}{\Gamma} + \overset{03}{\Gamma}\Big).$$

Thus the first type connection is average [4] with respect to the connections of the remaining types.

In [2] it is proved that this clothing of the manifold $\mathrm{Gr}^*(m,n)$ induces the first type bunch of connections in the fibering $G^*(\mathrm{Gr}^*(m,n))$. From this bunch a unique first type connection is distinguished. In [1], because the covariant derivatives are tensors, we construct the second type bunch of connections. A unique second type connection is distinguished from this bunch.

These bunches of connections [1, 2] have identical parameters. These parameters are the components of the subobject Γ_1. In consideration of the correlation between the group connections of three types, we can construct the third type bunch of connections with the object

$$\overset{3}{\Gamma} = 2\overset{1}{\Gamma} - \overset{2}{\Gamma}.$$

The dependent components of this bunch are

$$\overset{3}{\Gamma}{}^a_{\alpha\beta} = \lambda^a_\gamma \Gamma^\gamma_{\alpha\beta} - \lambda^b_\alpha \Gamma^a_{b\beta} - \lambda^a_{\alpha\beta} + 2\lambda_\alpha \mu^a_\beta,$$

$$\overset{3}{L}{}^{ab}_{\alpha\beta} = \lambda^a_\gamma L^{\gamma b}_{\alpha\beta} - \lambda^c_\alpha L^{ab}_{c\beta} - \lambda^{ab}_{\alpha\beta} + 2\lambda^b_\alpha \lambda^a_\beta - 2\lambda_\alpha \Lambda^{ab}_\beta,$$

$$\overset{3}{L}_{a\alpha} = \lambda_b \Gamma^b_{a\alpha} + 2\lambda_a \lambda_b \Lambda^b_\alpha + 2\lambda_a \eta_\alpha - \lambda_{a\alpha},$$

$$\overset{3}{\Pi}{}^b_{a\alpha} = \lambda_c L^{cb}_{a\alpha} + 2\lambda_a \lambda_c \Lambda^{cb}_\alpha + 2\delta^b_a \eta_\alpha - \lambda^b_{a\alpha},$$

$$\overset{3}{L}_{\alpha\beta} = \lambda_\gamma \Gamma^\gamma_{\alpha\beta} - \lambda^a_\alpha \lambda_b \Gamma^b_{a\beta} + 2\lambda_\alpha \lambda_\beta - \lambda_{\alpha\beta} + \lambda^a_\alpha \lambda_{a\beta} - 2\lambda^a_\alpha \lambda_a \eta_\beta - 2\lambda^a_\alpha \Lambda^b_\beta \lambda_a \lambda_b,$$

$$\overset{3}{\Pi}{}^a_{\alpha\beta} = \lambda_\gamma L^{\gamma a}_{\alpha\beta} - \lambda^b_\alpha \lambda_c L^{ca}_{b\beta} - 2\lambda^b_\alpha \lambda_b \lambda_c \Lambda^{ca}_\beta + 2\lambda^a_\alpha \lambda_b \lambda^b_\beta - \chi^a_{\alpha\beta} + \lambda^b_\alpha \lambda^a_{b\beta},$$

where

$$\mu^a_\alpha = \lambda^a_\alpha - \Lambda^a_\alpha,$$
$$\eta_\alpha = \lambda_\alpha - \lambda^a_\alpha \lambda_a.$$

Taking into account the covered subobject $\overset{0}{\Gamma}_1$ [3]

$$\overset{0}{\Gamma}{}^a_{b\alpha} = -\delta^a_b \lambda_\alpha + \mu^a_\alpha \lambda_b + \delta^a_b \mu^c_\alpha \lambda_c, \qquad \overset{0}{L}{}^{ac}_{b\alpha} = \delta^c_b \lambda^a_\alpha - (\delta^a_b \Lambda^{ec}_\alpha + \delta^e_b \Lambda^{ac}_\alpha)\lambda_e,$$

$$\overset{0}{\Gamma}{}^\alpha_{\beta\gamma} = -\delta^\alpha_\gamma \lambda_\beta - \delta^\alpha_\beta \lambda_\gamma + \delta^\alpha_\beta \mu^a_\gamma \lambda_a, \qquad \overset{0}{L}{}^{\alpha a}_{\beta\gamma} = -\delta^\alpha_\gamma \lambda^a_\beta - \delta^\alpha_\beta \Lambda^{ba}_\gamma \lambda_b,$$

we have a unique connection in the third type bunch of connections:

$$\begin{aligned}
\overset{03}{\Gamma}{}^a_{\alpha\beta} &= -\lambda^a_{\alpha\beta} - \mu^a_\beta \lambda^b_\alpha \lambda_b + \mu^a_\beta \lambda_\alpha - \Lambda^a_\beta \lambda_\alpha, \\
\overset{03}{L}{}^{ab}_{\alpha\beta} &= -\lambda^{ab}_{\alpha\beta} + \Lambda^{ab}_\beta \lambda^c_\alpha \lambda_c - 2\Lambda^{ab}_\beta \lambda_\alpha, \\
\overset{03}{L}_{a\alpha} &= -\lambda_{a\alpha} + \lambda_a \lambda_\alpha, \\
\overset{03}{\Pi}{}^b_{a\alpha} &= -\lambda^b_{a\alpha} - \delta^b_a \lambda_c \lambda^c_\alpha + 2\delta^b_a \lambda_\alpha, \\
\overset{03}{L}_{\alpha\beta} &= -\lambda_{\alpha\beta} + \lambda^a_\alpha \lambda_{a\beta} + \mu^a_\beta \lambda_a \lambda_\alpha - \lambda^a_\alpha \lambda_a \lambda_\beta, \\
\overset{03}{\Pi}{}^a_{\alpha\beta} &= -\chi^a_{\alpha\beta} - \Lambda^{ba}_\beta \lambda_b \lambda_\alpha + \lambda^a_{b\beta} \lambda^b_\alpha - \lambda_\beta \lambda^a_\alpha + \lambda^a_\alpha \lambda^b_\beta \lambda_b.
\end{aligned}$$

Theorem. *The analog of the strong Norden's normalization induces* $(n-m)(m+1)(m^2+(n-m)^2)$*-parameter third type bunch of group connections*

$$\overset{3}{\Gamma} = \left\{ \Gamma_1, \overset{3}{\Gamma}{}^a_{\alpha\beta}, \overset{3}{L}{}^{ab}_{\alpha\beta}, \overset{3}{L}_{a\alpha}, \overset{3}{\Pi}{}^b_{a\alpha}, \overset{3}{L}_{\alpha\beta}, \overset{3}{\Pi}{}^a_{\alpha\beta} \right\}$$

in the fibering $G^*(\mathrm{Gr}^*(m,n))$*. The unique third type connection*

$$\overset{03}{\Gamma} = \left\{ \overset{0}{\Gamma}{}^a_{b\alpha}, \overset{0}{L}{}^{ab}_{c\alpha}, \overset{0}{\Gamma}{}^\alpha_{\beta\gamma}, \overset{0}{L}{}^{\alpha a}_{\beta\gamma}, \overset{03}{\Gamma}{}^a_{\alpha\beta}, \overset{03}{L}{}^{ab}_{\alpha\beta}, \overset{03}{L}_{a\alpha}, \overset{03}{\Pi}{}^b_{a\alpha}, \overset{03}{L}_{\alpha\beta}, \overset{03}{\Pi}{}^a_{\alpha\beta} \right\}$$

is distinguished from this bunch.

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