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Jiří Tomáš

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JIŘÍ TOMÁŠ

Abstract. Let M be an m-dimensional manifold, P^rM the frame bundle, $A = \mathbb{D}^r_k/I = \mathbb{R} \oplus N_A$ a Weil algebra and $p : G^r_k \to \operatorname{Aut} A$ a Lie group homomorphism. For a Lie subgroup $G^r_{p,k} \subseteq G^r_k$ and for $m \geq k$, we define the concept of a (p, A)-covelocity extending restrictions $T^A_x f_{|\operatorname{Orb}(G^r_{p,k}, j^A\varphi_x)}$ of A-covelocities to $G^r_{p,k}$-orbits with respect to the left action $\ell : G^r_m \times N_A \to N_A$ determined by p. Further, we introduce bundle functors J^A_p of (p, A)-jets defined on $\mathcal{M}f_m \times \mathcal{M}f$ and give their geometrical description.

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1. Preliminaries

We give a contribution concerning the geometry of Weil functors and their morphisms. The starting points are the concepts of an r-jet, a jet space $J^r(M, N)$, and a bundle functor, all defined in [9]. As usual, we denote by M an m-dimensional manifold and by N a smooth manifold. All manifolds are considered to be C^∞-manifolds. We denote by $\mathcal{M}f_m$ the category of m-dimensional manifolds with local diffeomorphisms, by $\mathcal{M}f$ the category of smooth manifolds and smooth maps and by $\mathcal{FM}$ the category of fibered manifolds with smooth fibered maps. Analogously, we denote by $\mathcal{FM}_m$ the category of fibered manifolds with m-dimensional bases together with fibered maps covering local diffeomorphisms. Recall that a bundle functor defined on the category $\mathcal{M}f_m$ is said to be a natural bundle [9]. Further, we recall the jet functor defined on the category $\mathcal{M}f_m \times \mathcal{M}f$ which assigns the space of r-jets $J^r(M, N)$ of smooth maps $M \to N$ to any pair $(M, N) \in \operatorname{Obj}(\mathcal{M}f_m \times \mathcal{M}f)$ and the map $J^r(g, h) : J^r(M_1, N_1) \to J^r(M_2, N_2)$ defined by $j^r_x f \mapsto j^r_{f(x)}h \circ j^r_x f \circ (j^r_x g)^{-1}$ to any pair $(g, h) \in \operatorname{Morph}(\mathcal{M}f_m \times \mathcal{M}f)$.

Among bundle functors, we recall the product preserving bundle functors defined on the category $\mathcal{M}f$. The classical result by Kainz, Michor and others [6, 7, 9] reads that such functors coincide with Weil functors. Denoting them by T^A, we involve the associated Weil algebras to their notation. The restriction of a Weil functor to $\mathcal{M}f_m$ is

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said to be a Weil bundle. There are many authors studying their geometry, e. g., Kolář, Mikulski, Shurygin, Bushueva and many others ([5, 8–10, 14, 15]). Further, there are authors applying and connecting Weil functors with problems from the theory of Lie groups, e. g., Alonso, Muriel, and Muñoz Rodriguez [1, 2, 12]. Further, there are papers applying Weil functors in the theory of jets, e. g., Kureš [11], who also studies their applications in the theoretical mechanics.

From the algebraic point of view, Weil functors are studied by Bertram (e. g. [3, 4]) who generalizes their definition substituting general fields and rings for reals in the definition of the associated Weil algebra. Weil functors are also applied in the research of geometrical categories (e. g., Nishimura [13]).

Let us resume briefly the elementary concepts of the Weil theory. Consider the algebra $\mathcal{E}(k)$ of germs of smooth functions defined on $\mathbb{R}^k$ at zero. A Weil algebra can be defined either as $A = \mathbb{R} \oplus N_A$ for the ideal N_A of nilpotent elements (the so called nilpotent ideal) or as a quotient $A = \mathcal{E}(k)/I$ by an ideal I of finite codimension. It can also be defined as a quotient $A = \mathbb{D}^r_k/J$ of the so called jet algebra $\mathbb{D}^r_k$ or, in other words, as the algebra of polynomials of k indeterminates of order at most r factorized by some of its ideal J. Finally, we define width(A) as $\dim(N_A/N_A^2)$ and height(A) as the minimal r for which $A = \mathbb{D}^r_k/J$.

We apply the covariant approach to the definition of Weil functors. This comes out from the I-factorization of germs as follows. Two germs $\mathrm{germ}_0\, g : \mathbb{R}^k_0 \to M$ and $\mathrm{germ}_0\, h : \mathbb{R}^k_0 \to M$ taking the same value x at $0 \in \mathbb{R}^k$ are said to be I-equivalent if and only if $\mathrm{germ}_x\, \gamma \circ \mathrm{germ}_0\, g - \mathrm{germ}_x\, \gamma \circ \mathrm{germ}_0\, h \in I$ for any function $\gamma : M \to \mathbb{R}$ defined near x. Classes of such equivalence relation denoted by $j^A g$ form the space $T^A M$. For a smooth map $\varphi : M \to N$, the map $T^A\varphi : T^A M \to T^A N$ is defined by $T^A\varphi(j^A g) := j^A(\varphi \circ g)$.

We recall that natural transformations $\tilde{t}_M : T^B M \to T^A M$ bijectively correspond to homomorphisms $t : B \to A$ and consequentely to the so-called B-admissible A-velocities defined in [8] as follows. Let $A = \mathcal{E}(k)/I$ and $B = \mathcal{E}(p)/J$ be Weil algebras considered as quotiens of germ algebras. For a smooth map $f : \mathbb{R}^k_0 \to \mathbb{R}^p_0$ an A-velocity $j^A f$ is said to be B-admissible if and only if the condition of admissibility

$$\mathrm{germ}_0\, \varphi \in J \;\Rightarrow\; \mathrm{germ}_0(\varphi \circ f) \in I \tag{1.1}$$

is satisfied for every smooth function $\varphi : \mathbb{R}^p \to \mathbb{R}$. Further, every B-admissible A-velocity $j^A f$ is assigned bijectively a natural transformation $\tilde{t}_M : T^B M \to T^A M$ defined by $\tilde{t}_M(j^B\varphi) = t_M^{j^A f}(j^B\varphi) = j^A(\varphi \circ f)$. It follows that automorphisms of A are determined by reparametrizations of indeterminates $\tau_1, \ldots, \tau_k \in \mathbb{D}^r_k$ satisfying the admissibility condition (1.1).

Let $G^r_m = \mathrm{inv}\, J^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$ with the composition of jets be a general jet group. Recall the identification of G^r_m with the group $\mathrm{Aut}(\mathbb{D}^r_m)$ of Weil algebra automorphisms assigning to every $j^r_0 g \in G^r_m$ the automorphism of $\mathbb{D}^r_m$ defined by $j^r_0 \eta \mapsto$

$j_0^r \eta \circ (j_0^r g)^{-1}$ for any $j_0^r \eta \in \mathbb{D}_m^r$. For a Weil algebra $A = \mathbb{D}_m^r / I$ and the projection homomorphism $p : \mathbb{D}_m^r \to A$, Alonso defined the subgroup $G^A \subseteq G_m^r \simeq \mathrm{Aut}(\mathbb{D}_m^r)$ and its normal subgroup G_A ([1]) as follows

$$\begin{aligned} G_A &= \{j_0^r g \in G_m^r;\, p \circ j_0^r g = p\}, \\ G^A &= \{j_0^r g \in G_m^r;\, \mathrm{Ker}(p \circ j_0^r g) = \mathrm{Ker}(p)\}. \end{aligned} \tag{1.3}$$

He also proved the identification $G^A / G_A \simeq \mathrm{Aut}\, A$ of Lie groups.

In [16], we have defined the spaces $T^{A*}M$ of A-covelocities on M by $T^{A*}M = \{T_x^A f : T_x^A M \to T_0^A \mathbb{R} \simeq N_A; x \in M\}$ and for local diffeomorphisms $g : M \to N$ the maps $T^{A*}g : T^{A*}M \to T^{A*}N$ by $T^{A*}g(T_x^A f) = T_x^A f \circ (T_x^A g)^{-1}$. For $A = \mathbb{D}_k^r$, we write $T_k^{r*}M$ and $T_k^{r*}g$ (see [9]). In [16], we have proved the following statement.

(a) *Let $A = \mathbb{D}_k^r / I$ and $m = \dim M \geq k$. Then, the spaces $T^{A*}M$ with the maps $T^{A*}g$ form the natural bundle $P^r M[N_m^r, \ell]$ identified with $T_m^{r*}M$ where N_m^r is the nilpotent ideal of $\mathbb{D}_m^r$ and $\ell : G_m^r \times N_m^r$ is the left action defined by $\ell(j_0^r g, j_0^r \eta) = j_0^r \eta \circ (j_0^r g)^{-1}$*

For any $(M, N) \in \mathrm{Obj}(\mathcal{M}f_m) \times \mathrm{Obj}(\mathcal{M}f)$, define $J^A(M, N) = \{T_x^A f; f : M \to N\}$. For a local diffeomorphism $g : M_1 \to M_2$ and a smooth map $h : N_1 \to N_2$, define the map $J^A(g, h) : J^A(M_1, N_1) \to J^A(M_2, N_2)$ by

$$J^A(g, h)(T_x^A f) = T_{f(x)}^A h \circ T_x^A f \circ (T_x^A g)^{-1}.$$

Then, it holds

(b) *For $m \geq k$ and $A = \mathbb{D}_k^r / I$ the space $J^A(M, N)$ is identified with $J^r(M, N)$.*

Proof. For any $M \in \mathrm{Obj}(\mathcal{M}f_m)$, consider the bundle functor $G_M : \mathcal{M}f \to \mathcal{FM}$ defined by $G_M N = J^r(M, N)$ on objects and by $G_M h = J^r(\mathrm{id}_M, h)$ on morphisms.

The assertion *(a)* implies

$$J^r(M, N) = G_M N = P^r N[(T_k^{r*}M)^n, \ell_M] = P^r N[(T^{A*}M)^n, \ell_M],$$

where $\ell_M : G_n^r \times (T_k^{r*}M)^n \to (T_k^{r*}M)^n$ is defined by $\ell_M(j_0^r g, (T_k^r)_x f) = (T_k^r)_0 g \circ (T_k^r)_x f = T_0^A g \circ T_x^A f$. It follows $J^r(M, N) = J^A(M, N)$. For morphisms, we have $J^A(g, h)(T_x^A f) = T_{g(x)}^A (h \circ f \circ g^{-1}) = j_{g(x)}^r (h \circ f \circ g^{-1}) = J^r(g, h)(j_x^r f)$ if we denote by g^{-1} the inverse to g considered near x. □

2. BUNDLES OF (p, A)-COVELOCITIES AND (p, A)-JETS

Let $\hat{p} : G_k^r \to G^A$ be a G_A-stabilizing Lie group homomorphism and $p : G_k^r \to \mathrm{Aut}(A)$ its corestriction in the sense of (1.3). For $m \geq k$, we define and investigate the spaces $T_p^{A*}M$ of (p, A)-covelocities.

For the present, suppose $m = k$. Consider the left action ℓ of G^A with its effective left action $\bar{\ell}$ of $\operatorname{Aut} A$ on $T_x^A M$

$$\ell(j_0^r g, j^A \varphi_x) = \bar{\ell}(\pi(j_0^r g), j^A \varphi_x) = j^A \varphi_x \circ (j_0^r g)^{-1} \tag{2.1}$$

if $\pi : G^A \to \operatorname{Aut} A$ denotes the projection Lie group homomorphism. For a Lie subgroup $H \subseteq \operatorname{Aut} A$, let us denote by $\operatorname{Orb}(H, j^A \varphi_x)$ the H-orbit of $j^A \varphi_x$ with respect to $\bar{\ell}$ restricted to $H \times T_x^A M$. Since the elements of $\operatorname{Aut} A$ determine natural tranformations over $T^A M$ determined by (2.1), the values of $T_x^A f$ on the whole $\operatorname{Orb}(\operatorname{Aut} A, j^A \varphi_x)$ are determined by the value $T_x^A f(j^A \varphi_x)$. Indeed, we have

$$\begin{aligned} T_x^A f \circ \ell(j_0^r g, j^A \varphi_x) &= T_x^A f \circ \bar{\ell}(\pi(j_0^r g), j^A \varphi_x) = \\ &= \bar{\ell}(\pi(j_0^r g), T_x^A f(j^A \varphi_x)) = \ell(j_0^r g, T_x^A f(j^A \varphi_x)). \end{aligned} \tag{2.2}$$

We are searching for the greatest subgroup $H \subseteq \operatorname{Aut} A$ satisfying the existence of a map $X : \operatorname{reg} T_x^A M \to N_A$ prolongating for $j^A \varphi_x \in \operatorname{reg} T_x^A M$ the restrictions $T_x^A f_{| \operatorname{Orb}(H, j^A \varphi_x)}$ in the sense of the following formula

$$X(j^A \psi_x) = T_x^A \widehat{f}(j^A \psi_x) = T_x^A f(j^A \varphi_x) \circ p((j_0^r \widehat{\varphi}_x)^{-1} \circ j_0^r \widehat{\psi}_x)) \tag{2.3}$$

for suitable $T_x^A \widehat{f} \in T_x^{A*} M$ and any $j_0^r \widehat{\varphi}_x \in j^A \varphi_x$ and $j_0^r \widehat{\psi}_x \in j^A \psi_x$. It is easy to see that

$$j^A \varphi_{x,1} = j^A \varphi_{x,2} \text{ implies } (j_0^r \varphi_{x,1})^{-1} \circ j_0^r \varphi_{x,2} \in G_A \tag{2.4}$$

and thus, the formula (2.3) is correct.

It follows from (2.2) and (2.3) that $H \subseteq p(G_k^r)$. Insisting on the condition $H = p(G_k^r)$, one deduces easily that (2.3) works only if an additional assumption concerning p is satisfied. It reads as follows

$$\pi \circ (\widehat{p} \circ \widehat{p})(j_0^r g) = p(j_0^r g) \text{ for any } \widehat{p} \in p. \tag{2.5}$$

We remark that this case leads to the concept of a p-vertical A-covelocity, which was presented in [16].

Coming back to (p, A)-covelocities, we define the Lie subgroups

$$\begin{aligned} \bar{G}_k^{r,p} &= \{j_0^r g \in G_k^r, \pi \circ (\widehat{p} \circ \widehat{p})(j_0^r g) = p(j_0^r g) \text{ for any } \widehat{p} \in p\}, \\ G_k^{r,p} &= \bar{G}_k^{r,p} \cap p(G_k^r) \end{aligned} \tag{2.6}$$

of G_k^r determined by $p : G_k^r \to \operatorname{Aut} A$ under discussion. We put $H = G_k^{r,p}$.

On the other hand, a (p, A)-covelocity will be a map $X : \operatorname{reg} T_x^A M \to N_A$ sharing the same value with some A-covelocity over a selected element from $\operatorname{reg} T_x^A M$ and uniquely extended to the whole $\operatorname{reg} T_x^A M$ by (2.3). We give a definition connecting both of those approaches.

Definition 1. Let $A = \mathbb{D}_k^r / I$, $p : G_k^r \to \operatorname{Aut} A$ be a Lie group homomorphism and $k = m = \dim M$. A map $X : \operatorname{reg} T_x^A M \to T_0^A \mathbb{R} = N_A$ is said to be a

(p, A)-covelocity at $x \in M$, i. e., an element of $(T_p^{A*})_x M$ if some of the following conditions is satisfied

(i) for every $j^A\varphi_x \in \operatorname{reg} T_x^A M$, there is a covelocity $T_x^A f \in T_x^{A*} M$ satisfying $X(j^A\varphi_x) = T_x^A f(j^A\varphi_x)$ and for any $j^A\psi_x \in \operatorname{reg} T_x^A M$, the value $X(j^A\psi_x)$ is obtained from $X(j^A\varphi_x)$ by the following extension

$$X(j^A\psi_x) = X(j^A\varphi_x) \circ p((j_0^r\widehat{\varphi}_x)^{-1} \circ j_0^r\widehat{\psi}_x),\ j_0^r\widehat{\varphi}_x \in j^A\varphi_x \text{ and } j_0^r\widehat{\psi}_x \in j^A\psi_x. \quad (2.7)$$

(ii) for every $j^A\varphi_x \in \operatorname{reg} T_x^A M$, there is $T_x^A f \in T_x^{A*} M$ such that the restrictions of X and $T_x^A f$ to $\operatorname{Orb}(G_k^{r,p}, j^A\varphi_x)$ coincide and the extensions of X to other $G_k^{r,p}$-orbits of $\operatorname{reg} T_x^A M$ are obtained by (2.7).

Then, $T_p^{A*} M = \cup_{x\in M}(T_p^{A*})_x M$ is said to be the space of (p, A)-covelocities on M. For a local diffeomorphism $g : M \to N$, there is a map $T_p^{A*} g : T_p^{A*} M \to T_p^{A*} N$ defined as follows

$$(T_p^{A*})g(X) = X \circ (T_x^A g)^{-1} \text{ for } X \in (T_p^{A*})_x M \quad (2.8)$$

We note that (2.4) implies the correctness of (2.7) in the sense of its independency on the choice of $j_0^r\widehat{\varphi}_x \in j^A\varphi_x$ and $j_0^r\widehat{\psi}_x \in j^A\psi_x$.

It is easy to see that for any $p : G_k^r \to \operatorname{Aut} A$ satisfying the assumption (2.5), p-vertical A-covelocities coincide with more general objects of (p, A)-covelocities. In other cases, the difference corresponds to the difference between (2.1) and (2.7), namely in the transformations of values within $p(G_k^r)$-orbits. One can consider the substitution of $p(j_0^r g) = p((j_0^r\varphi_x)^{-1} \circ j_0^r\psi_x)$ for $j_0^r g = (j_0^r\varphi_x)^{-1} \circ j_0^r\psi_x$ in (2.1) as an addition of a deformation within orbits expressed by p.

Proposition 1. *For $m = k = \operatorname{width}(A)$, the system of spaces $T_p^{A*} M$ with T_p^{A*}-maps forms the structure of a natural bundle $P^r M[N_A, \ell]$ with the standard fiber formed by N_A. The left action $\ell : G_m^r \times N_A \to N_A$ is defined as follows*

$$\ell(j_0^r g, j^A\alpha) = p_A \circ r(j_0^r\alpha, (p(j_0^r g))^{-1}) = j^A\alpha \circ (p(j_0^r g))^{-1}) \quad (2.9)$$

where r is obtained in the obvious way from the standard right action of G_m^r on $P_x^r M$ considered on $M_x = \mathbb{R}_0^m$. Further, $p_A : \mathbb{D}_m^r \to A$ denotes the projection homomorphism of Weil algebras.

Proof. The identification $H : T_p^{A*} M \to P^r M[N_A, \ell]$ is given by the assignement $X \mapsto \{j_0^r\alpha_x, X(j_0^r\alpha_x)\}$ and H^{-1} by the assignement $\{j_0^r\alpha_x, j^A\eta\} \mapsto (j_0^r\alpha_x \mapsto j^A\eta)$. We must verify that the map H is well defined. Indeed, we have

$$\begin{aligned}\{j_0^r\alpha_x \circ j_0^r g, X(j_0^r\alpha_x \circ j_0^r g)\} &= \{j_0^r\alpha_x \circ j_0^r g, X(j_0^r\alpha_x) \circ p(j_0^r g)\}\\ &= \{r(j_0^r\alpha_x, j_0^r g), \ell((j_0^r g)^{-1}, X(j_0^r\alpha_x))\}.\end{aligned}$$

Clearly, $\{j_0^r\alpha_x, j^A\eta\}$ and $\{j_0^r\alpha_x \circ j_0^r g, j^A\eta \circ p(j_0^r g)\}$ determine the same (p, A)-covelocity by its definition.

Furthermore, morphisms of associated bundles $\{P^r g, \mathrm{id}_{N_A}\}$ are identified with maps $T_p^{A*}g$ since every $\{j_0^r\alpha_x, T_x^A f(j_0^r\alpha_x)\}$ is assigned $\{j_0^r(g\circ\alpha_x), T_x^A f(j_0^r\alpha_x)\} = \{j_0^r(g\circ\alpha_x), T_x^A f \circ (T_x^A g)^{-1})(j_0^r(g\circ\alpha_x))\}$. This completes the proof. □

There is an easy extension of the result from $m = k =$ width A to the cases of $m > k$. We can replace $A = \mathbb{D}_k^r/I$ by $\tilde{A} = \mathbb{D}_m^r/(I \vee \langle\tau_{k+1}, \dots, \tau_m\rangle)$ adding formally $m-k$ indeterminates. As for the subgroups G^A or $G_A \subseteq G_k^r$, replace them by $G^{\tilde{A}}$ and $G_{\tilde{A}} \subset G_m^r$ as follows. For any $j_0^r\tilde{g} \in G_m^r$, put

$$j_0^r\tilde{g} \in G^{\tilde{A}} \text{ or } j_0^r\tilde{g} \in G_{\tilde{A}} \text{ if and only if } \mathrm{pr}_1 \circ j_0^r\tilde{g} \circ j_0^r i_1 \in G^A$$
$$\text{or } \mathrm{pr}_1 \circ j_0^r\tilde{g} \circ j_0^r i_1 \in G_A \text{ and } \mathrm{pr}_2 \circ j_0^r\tilde{g} \circ j_0^r i_1 = j_0^r 0^{m-k} \tag{2.10}$$

for the zero function on $\mathbb{R}^k$, the inclusion $i_1 : \mathbb{R}^k \to \mathbb{R}^m$ defined by $(x^1, \dots, x^k) \mapsto (x^1, \dots, x^k, 0, \dots, 0)$ and the projection $\mathrm{pr}_2 : \mathbb{R}^k \times \mathbb{R}^{m-k} \to \mathbb{R}^{m-k}$.

Let us replace $p : G_k^r \to \mathrm{Aut}(A)$ by a Lie group homomorphism $\tilde{p} : G_m^r \to \mathrm{Aut}(\tilde{A})$. Consider any free basis $\mathcal{B}_1$ of generators of $G_k^r \simeq G_k^r \times \{j_0^r \,\mathrm{id}_{\mathbb{R}^{m-k}}\} \subseteq G_m^r$ and complete it to a free basis of generators $\mathcal{B}$ of G_m^r. Then, we define $\tilde{p} : G_m^r \to \mathrm{Aut}(\tilde{A})$ by $j_0^r g \times \{j_0^r \,\mathrm{id}_{\mathbb{R}^{m-k}}\} \mapsto p(j_0^r g)$ for $j_0^r g \in G_k^r$ and by $j_0^r\tilde{g} \mapsto j_0^r \,\mathrm{id}_{\mathbb{R}^m}$ in case of $j_0^r\tilde{g} \in \mathcal{B} - \mathcal{B}_1$. Further, the left action from (2.9) is replaced by $\ell : G_m^r \times N_{\tilde{A}} \to N_{\tilde{A}}$, which is defined as follows

$$\ell(j_0^r g, j^A\alpha) = \ell(j_0^r g, j^{\tilde{A}}\alpha) \tag{2.11}$$
$$= p_{\tilde{A}} \circ r(j_0^r\alpha, (\tilde{p}(j_0^r g))^{-1})$$
$$= j^{\tilde{A}}\alpha \circ (\tilde{p}(j_0^r g))^{-1}). \tag{2.12}$$

So, we can modify the extension of (p, A)-covelocity X from $j^A\varphi_x \in \mathrm{reg}\, T_x^A M$ to another value $j^A\psi_x \in \mathrm{reg}\, T_x^A M$ from (2.7) as follows

$$X(j^A\psi_x) = X(j^A\varphi_x) \circ (\tilde{p}(j_0^r(\widehat{\psi}_x^{-1} \circ \widehat{\varphi}_x))^{-1} \tag{2.13}$$

where $j_0^r\widehat{\varphi}_x$ and $j_0^r\widehat{\psi}_x$ are formally considered as arbitrary elements of $j^{\tilde{A}}\varphi_x$ and $j^{\tilde{A}}\psi_x \in \mathrm{reg}\, T_x^{\tilde{A}} M$. Since the definition (2.8) of T_p^{A*}-maps remains unchanged, we can reformulate Proposition 1 as follows

Proposition 2. *For $m \geq k$ and a Lie group homomorphism $p : G_k^r \to \mathrm{Aut}(A)$, the system of spaces $T_p^{A*}M$ of (p, A)-covelocities with their T_p^{A*}-maps is identified with the system of spaces $T_{\tilde{p}}^{\tilde{A}*}M$ with their $T_{\tilde{p}}^{\tilde{A}*}$-maps and consequentely to a natural bundle $T_p^{A*}M = P^r M[N_A, \ell]$ with the standard fiber formed by $N_A \simeq N_A \vee \langle\tau_{k+1}, \dots, \tau_m\rangle$ and the left action ℓ defined by* (2.12).

Let us define the space $J_p^A(M, N)$ of (p, A)-jets.

Definition 2. Let $A = \mathbb{D}_k^r/I$ and $p : G_k^r \to \mathrm{Aut}(A)$ be as above, $m \geq k = \mathrm{width}(A)$. For $(M, N) \in \mathrm{Obj}(\mathcal{M}f_m) \times \mathrm{Obj}(\mathcal{M}f)$, define $(J_p^A)_x(M, N)_y = \{X : \mathrm{reg}\, T_x^A M \to T_y^A N$; for all $j^A\varphi_x \in \mathrm{reg}\, T_x^A M$, there is $T_x^A f \in J_x^A(M, N)_y$ satisfying $X(j^A\varphi_x) = T_x^A f(j^A\varphi_x)$ and $X(j^A\psi_x) = X(j^A\varphi_x) \circ \tilde{p}((j_0^r\widehat{\varphi}_x)^{-1} \circ j_0^r\widehat{\psi}_x)$ for any $j_0^r\widehat{\varphi}_x \in j^{\tilde{A}}\varphi_x$, $j_0^r\widehat{\psi}_x \in j^{\tilde{A}}\psi_x\}$. We define the space $J_p^A(M, N)$ of (p, A)-jets setting

$$J_p^A(M, N) := \bigcup_{x \in M, y \in N} (J_p^A)_x(M, N)_y.$$

For a local diffeomorphism $g : M_1 \to M_2$ and a smooth map $h : N_1 \to N_2$, define the map $J^A(g, h) : J_p^A(M_1, N_1) \to J_p^A(M_2, N_2)$ by

$$J^A(g, h)(X) = T_{f(x)}^A h \circ X \circ (T_x^A g)^{-1}$$

for all $X \in (J_p^A)_x(M, N)_y$.

In the very end, we give the geometrical description of the spaces $J^A(M, N)$. Denote by $\mathcal{N}\mathcal{E}(A)$ the group of natural equivalences over T^A identified with the group $\mathrm{Aut}(A)$.

Proposition 3 (Kolář and Mikulski [10]). *Every bundle functor F defined on the product category $\mathcal{M}f_m \times \mathcal{M}f$ of order r in the first factor and preserving products in the second factor is of the form $F(M, N) = P^r M[G^F N, H_N^F] = P^r M[T^A N, H_N^F]$ for a product preserving bundle functor $G^F = T^A$ defined on objects by $G^F N = F_0(\mathbb{R}^m, N) = T^A N$, on morphisms by $G^F f = F_0(\mathrm{id}_{\mathbb{R}^m}, f) = T^A f$ and for a Lie group homomorphism $H^F : G_m^r \to \mathcal{N}\mathcal{E}(A) \simeq \mathrm{Aut}(A)$ defined by $H_N^F(j_0^r g)(a) = F_0(g, \mathrm{id}_N)(a)$ for every $a \in G^F N \simeq T^A N$.*

Proposition 4. *For $m \geq k$, the spaces $J_p^A(M, N)$ with J_p^A-maps form a bundle functor on the product category $\mathcal{M}f_m \times \mathcal{M}f$ satisfying $J_p^A(M, \mathbb{R})_0 = T_p^{A*}M$ and $J_p^A(g, \mathrm{id}_{\mathbb{R}})_{|\,\mathrm{Obj}(\mathcal{M}f_m)\times\{\mathbb{R}_0\}} = T_p^{A*}g$. Moreover, J_p^A preserves products in the second factor and $J_p^A(M, N) \simeq P^r M[G^F N, H_N^F]$ for $G^F = T^A$ and $H^F \simeq \ell$ for ℓ defined by* (2.9) *or* (2.12).

Proof. We are searching for a bundle functor of the required kind on J_p^A extending T_p^{A*}. We prove

$$J_p^A(M, N) \simeq P^r N[(T_p^{A*}M)^n, \bar{\ell}] \to M \times N,$$

where $\bar{\ell} : G_n^r \times (T_p^{A*}M)^n \to (T_p^{A*}M)^n$ is defined in the following way. Let us identify $(\{j_0^r\alpha_x^1, j^A\theta^1\}, \ldots, \{j_0^r\alpha_x^n, j^A\theta^n\}) \in (T_p^{A*}M)_x^n \simeq (P_x^r M[N_A, \ell])^n$ with $\{j_0^r\alpha_x, j^A\eta^1, \ldots, j^A\eta^n\} \in P_x^r M \times_{G_m^r} N_A^n$ where $j_0^r\alpha_x \in P_x^r M$ is arbitrary and

$j^A\eta^1, \dots, j^A\eta^n \in N_A$ are obtained by $j^A\eta^i = \ell(j_0^r h_i^{-1}, j^A\theta^i)$ for $j_0^r h_i \in G_m^r$ satisfying $j_0^r \alpha_x = r(j_0^r \alpha_x^i, j_0^r h_i)$. In this identification, we put

$$\bar{\ell}(j_0^r g, \{j_0^r \alpha_x, j^A\eta^1, \dots, j^A\eta^n\}) = \\ = \{j_0^r \alpha_x, T_0^A g^1(j^A\eta^1, \dots, j^A\eta^n), \dots, T_0^A g^n(j^A\eta^1, \dots, j^A\eta^n)\}. \quad (2.14)$$

It is clear that, for any $X \in J_x^A(M, N)_y$ and $j_0^r \gamma_y \in P_y^r N$, we have $(T_0^A \gamma_y)^{-1} X \in ((T_p^{A*})_x M)^n$. Thus, $X \simeq \{j_0^r \gamma_y, (T_0^A \gamma_y)^{-1}(X)\} \in P_y^r N \times_{G_n^r} ((T_p^{A*})_x M)^n$ with respect to the standard right action of G_n^r on $P^r N$ and the left action $\bar{\ell}$ of G_n^r on $(T_p^{A*} M)^n$ given by the composition of maps $T_0^A g : T_0^A \mathbb{R}^n \to T_0^A \mathbb{R}^n$ determined by $j_0^r g \in G_n^r$ with elements of $(T_p^{A*} M)^n$, which can be also expressed by (2.14).

One can observe easily the smoothness of $\bar{\ell}$ from (2.14). Further, there is the obvious projection $P^r N[(T_p^{A*} M)^n, \bar{\ell}] \to M \times N$ compatible with that of $J_p^A(M, N) \to M \times N$, which proves its smoothness.

The maps $J_p^A(g, h)$ coincide with the compositions of the pairs of smooth maps $J_p^A(\mathrm{id}_M, h) \simeq \{P^r h, \mathrm{id}_{(T_p^{A*} M)^n}\}$ and $J_p^A(g, \mathrm{id}_N) \simeq \{\mathrm{id}_{P^r N}, T_p^{A*} g\}$. The verification of the localization condition from the definition of a bundle functor (see [9]) is easy. The last assertion is a direct consequence of Proposition 3. □

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Author's address

Jiří Tomáš
Brno University of Technology, Faculty of Chemistry, Department of Physical and Applied Chemistry, Purkyňova 118, 612 00 Brno, Czech Republic
E-mail address: Tomas@fch.vutbr.cz