On the structure of Finsler and areal spaces

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Abstract. We study underlying geometric structures for integral variational functionals, depending on submanifolds of a given manifold. Applications include (first order) variational functionals of Finsler and areal geometries with integrand the Hilbert $1$-form, and admit immediate extensions to higher-order functionals.

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1. INTRODUCTION

This paper is a contribution to the theory of integral variational functionals, depending on submanifolds of a given manifold $X$. The theory is based on geometric notions such as the bundles of (skew-symmetric) multivectors, and Grassmann fibrations. Conceptually, it extends local parametric integrals of Finsler–Kawaguchi and areal geometries (see, e.g., Chern, Chen, Lam [1], Davies [3], Kawaguchi [4], and Tamassy [6]) to global functionals, depending on (global) submanifolds. In Section 2 we summarize integration theory of differential forms along submanifolds. Section 3 is devoted to vector bundles of $k$-vectors; we show how mappings of Euclidean spaces into manifolds (parametrisations) can be lifted to the bundles of $k$-vectors. In Section 4 we introduce, using the Plücker embedding, underlying spaces for parameter-invariant variational problems, the Grassmann fibrations. In Section 5 we show that any $k$-form on the Grassmann fibration defines an integral variational functional, depending on $k$-dimensional submanifolds. An example is the Hilbert form, a well-known first-order construction in Finsler geometry and its generalisations (Chern, Chen, Lam [1], Crampin, Saunders [2]).

It should be pointed out that the theory can be further generalised. To this end, one should consider higher-order Grassmann fibrations endowed with Lagrangians satisfying the relevant homogeneity conditions (Zermelo conditions, see, e.g., Saunders [5], and Urban and Krupka [8]).

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where we write $f$ unity, subordinate to this covering. Then, from now on we suppose that $S$. The following basic properties of the integral are needed in the calculus of variations.

Let $X$ be an $n$-dimensional manifold, $S$ a subset of $X$, $x_0 \in S$ a point. A chart $(U, \varphi) \varphi = (x^i)$, at $x_0$ is a submanifold chart for $S$, if there exists a non-negative integer $k \leq n$ such that $\varphi(U \cap S) = \{x \in U | x^{k+1}(x) = \cdots = c_{n-k} \}$. If such a chart exists, we say that $S$ is a submanifold of $X$ at the point $x_0$; $k$ is the dimension of $S$ at $x_0$. If such a submanifold chart exists at every point of $X$, we say $S$ is a submanifold of $X$ and call $k$ the dimension of $S$.

Denote by $(t^1, t^2, \ldots, t^n)$ the canonical coordinates on the Euclidean space $\mathbb{R}^n$, and $\mathbb{R}^n_{(\nu)} = \{t_0 \in \mathbb{R}^n | t^n(t_0) \leq 0\}$, $\partial \mathbb{R}^n_{(\nu)} = \{t_0 \in \mathbb{R}^n | t^n(t_0) = 0\}$. $\mathbb{R}^n_{(\nu)}$ is the halfspace of $\mathbb{R}^n$, $\partial \mathbb{R}^n_{(\nu)}$ is the boundary of $\mathbb{R}^n_{(\nu)}$. Let $\Omega$ be a non-void subset of $X$, and $x_0 \in \Omega$ a point. A chart $(U, \varphi)$ at $x_0$ is said to be adapted to $\Omega$, if the set $\varphi(U \cap \Omega)$ is an open set in $\mathbb{R}^n_{(\nu)}$. $\Omega$ is a piece of $X$, if it is compact and each point $x \in \Omega$ admits a chart, adapted to $\Omega$. Let $\eta$ be a $k$-form on $X$. Our aim now will be to introduce an integral of $\eta$ on a piece of a $k$-dimensional submanifold $S$ ($k$-piece of a $X$). Express $\eta$ in a submanifold chart $(U, \varphi)$, $\varphi = (x^i)$, as $\eta = \eta_{i_{1i_2\ldots i_k}} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$. Then, restricting $\eta$ to $S$ we get from the equations $x^{k+1} = 0, x^{k+2} = 0, \ldots, x^n = 0$

$\eta = f dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$,

where we write $f = f(x^{i_1}, x^{i_2}, \ldots, x^{i_k})$ for the component of $\eta$ restricted to $S$. From now on we suppose that $S$ is orientable, and is endowed with an orientation $\text{Or}_S X$; only submanifold charts on $X$ belonging to $\text{Or}_S X$ are used. The integral of $\eta$ on a compact set $\Omega \subset S$ is defined in a standard way. There exist a finite family \{(U_1, \varphi_1), (U_2, \varphi_2), \ldots, (U_N, \varphi_N)\} of submanifold charts on $X$, such that the family \{U_1 \cap S, U_2 \cap S, \ldots, U_N \cap S\} covers $\Omega$. Let \{\chi_1, \chi_2, \ldots, \chi_N\} be a partition of unity, subordinate to this covering. Then,

$$\int_{\Omega} \eta = \sum_{j=1}^{N} \int_{\text{supp} \chi_j \cap \Omega} \chi_j \eta.$$ 

The following basic properties of the integral are needed in the calculus of variations.

**Lemma 1** (transformation of integration domain). Let $X$ and $Y$ be two smooth $n$-dimensional oriented manifolds, $\alpha : X \rightarrow Y$ an orientation-preserving diffeomorphism. Then

$$\int_{\Omega} \eta = \int_{\alpha^{-1}(\Omega)} \alpha \ast \eta$$

for any compact set $\Omega \subset S$ and any continuous differential $n$-form on $Y$.

**Lemma 2** (Leibniz rule). Let $X$ be an oriented $n$-dimensional manifold, $\eta_t$ a family of $n$-forms on $X$, differentiable on a real parameter $t$, $\Omega \subset S$ a compact set.
Then, the function \( I \equiv t \mapsto \int_{\Omega} \eta_t \in \mathbb{R} \) is differentiable, and
\[
\frac{d}{dt} \int_{\Omega} \eta_t = \int_{\Omega} \frac{d \eta_t}{dt}.
\]

**Lemma 3** (Stokes formula). Let \( X \) be an \( n \)-dimensional manifold, \( S \) a \( k \)-dimensional oriented submanifold of \( X \), \( \eta \) a \((k - 1)\)-form on \( X \). Let \( \Omega \) be a piece of \( S \) with boundary \( \partial \Omega \), endowed with induced orientation. Then
\[
\int_{\partial \Omega} \eta = \int_{\Omega} \frac{d \eta}{dt}.
\]

3. ** Bundles of \( k \)-vectors

Let \( X \) be an \( n \)-dimensional manifold, \( \Lambda^k T_x X \) the \( k \)-exterior product of the tangent space \( T_x X \), \( x \in X \) a point. We put
\[
\Lambda^k T X = \bigcup_{x \in X} \Lambda^k T_x X.
\]
This set has a natural vector bundle structure over \( X \), with type fibre \( \Lambda^k \mathbb{R}^n \). We denote by \( \tau^k \) the vector bundle projection of \( \Lambda^k T X \).

Let \( X \) (resp. \( Y \)) be a smooth manifold of dimension \( n \) (resp. \( m \)), and let \( f : X \to Y \) be a differentiable mapping. Choose a point \( x \in X \) and a \( k \)-vector \( \Sigma \in \Lambda^k T_x X \). Then, choose a chart \((U, \varphi), \varphi = (x^i), \) at \( x \) and a chart \((V, \psi), \psi = (y^\sigma), \) at \( f(x) \in Y \) such that \( f(U) \subset V \). Expressing \( \Sigma \) in components and setting
\[
\Lambda^k T_x f : \Sigma = \frac{1}{(k!)^2} \left( \frac{\partial y^{\sigma_1} f}{\partial x^1} \right) \varphi(x) \left( \frac{\partial y^{\sigma_2} f}{\partial x^2} \right) \varphi(x) \ldots \left( \frac{\partial y^{\sigma_k} f}{\partial x^k} \right) \varphi(x),
\]
we get a \( k \)-vector \( \Lambda^k T_x f : \Sigma \in \Lambda^k T_{f(x)} Y \), and a vector bundle homomorphism \( \Lambda^k T f : \Lambda^k T X \to \Lambda^k T Y \) over \( f \) (the lift of \( f \)).

It is easily seen that differentiable mappings of a Euclidean space into a manifold can be canonically lifted to the bundles of \( k \)-vectors. For this purpose we use the canonical \( k \)-vector field on \( \mathbb{R}^n \)
\[
\mathbb{R}^n \ni t \mapsto \theta(t) = \frac{1}{k!} \epsilon^{ij_2 \ldots i_k} \left( \frac{\partial}{\partial y^{i_1}} \right)_t \wedge \left( \frac{\partial}{\partial y^{i_2}} \right)_t \wedge \ldots \wedge \left( \frac{\partial}{\partial y^{i_k}} \right)_t \in \Lambda^k T \mathbb{R}^n.
\]
Identifying \( \Lambda^k T \mathbb{R}^n \) with \( \mathbb{R}^n \times \Lambda^k \mathbb{R}^n \), the canonical section becomes the mapping
\[
t \mapsto (t, \epsilon^{i_2 \ldots i_k} \left( \frac{\partial}{\partial y^{i_1}} \right)_t \wedge \left( \frac{\partial}{\partial y^{i_2}} \right)_t \wedge \ldots \wedge \left( \frac{\partial}{\partial y^{i_k}} \right)_t).
\]
Consider a differentiable mapping \( f : U \to Y \), where \( U \) is an open subset of \( \mathbb{R}^n \). For any point \( t \in U \), \( \Lambda^k T_t f : \theta(t) \) is an element of the vector space \( \Lambda^k T_{f(t)} Y \). We
get the canonical lift $\Lambda^k f$ of $f$ to $\Lambda^k TY$, defined by

$$\Lambda^k f = \Lambda^k T f \cdot \theta.$$ 

The canonical lift of the parametrisation $U \ni t \to (\psi^{-1} \circ (k,m))(t) \in V \cap S$ is expressed in a chart $(V, \psi)$, $\psi = (y^\sigma)$, as

$$\Lambda^k (\psi^{-1} \circ (k,m))(t) = \left( \frac{\partial}{\partial y^1} \right) \psi^{-1} \circ (k,m)(t) \wedge \left( \frac{\partial}{\partial y^2} \right) \psi^{-1} \circ (k,m)(t) \wedge \ldots \wedge \left( \frac{\partial}{\partial y^k} \right) \psi^{-1} \circ (k,m)(t).$$  \hfill (3.1)

Formula (3.1) also defines the mapping $V \ni y \to (\Lambda^k \psi)(y) \in (\tau^k)^{-1}(V)$ by

$$\Lambda^k \psi = \Lambda^k (\psi^{-1} \circ (k,m) \circ \text{pr}_{m,k} \psi),$$  \hfill (3.2)

the canonical section along $S$, associated with $(V, \psi)$. $\Lambda^k \psi$ is expressed by

$$(y^1, y^2, \ldots, y^k, y^{k+1}, y^{k+2}, \ldots, y^m) \mapsto \Lambda^k (\psi^{-1} \circ (k,m))(y^1, y^2, \ldots, y^k) = ((y^1, y^2, \ldots, y^k, 0, 0, \ldots, 0), (1,0,0,\ldots,0)).$$

Writing in the multi-index notation $((\tau^r)^{-1}(V), \Phi)$, $\Phi = (\tilde{y}^I)$, and setting $I_0 = (1,2,\ldots,k)$, we get the image of this mapping as a subset of $((\tau^r)^{-1}(V), \Phi)$, defined by the equations $y^{k+1} = 0, y^{k+2} = 0, \ldots, y^m = 0, \tilde{y}^{I_0} = 1, \tilde{y}^I = 0, I \neq I_0$.

**Lemma 4.** Let $(V, \psi)$, $\psi = (y^\sigma)$, and $(\tilde{V}, \tilde{\psi})$, $\tilde{\psi} = (\tilde{y}^\tau)$, be two charts on $Y$, adapted to $S$, such that $V \cap \tilde{V} \neq \emptyset$.

1. The canonical sections along $S$ satisfy

$$\Lambda^k \tilde{\psi} = \text{det} \left( \frac{\partial \tilde{y}^j}{\partial y^i} \right)_{\tilde{\psi}(y)} \Lambda^k \psi.$$  \hfill (3.3)

2. The differential forms $d\tilde{y}^\tau$ and $d\psi^\sigma$ satisfy $(\Lambda^k \psi)^* d\psi^\sigma = d\psi^\sigma$, $1 \leq i \leq k$.

(3.4)

4. **Grassmann Fibrations**

Consider the vector bundle $\Lambda^k TY$ and the subset $\Lambda^k_0 TY \subset \Lambda^k TY$, consisted of non-zero $k$-vectors. We have an equivalence relation on $\Lambda^k_0 TY$ “$\Xi_1$ is equivalent with $\Xi_2$, if there exists a real number $\lambda > 0$ such that $\Xi_1 = \lambda \Xi_2$”. The quotient set has the structure of a fibration over $Y$, called the Grassmann fibration of degree $k$, and is denoted by $G^k Y$. 


To describe the structure of the set $G^k Y$, we proceed similarly as in the case of classical projective spaces. If in a chart $(V, \psi)$, $\psi = (y^\sigma)$,

$$\Sigma_i = \frac{1}{k!} \Sigma^\sigma \cdot \sigma_2 \cdot \sigma_k \left( \frac{\partial}{\partial y^{\sigma_1}} \right) \wedge \left( \frac{\partial}{\partial y^{\sigma_2}} \right) \wedge \ldots \wedge \left( \frac{\partial}{\partial y^{\sigma_k}} \right), \quad i = 1, 2,$$

are two nonzero $k$-vectors, then $\Sigma_1$ is equivalent with $\Sigma_2$ if and only if in this chart, $\Sigma^\sigma \cdot \sigma_2 \cdot \sigma_k = \lambda \Sigma^\sigma \cdot \sigma_2 \cdot \sigma_k$ for some $\lambda > 0$ and all $\sigma_1, \sigma_2, \ldots, \sigma_k$. We denote $V^{v_1 v_2 \ldots v_k} = \{ \Sigma \in (\tau^k)^{-1}(V) \mid \Sigma^v_{v_1 v_2 \ldots v_k} > 0 \}$. Then, a $k$-vector belonging to the set $V^{v_1 v_2 \ldots v_k} \subset \Lambda^k_0 T Y$ can be expressed by

$$\Sigma = \Sigma^v_{v_1 v_2 \ldots v_k} \left( \frac{\partial}{\partial y^{v_1}} \right) \wedge \left( \frac{\partial}{\partial y^{v_2}} \right) \wedge \ldots \wedge \left( \frac{\partial}{\partial y^{v_k}} \right) + \frac{1}{k!} \sum_{(t_1 t_2 \ldots t_k) \neq (v_1 v_2 \ldots v_k)} \Sigma^{t_1 t_2 \ldots t_k} \left( \frac{\partial}{\partial y^{t_1}} \right) \wedge \left( \frac{\partial}{\partial y^{t_2}} \right) \wedge \ldots \wedge \left( \frac{\partial}{\partial y^{t_k}} \right),$$

(no summation through $v_1, v_2, \ldots, v_k$). Denoting by $\text{sgn} \Sigma^v_{v_1 v_2 \ldots v_k}$ the sign of the component $\Sigma^v_{v_1 v_2 \ldots v_k}$, we can write $\Sigma^v_{v_1 v_2 \ldots v_k} = \text{sgn} \Sigma^v_{v_1 v_2 \ldots v_k} |\Sigma^v_{v_1 v_2 \ldots v_k}|$ and

$$\Sigma^v_{v_1 v_2 \ldots v_k} = \text{sgn} \Sigma^v_{v_1 v_2 \ldots v_k} \cdot |\Sigma^v_{v_1 v_2 \ldots v_k}| \left( \frac{\partial}{\partial y^{v_1}} \right) \wedge \left( \frac{\partial}{\partial y^{v_2}} \right) \wedge \ldots \wedge \left( \frac{\partial}{\partial y^{v_k}} \right) + \frac{1}{k!} \sum_{(t_1 t_2 \ldots t_k) \neq (v_1 v_2 \ldots v_k)} |\Sigma^v_{v_1 v_2 \ldots v_k}| \left( \frac{\partial}{\partial y^{t_1}} \right) \wedge \left( \frac{\partial}{\partial y^{t_2}} \right) \wedge \ldots \wedge \left( \frac{\partial}{\partial y^{t_k}} \right),$$

with the summation through $(t_1 t_2 \ldots t_k) \neq (v_1 v_2 \ldots v_k)$. But $\text{sgn} \Sigma^v_{v_1 v_2 \ldots v_k} = 1$, so we see the class of $\Sigma$ can be represented as

$$[\Sigma] = \left( \frac{\partial}{\partial y^{v_1}} \right) \wedge \left( \frac{\partial}{\partial y^{v_2}} \right) \wedge \ldots \wedge \left( \frac{\partial}{\partial y^{v_k}} \right) + \frac{1}{k!} \sum \Sigma^{t_1 t_2 \ldots t_k} \left( \frac{\partial}{\partial y^{t_1}} \right) \wedge \left( \frac{\partial}{\partial y^{t_2}} \right) \wedge \ldots \wedge \left( \frac{\partial}{\partial y^{t_k}} \right).$$

We set for any $\Sigma \in V^{v_1 v_2 \ldots v_k}$

$$w^\sigma(\Sigma) = y^\sigma(\Sigma), \quad w^{v_1 v_2 \ldots v_k}(\Sigma) = \hat{y}^{v_1 v_2 \ldots v_k}(\Sigma),$$

$$w^{\sigma_1 \sigma_2 \ldots \sigma_k}(\Sigma) = \hat{y}^{\sigma_1 \sigma_2 \ldots \sigma_k}(\Sigma), \quad (\sigma_1 \sigma_2 \ldots \sigma_k) \neq (v_1 v_2 \ldots v_k).$$

(4.1)

The pair $(V^{v_1 v_2 \ldots v_k}, \psi_{v_1 v_2 \ldots v_k}, \psi^{v_1 v_2 \ldots v_k} = (w^\sigma, w^{v_1 v_2 \ldots v_k}, w^{\sigma_1 \sigma_2 \ldots \sigma_k})$, where the indices satisfy $(\sigma_1 \sigma_2 \ldots \sigma_k) \neq (v_1 v_2 \ldots v_k)$, is a chart on $\Lambda^k_0 T Y$; we call this chart $(v_1 v_2 \ldots v_k)$-associated with $(V, \psi)$. The pair $(V^{v_1 v_2 \ldots v_k}, W^{v_1 v_2 \ldots v_k})$, $W^{v_1 v_2 \ldots v_k} = (w^\sigma, w^{\sigma_1 \sigma_2 \ldots \sigma_k})$, $(\sigma_1 \sigma_2 \ldots \sigma_k) \neq (v_1 v_2 \ldots v_k)$, is a fibred chart on
\( G^k Y \). Writing formulas (4.1) in a different way, we have the transformation equations
\[
\begin{align*}
\psi^\sigma &= \eta^\sigma, \\
\psi^{v_1 v_2 \ldots v_k} &= \eta^{v_1 v_2 \ldots v_k}, \\
\psi_{\sigma_1 \sigma_2 \ldots \sigma_k} &= \eta_{\sigma_1 \sigma_2 \ldots \sigma_k}.
\end{align*}
\]

The projection \( \kappa^k : \Lambda^k TY \to G^k Y \) of \( \Lambda^k TY \) onto \( G^k Y \) is the Cartesian projection \((\psi^\sigma, \psi^{v_1 v_2 \ldots v_k}, \psi_{\sigma_1 \sigma_2 \ldots \sigma_k}) \to (\psi^\sigma, \psi_{\sigma_1 \sigma_2 \ldots \sigma_k})\). Combining \( \Lambda^k (\psi^{-1}_{(k,m)}) \) and \( \kappa^k \) we get the canonical lift of \( \psi^{-1}_{(k,m)} \) to the Grassmann fibration,
\[
G^k (\psi^{-1}_{(k,m)}) = \kappa^k \circ \Lambda^k (\psi^{-1}_{(k,m)}). \tag{4.2}
\]

**Lemma 5.** Let \((V, \psi), \psi = (\psi^\sigma), \) and \((\tilde{V}, \tilde{\psi}), \tilde{\psi} = (\tilde{\psi}^\sigma), \) be two rectangle charts, adapted to \( S \) at a point \( y \in Y \). Suppose that \((V, \psi) \) and \((\tilde{V}, \tilde{\psi}) \) are consistently oriented. Then
\[
G^k (\tilde{\psi}^{-1}_{(k,m)}) = G^k (\psi^{-1}_{(k,m)}). \tag{4.3}
\]

We set
\[
G^k S = \{ [\Xi] \in G^k Y | [\Xi] = G^k (\psi^{-1}_{(k,m)})(\text{pr}_{m,k}(\eta(y))), y \in S \}. \tag{4.4}
\]

To a given chart \((V, \psi), \psi = (\psi^\sigma), \) we associate the induced chart \((\tau^k)^{-1}(V), \Phi), \Phi = (\psi^\sigma, \tilde{\psi}^\sigma_{\sigma_1 \sigma_2 \ldots \sigma_k}),\) on \( \Lambda^k TY \); the associated charts on the Grassmann fibration \( G^k Y \) are \((V_0^{v_1 v_2 \ldots v_k}, W^{v_1 v_2 \ldots v_k}),\)
\[
W^{v_1 v_2 \ldots v_k} = (\psi^\sigma, \psi_{\sigma_1 \sigma_2 \ldots \sigma_k}),
\]
with \((\sigma_1 \sigma_2 \ldots \sigma_k) \neq (v_1 v_2 \ldots v_k). \) Then, it is easily seen that each of the charts \((V_0^{v_1 v_2 \ldots v_k}, W^{v_1 v_2 \ldots v_k})\) is adapted to the submanifold \( G^k S. \)

**Theorem 1.** Suppose \( S \) is oriented. Then, the subset \( G^k S \) of the Grassmann fibration \( G^k Y \) is a \( k \)-dimensional oriented submanifold, diffeomorphic with \( S. \)

Theorem 1 allows us to integrate over \( k \)-dimensional submanifolds of \( Y \) directly on the Grassmann fibration \( G^k Y. \)

5. **Variational functionals depending on submanifolds**

As before, we write \( G^k S \) (resp., \( G^k \Omega \)) for the canonical lift of a \( k \)-dimensional submanifold \( S \subset Y \) (resp., \( k \)-piece \( \Omega \subset Y \)) to the Grassmann fibration \( G^k Y \). Denote by \( H^k Y \) the set of all \( k \)-pieces \( \Omega \) of the manifold \( Y. \)

Let \( \eta \) be a \( k \)-form on the Grassmann fibration \( G^k Y. \) The form \( \eta \) defines the variational functional
\[
H^k Y \ni \Omega \to \eta(\Omega(S)) = \int_{G^k \Omega} \eta \in \mathbb{R}, \tag{5.1}
\]
We roughly describe in this paper this construction for \( k = 1, \) representing variational functionals of **Finsler geometry** in terms of differential forms (cf. Urban and
Krupka [7]). Consider the tangent bundle $\Lambda^1 TY = TY$, a chart $(V, \psi, \psi = (y^\alpha))$, on $Y$, and the associated chart $(\epsilon^1)^{-1}(V), \Psi = (y^\alpha, \dot{y}^\alpha)$, on $TY$. A function $F : TY \to \mathbb{R}$ satisfies the homogeneity condition, if it satisfies
\[ F(\lambda, \xi) = \lambda F(\xi) \]
for all tangent vectors $\xi$ and every positive $\lambda \in \mathbb{R}$. The same can be stated in coordinates, requiring that
\[ F(y^\nu, \lambda \dot{y}^\nu) = \lambda F(y^\nu, \dot{y}^\nu). \]

**Theorem 2.**

1. For any function $F : TY \to \mathbb{R}$, the chart expressions
\[ \eta = \frac{\partial F}{\partial y^\nu} d y^\nu \quad (5.2) \]
define a global 1-form on $TY$.

2. If $F$ satisfies the homogeneity condition, then $\eta$ is projectable on the Grassmann fibration $G^1 TY$.

3. If $F$ satisfies the homogeneity condition, then, for any curve $\zeta : I \to Y$
\[ (\Lambda^1 \zeta) \ast \eta = (F \circ \Lambda^1 \zeta) dt. \quad (5.3) \]

The form $\eta$ (5.2) is known as the Hilbert form (Chern, Chen and Lam [1], Crampin and Saunders [2]). Theorem 2 (2) characterizes its basic property when $F$ is positive homogeneous: namely, in this case the Hilbert form is defined on the Grassmann fibration $G^1 TY$. One can also easily verify that $\eta$ is a special case of the Lepage-Cartan form. This fact completely determines the behaviour of the variational functional (5.1) under variations of submanifolds, extremal submanifolds, and their invariance properties.

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