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UPPER AND LOWER SLIGHTLY α-CONTINUOUS MULTIFUNCTIONS

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ABSTRACT. The aim of this paper is to introduce and study upper and lower slightly α-continuous multifunctions as a generalization of upper and lower α-continuous multifunctions, respectively, due to Neubrunn (1988). Some characterizations and several properties concerning upper (lower) slightly α-continuous multifunctions are obtained. The relationships between upper (lower) slightly α-continuous multifunctions and upper (lower) α-continuous multifunctions are also discussed.

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1. INTRODUCTION

Various types of functions play a significant role in the theory of classical point set topology. A great number of papers dealing with such functions have appeared, and a good many of them have been extended to the setting of multifunctions. In 1988, Neubrunn introduced the concept of α-continuous multifunctions [7]. α-continuity which is a weaker form of continuity, in ordinary topology was extended to multifunctions.

The purpose of the present paper is to define upper (lower) slightly α-continuous multifunctions and to obtain several characterizations of upper (lower) slightly α-continuous multifunctions and basic properties of such multifunctions. Moreover, the relationships between upper (lower) slightly α-continuous multifunctions and upper (lower) α-continuous multifunctions are also discussed.

2. PRELIMINARIES

Throughout this paper, spaces (X, τ) and (Y, υ) (or simply X and Y) always mean topological spaces in which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. For a subset A of (X, τ), $\mathrm{cl}(A)$ and $\mathrm{int}(A)$ represent the closure of A with respect to τ and the interior of A with respect to τ, respectively.

A subset A is said to be α-open [8] (resp., preopen [4]) if $A \subset \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$, (resp., $A \subset \mathrm{int}(\mathrm{cl}(A))$). The complement of a α-open (resp., preopen) set is said to be

α-closed (preclosed). The family of all α-open (resp., preopen) sets of X is denoted by $\alpha O(X)$ (resp., $PO(X)$).

It is shown in (Njastad, 1965) that $\alpha O(X)$ is a topology for X.

By a multifunction $F : X \to Y$, we mean a point-to-set correspondence from X into Y, and always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction $F : X \to Y$, following [1, 2] we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X : F(x) \subset B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X : y \in F(x)\}$ for each point $y \in Y$. For each $A \subset X$, $F(A) = \bigcup_{x \in A} F(x)$. Then F is said to be a surjection if $F(X) = Y$, or equivalently if for each $y \in Y$ there exists an $x \in X$ such that $y \in F(x)$.

Moreover, $F : X \to Y$ is called upper semi-continuous (resp., lower semi-continuous) if $F^+(V)$ (resp., $F^-(V)$) is open in X for every open set V of Y [10].

For a multifunction $F : X \to Y$, the graph multifunction $G_F : X \to X \times Y$ is defined as follows $G_F(x) = \{x\} \times F(x)$ for every $x \in X$ and the subset $\{\{x\} \times F(x) : x \in X\} \subset X \times Y$ is called the multigraph of F and is denoted by $G(F)$.

Definition 1. A multifunction $F : X \to Y$ is said to be:

(1) Upper α-continuous at $x \in X$ [7] if for each open set V of Y containing $F(x)$, there exists $U \in \alpha O(X)$ containing x such that $F(U) \subset V$;
(2) Lower α-continuous at $x \in X$ [7] if for each open set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in \alpha O(X)$ containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$;
(3) Upper (lower) α-continuous if it has this property at each point of X.

3. Slightly α-continuous multifunctions

Definition 2. A multifunction $F : X \to Y$ is said to be:

(1) Upper slightly α-continuous at $x \in X$ if for each clopen set V of Y containing $F(x)$, there exists $U \in \alpha O(X)$ containing x such that $F(U) \subset V$;
(2) Lower slightly α-continuous at $x \in X$ if for each clopen set V of Y such that $F(x) \cap V \neq \emptyset$, there exists $U \in \alpha O(X)$ containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$;
(3) Upper (lower) slightly α-continuous if it has this property at each point of X.

The following theorem states some characterizations of upper slightly α-continuous multifunctions.

We know that a net (x_α) in a topological space (X, τ) is called eventually in the set $U \subset X$ if there exists an index $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$.

Definition 3. A sequence (x_n) is said to α-converge to a point x if for every α-open set V containing x, there exists an index n_0 such that for $n \geq n_0$, $x_n \in V$. This is denoted by $x_n \to_\alpha x$.

Theorem 1. *Let $F : X \to Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, υ). Then the following statements are equivalent.*

(i) *F is upper slightly α-continuous;*
(ii) *For each $x \in X$ and for each clopen set V such that $x \in F^+(V)$, there exists an α-open set U containing x such that $U \subset F^+(V)$;*
(iii) *For each $x \in X$ and for each clopen set V such that $x \in F^+(Y \setminus V)$, there exists an α-closed set H such that $x \in X \setminus H$ and $F^-(V) \subset H$;*
(iv) *$F^+(V)$ is an α-open set for any clopen set $V \subset Y$;*
(v) *$F^-(V)$ is an α-closed set for any clopen set $V \subset Y$;*
(vi) *$F^-(Y \setminus V)$ is an α-closed set for any clopen set $V \subset Y$;*
(vii) *$F^+(Y \setminus V)$ is an α-open set for any clopen set $V \subset Y$;*
(viii) *For each $x \in X$ and for each net (x_α) which α-converges to x in X and for each clopen set $V \subset Y$ such that $x \in F^+(V)$, the net (x_α) is eventually in $F^+(V)$.*

Proof. (i) $\Leftrightarrow$ (ii). Clear.

(ii) $\Leftrightarrow$ (iii). Let $x \in X$ and let V be a clopen such that $x \in F^+(Y \setminus V)$. By (ii), there exists an α-open set U containing x such that $U \subset F^+(Y \setminus V)$. Then $F^-(V) \subset X \setminus U$. Take $H = X \setminus U$. We have $x \in X \setminus H$ and H is α-closed.

The converse is similar.

(i) $\Leftrightarrow$ (iv). Let $x \in F^+(V)$ and let V be a clopen set. From (i), there exists an α-open set U_x containing x such that $U_x \subset F^+(V)$. It follows that $F^+(V) = \bigcup_{x \in F^+(V)} U_x$ and hence $F^+(V)$ is α-open.

The converse can be shown easily.

(iv) $\Rightarrow$ (v). Let $V \subset Y$ be a clopen set. We have that $Y \setminus V$ is a clopen set. From (iv), $F^+(Y \setminus V) = X \setminus F^-(V)$ is an α-open set. Then it is obtained that $F^-(V)$ is an α-closed set.

(v) $\Rightarrow$ (iv). similar to the above.

(iv) $\Leftrightarrow$ (vi), (v) $\Leftrightarrow$ (vii). Since $F^-(Y \setminus V) = X \setminus F^+(V)$ and $F^+(Y \setminus V) = X \setminus F^-(V)$, the proof is clear.

(i) $\Rightarrow$ (viii). Let (x_α) be a net which α-converges to x in X and let $V \subset Y$ be any clopen set such that $x \in F^+(V)$. Since F is an upper slightly α-continuous multifuction, it follows that there exists an α-open set $U \subset X$ containing x such that $U \subset F^+(V)$. Since (x_α) α-converges to x, it follows that there exists an index $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. From here, we obtain that $x_\alpha \in U \subset F^+(V)$ for all $\alpha \geq \alpha_0$. Thus, the net (x_α) is eventually in $F^+(V)$.

(viii) $\Rightarrow$ (i). Suppose that (i) is not true. There exists a point x and a clopen set V with $x \in F^+(V)$ such that $U \not\subseteq F^+(V)$ for each α-open set $U \subset X$ containing x. Let $x_U \in U$ and $x_U \notin F^+(V)$ for each α-open set $U \subset X$ containing x. Then for the α-neighbourhood net (x_U), $x_U \to_\alpha x$, but (x_U) is not eventually in $F^+(V)$. This is a contradiction. Thus, F is an upper slightly α-continuous multifunction. □

Remark 1. For a multifunction $F : X \to Y$ from a topological space (X, τ) to a topological space (Y, υ), the following implications hold:

upper semi-continuity $\Longrightarrow$ upper α-continuity $\Longrightarrow$ upper slightly α-continuity

However, the converse implications are not true in general.

Example 1. Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4, 5\}$. Let τ and υ, respectively, be the topologies on X and on Y given by $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\upsilon = \{\emptyset, Y, \{1, 2\}, \{3, 4\}, \{3, 4, 5\}, \{1, 2, 3, 4\}\}$. Define the multifunction $F : X \to Y$ by $F(a) = \{1, 3, 4, 5\}$, $F(b) = \{1, 5\}$ and $F(c) = \{1, 2, 3\}$. Then F is a lower slightly α-continuous multifunction but F is not lower α-continuous.

Example 2. Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4, 5\}$. Let τ and υ, respectively, be the topologies in X and Y given by the formulae $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $\upsilon = \{\emptyset, Y, \{1, 2\}, \{3, 4\}, \{3, 4, 5\}, \{1, 2, 3, 4\}\}$. Define the multifunction $F : X \to Y$ by $F(a) = \{3, 4, 5\}$, $F(b) = \{1, 2\}$ and $F(c) = \{1, 2, 3\}$. Then F is an upper slightly α-continuous multifunction but F is not upper α-continuous.

The other implication is not reversible as shown in [7].

Recall that a space is 0-dimensional if its topology has a base consisting of clopen sets.

Theorem 2. *Let $F : X \to Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, υ). Suppose that Y is a 0-dimensional space. If F is an upper (lower) slightly α-continuous multifunction, then F is an upper (lower) α-continuous.*

Proof. Let $x \in X$ and let V be any open set such that $x \in F^+(V)$. Since Y is a 0-dimensional space, then there exists a clopen set G containing x such that $x \in F^+(G) \subset F^+(V)$. Since F is an upper slightly α-continuous multifunction, there exists an α-open set U containing x such that $U \subset F^+(G) \subset F^+(V)$. Thus, we obtain that F is upper α-continuous.

The proof of lower continuity is similar. □

The following theorem states some characterizations of a lower slightly α-continuous multifunction.

Theorem 3. *Let $F : X \to Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, υ). Then the following statements are equivalent.*

(i) *F is lower slightly α-continuous;*

(ii) *For each $x \in X$ and for each clopen set V such that $x \in F^-(V)$, there exists an α-open set U containing x such that $U \subset F^-(V)$;*

(iii) *For each $x \in X$ and for each clopen set V such that $x \in F^-(Y \setminus V)$, there exists an α-closed set H such that $x \in X \setminus H$ and $F^+(V) \subset H$;*

(iv) *$F^-(V)$ is an α-open set for any clopen set $V \subset Y$;*

(v) *$F^+(V)$ is an α-closed set for any clopen set $V \subset Y$;*

(vi) *$F^+(Y \setminus V)$ is an α-closed set for any clopen set $V \subset Y$;*

(vii) *$F^-(Y \setminus V)$ is an α-open set for any clopen set $V \subset Y$;*
(viii) *For each $x \in X$ and for each net (x_α) which α-converges to x in X and for each clopen set $V \subset Y$ such that $x \in F^-(V)$, the net (x_α) is eventually in $F^-(V)$.*

Proof. It can be obtained similarly as Theorem 4. □

Theorem 4. *Let $F : X \to Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, υ) and let $F(X)$ be endowed with subspace topology. If F is an upper slightly α-continuous multifunction, then $F : X \to F(X)$ is an upper slightly α-continuous multifunction.*

Proof. Since F is an upper slightly α-continuous, $F^+(V \cap F(X)) = F^+(V) \cap F^+(F(X)) = F^+(V)$ is α-open for each clopen subset V of Y. Hence $F : X \to F(X)$ is an upper slightly α-continuous multifunction. □

Suppose that (X, τ), (Y, υ) and (Z, ω) are topological spaces. It is known that if $F_1 : X \to Y$ and $F_2 : Y \to Z$ are multifunctions, then the multifunction $F_2 \circ F_1 : X \to Z$ is defined by $(F_2 \circ F_1)(x) = F_2(F_1(x))$ for each $x \in X$.

Theorem 5. *Let (X, τ), (Y, υ), (Z, ω) be topological spaces and let $F : X \to Y$ and $G : Y \to Z$ be multifunctions. If $F : X \to Y$ is an upper (lower) α-continuous multifunction and $G : Y \to Z$ is an upper (lower) semi-continuous multifunction, then $G \circ F : X \to Z$ is an upper (lower) slightly α-continuous multifunction.*

Proof. Let $V \subset Z$ be any clopen set. From the definition of $G \circ F$, we have $(G \circ F)^+(V) = F^+(G^+(V))$ $((G \circ F)^-(V) = F^-(G^-(V)))$. Since G is an upper (lower) semi-continuous multifunction, it follows that $G^+(V)$ $(G^-(V))$ is an open set. Since F is an upper (lower) α-continuous multifunction, it follows that $F^+(G^+(V))$ $(F^-(G^-(V)))$ is an α-open set. It shows that $G \circ F$ is an upper (lower) slightly α-continuous multifunction. □

Lemma 1. *If $A \in PO(X)$ and $B \in \alpha O(X)$, then $A \cap B \in \alpha O(A)$* [5].

Theorem 6. *Let $F : X \to Y$ be a multifunction and let $U \in PO(X)$. If F is a lower (upper) slightly α-continuous multifunction, then the restriction multifunction $F|_U : U \to Y$ is a lower (upper) slightly α-continuous multifunction.*

Proof. Suppose that $V \subset Y$ is a clopen set. Let $x \in U$ and let $x \in F^-|_U(V)$. Since F is a lower slightly α-continuous multifunction, it follows that there exists $G \in \alpha O(X)$ containing x such that $G \subset F^-(V)$. From here we obtain that $x \in G \cap U \in \alpha O(U)$ and $G \cap U \subset F^-|_U(V)$. Thus, we show that the restriction multifunction $F|_U$ is a lower slightly α-continuous.

The proof of the upper slightly α-continuity of $F|_U$ is similar to the above. □

Lemma 2. *If $A \in \alpha O(Y)$ and $Y \in \alpha O(X)$, then $A \in \alpha O(X)$* [5].

Theorem 7. *Let $\{U_\lambda : \lambda \in \Lambda\}$ be an α-open cover of a space X. Then a multifunction $F : X \to Y$ is upper slightly α-continuous (resp., lower slightly α-continuous) if and only if the restriction $F|_{U_\lambda} : U_\lambda \to Y$ is upper slightly α-continuous (resp., lower slightly α-continuous) for each $\lambda \in \Lambda$.*

Proof. We prove only the case when F is an upper slightly α-continuous.

($\Rightarrow$) Let $\lambda \in \Lambda$ and V be any clopen set of Y. Since F is upper slightly α-continuous, $F^+(V)$ is α-open in X. By Lemma 12, $(F|_{U_\lambda})^+(V) = F^+(V) \cap U_\lambda$ is α-open in U_λ and hence $F|_{U_\lambda}$ is slightly α-continuous.

($\Leftarrow$) Let V be any clopen set of Y. Since $F|_{U_\lambda}$ is slightly α-continuous for each $\lambda \in \Lambda$, $(F|_{U_\lambda})^+(V) = F^+(V) \cap U_\lambda$ is α-open in U_λ. By Lemma 14, $(F|_{U_\lambda})^+(V)$ is α-open in X for each $\lambda \in \Lambda$. We obtain that $F^+(V) = \bigcup_{\lambda\in\Lambda}(F|_{U_\lambda})^+(V)$ is α-open in X. Hence F is upper slightly α-continuous. □

Lemma 3. *For a multifunction $F : X \to Y$, the following holds:*

(1) $G_F^+(A \times B) = A \cap F^+(B)$;
(2) $G_F^-(A \times B) = A \cap F^-(B)$

for any subsets $A \subset X$ and $B \subset Y$ [9].

Theorem 8. *Let $F : X \to Y$ be a multifunction from a topological space (X, τ) to a topological space (Y, υ). If the graph multifunction of F is an upper slightly α-continuous multifunction, then F is an upper slightly α-continuous multifunction.*

Proof. Let $x \in X$ and let $V \subset Y$ be a clopen set such that $x \in F^+(V)$. We obtain that $x \in G_F^+(X \times V)$ and that $X \times V$ is a clopen set. Since graph multifunction G_F is an upper slightly α-continuous, it follows that there exists an α-open set $U \subset X$ containing x such that $U \subset G_F^+(X \times V)$. Since $U \subset G_F^+(X \times V) = X \cap F^+(V) = F^+(V)$, we obtain that $U \subset F^+(V)$. Thus, F is upper slightly α-continuous multifunction. □

Theorem 9. *A multifunction $F : X \to Y$ is lower slightly α-continuous if $G_F : X \to X \times Y$ is lower slightly α-continuous.*

Proof. Suppose that G_F is lower slightly α-continuous. Let $x \in X$ and V be any clopen set of Y such that $x \in F^-(V)$. Then $X \times V$ is clopen in $X \times Y$ and $G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$. Since G_F is lower slightly α-continuous, there exists an α-open set U containing x such that $U \subset G_F^-(X \times V)$. By the previous lemma, we have $U \subset F^-(V)$. This shows that F is lower slightly α-continuous. □

Theorem 10. *Suppose that (X, τ) and (X_α, τ_α) are topological spaces where $\alpha \in J$. Let $F : X \to \prod_{\alpha\in J} X_\alpha$ be a multifunction from X to the product space $\prod_{\alpha\in J} X_\alpha$ and let $P_\alpha : \prod_{\alpha\in J} X_\alpha \to X_\alpha$ be the projection multifunction for each $\alpha \in J$ which is defined by $P_\alpha((x_\alpha)) = \{x_\alpha\}$. If F is an upper (lower) slightly α-continuous multifunction, then $P_\alpha \circ F$ is an upper (lower) slightly α-continuous multifunction for each $\alpha \in J$.*

Proof. Take any $\alpha_0 \in J$. Let V_{α_0} be a clopen set in $(X_{\alpha_0}, \tau_{\alpha_0})$. Then $(P_{\alpha_0} \circ F)^+(V_{\alpha_0}) = F^+(P^+_{\alpha_0}(V_{\alpha_0})) = F^+(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$ (respectively, $(P_{\alpha_0} \circ F)^-(V_{\alpha_0}) = F^-(P^-_{\alpha_0}(V_{\alpha_0})) = F^-(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$). Since F is an upper (lower) slightly α-continuous multifunction and since $V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha$ is a clopen set, it follows that $F^+(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$ (respectively, $F^-(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$) is α-open in (X, τ). It shows that $P_{\alpha_0} \circ F$ is an upper (lower) slightly α-continuous multifunction.

Hence, we obtain that $P_\alpha \circ F$ is an upper (lower) slightly α-continuous multifunction for each $\alpha \in J$. □

Theorem 11. *Suppose that for each $\alpha \in J$, (X_α, τ_α), $(Y_\alpha, \upsilon_\alpha)$ are topological spaces. Let $F_\alpha : X_\alpha \to Y_\alpha$ be a multifunction for each $\alpha \in J$ and let $F : \prod_{\alpha \in J} X_\alpha \to \prod_{\alpha \in J} Y_\alpha$ be defined by $F((x_\alpha)) = \prod_{\alpha \in J} F_\alpha(x_\alpha)$ from the product space $\prod_{\alpha \in J} X_\alpha$ to the product space $\prod_{\alpha \in J} Y_\alpha$. If F is an upper (lower) slightly α-continuous multifunction, then each F_α is an upper (lower) slightly α-continuous multifunction for each $\alpha \in J$.*

Proof. Let $V_\alpha \subset Y_\alpha$ be a clopen set. Then $V_\alpha \times \prod_{\alpha \neq \beta} Y_\beta$ is a clopen set. Since F is an upper (lower) slightly α-continuous multifunction, it follows that $F^+(V_\alpha \times \prod_{\alpha \neq \beta} Y_\beta) = F^+_\alpha(V_\alpha) \times \prod_{\alpha \neq \beta} X_\beta$ ($F^-(V_\alpha \times \prod_{\alpha \neq \beta} Y_\beta) = F^-_\alpha(V_\alpha) \times \prod_{\alpha \neq \beta} X_\beta$) is an α-open set. Consequently, we obtain that $F^+_\alpha(V_\alpha)$ ($F^-_\alpha(V_\alpha)$) is an α-open set. Thus, we show that F_α is an upper (lower) slightly α-continuous multifunction. □

For two multifunctions $F_1 : X_1 \to Y_1$ and $F_2 : X_2 \to Y_2$, the product multifunction $F_1 \times F_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is defined as follows: $(F_1 \times F_2)(x_1, x_2) = F_1(x_1) \times F_2(x_2)$ for every $x_1 \in X_1$ and $x_2 \in X_2$.

Theorem 12. *Suppose that $F_1 : X_1 \to Y_1$, $F_2 : X_2 \to Y_2$ are multifunctions. If $F_1 \times F_2$ is an upper (lower) slightly α-continuous multifunction, then F_1 and F_2 are upper (lower) slightly α-continuous multifunctions.*

Proof. Let $K \subset Y_1$, $H \subset Y_2$ be clopen sets. It is known that $K \times H$ is a clopen set and $(F_1 \times F_2)^+(K \times H) = F^+_1(K) \times F^+_2(H)$. Since $F_1 \times F_2$ is an upper slightly α-continuous multifunction, it follows that $F^+_1(K) \times F^+_2(H)$ is an α-open set. From here, $F^+_1(K)$ and $F^+_2(H)$ are α-open sets. Hence, it is obtained that F_1 and F_2 are upper slightly α-continuous multifunctions.

The proof of the lower slightly α-continuity of F_1 and F_2 is similar to the above argument. □

Theorem 13. *Suppose that (X, τ), (Y, υ), (Z, ω) are topological spaces and $F_1 : X \to Y$, $F_2 : X \to Z$ are multifunctions. Let $F_1 \times F_2 : X \to Y \times Z$ be a multifunction which is defined by $(F_1 \times F_2)(x) = F_1(x) \times F_2(x)$ for each $x \in X$. If $F_1 \times F_2$ is an upper (lower) slightly α-continuous multifunction, then F_1 and F_2 are upper (lower) slightly α-continuous multifunctions.*

Proof. Let $x \in X$ and let $K \subset Y$, $H \subset Z$ be clopen sets such that $x \in F^+_1(K)$ and $x \in F^+_2(H)$. Then we obtain that $F_1(x) \subset K$ and $F_2(x) \subset H$ and from here, $F_1(x) \times F_2(x) =$

$(F_1 \times F_2)(x) \subset K \times H$. We have $x \in (F_1 \times F_2)^+(K \times H)$. Since $F_1 \times F_2$ is upper slightly α-continuous multifunction, it follows that there exists an α-open set U containing x such that $U \subset (F_1 \times F_2)^+(K \times H)$. We obtain that $U \subset F_1^+(K)$ and $U \subset F_2^+(H)$. Thus, we obtain that F_1 and F_2 are upper slightly α-continuous multifunctions.
The proof of the lower slightly α-continuity of F_1 and F_2 is similar to the above. □

Definition 4. Let (X, τ) be a topological space. X is said to be a strongly normal space if for any disjoint closed subsets K and F of X, there exists two clopen sets U and V such that $K \subset U$, $F \subset V$ and $U \cap V = \emptyset$.

Example 3. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then (X, τ) is a strongly normal space.

Recall that a multifunction $F : X \to Y$ is said to be punctually closed if, for each $x \in X$, $F(x)$ is closed.

Theorem 14. *If Y is strongly normal space and $F_i : X_i \to Y$ is an upper slightly α-continuous multifunction such that F_i is punctually closed for $i = 1, 2$, then a set $\{(x_1, x_2) \in X_1 \times X_2 : F_1(x_1) \cap F_2(x_2) \neq \emptyset\}$ is an α-closed set in $X_1 \times X_2$.*

Proof. Let $A = \{(x_1, x_2) \in X_1 \times X_2 : F_1(x_1) \cap F_2(x_2) \neq \emptyset\}$ and $(x_1, x_2) \in (X_1 \times X_2) \setminus A$. Then $F_1(x_1) \cap F_2(x_2) = \emptyset$. Since Y is strongly normal and F_i is punctually closed for $i = 1, 2$, there exist disjoint clopen sets V_1, V_2 such that $F_i(x_i) \subset V_i$ for $i = 1, 2$. Since F_i is upper slightly α-continuous, $F_i^+(V_i)$ is α-open for $i = 1, 2$. Put $U = F_1^+(V_1) \times F_2^+(V_2)$, then U is α-open and $(x_1, x_2) \in U \subset (X_1 \times X_2) \setminus A$. This shows that $(X_1 \times X_2) \setminus A$ is α-open and hence A is α-closed in $(X_1 \times X_2)$. □

Theorem 15. *Let F and G be upper slightly α-continuous and punctually closed multifunctions from a topological space X to a strongly normal topological space Y. Then the set $K = \{x : F(x) \cap G(x) \neq \emptyset\}$ is α-closed in X.*

Proof. Let $x \in X \setminus K$. Then $F(x) \cap G(x) = \emptyset$. Since F and G are punctually closed multifunctions and Y is a strongly normal space, it follows that there exist disjoint clopen sets U and V containing $F(x)$ and $G(x)$, respectively. Since F and G are upper slightly α-continuous, then the sets $F^+(U)$ and $G^+(V)$ are α-open and contain x. Let $H = F^+(U) \cup G^+(V)$. Then H is an α-open set containing x and $H \cap K = \emptyset$. Hence, K is α-closed in X. □

Definition 5. A space X is said to be α-Hausdorff if for each pair of distinct points x and y in X, there exist disjoint α-open sets U and V in X such that $x \in U$ and $y \in V$ [6].

Theorem 16. *Let $F : X \to Y$ be an upper slightly α-continuous multifunction and punctually closed from a topological space X to a strongly normal topological space Y and let $F(x) \cap F(y) = \emptyset$ for each distinct pair $x, y \in X$. Then X is a α-Hausdorff space.*

PROOF. Let x and y be any two distinct points in X. Then we have $F(x) \cap F(y) = \emptyset$. Since Y is a strongly normal space, it follows that there exist disjoint clopen sets U and V containing $F(x)$ and $F(y)$ respectively. Thus $F^+(U)$ and $F^+(V)$ are disjoint α-open sets containing x and y, respectively. Thus, it is obtained that X is α-Hausdorff. □

4. SOME PROPERTIES

Definition 6. A space X is said to be mildly compact if every clopen cover of X has a finite subcover [11].

Definition 7. A space X is said to be α-compact if every α-open cover of X has a finite subcover [3].

Theorem 17. *Let $F : X \to Y$ be an upper slightly α-continuous surjective multifunction such that $F(x)$ is mildly compact for each $x \in X$. If X is an α-compact space, then Y is mildly compact.*

PROOF. Let $\{V_\lambda : \lambda \in \Lambda\}$ be a clopen cover of Y. Since $F(x)$ is mildly compact for each $x \in X$, there exists a finite subset $\Lambda(x)$ of Λ such that $F(x) \subset \bigcup\{V_\lambda : \lambda \in \Lambda(x)\}$. Put

$$V(x) = \bigcup\{V_\lambda : \lambda \in \Lambda(x)\}.$$

Since F is an upper slightly α-continuous, there exists an α-open set $U(x)$ of X containing x such that $F(U(x)) \subset V(x)$. Then the family $\{U(x) : x \in X\}$ is a α-open cover of X and since X is α-compact, there exists a finite number of points, say, x_1, x_2, x_3, $\ldots, x_n$ in X such that $X = \bigcup\{U(x_i) : i = 1, 2, 3, \ldots, n\}$. Hence we have

$$Y = F(X) = F\left(\bigcup_{i=1}^{n} U(x_i)\right) = \bigcup_{i=1}^{n} F(U(x_i)) \subset \bigcup_{i=1}^{n} V(x_i) = \bigcup_{i=1}^{n} \bigcup_{\lambda \in \Lambda(x_i)} V_\lambda.$$

This shows that Y is mildly compact. □

Definition 8. Let $F : X \to Y$ be a multifunction. The multigraph $G(F)$ is said to be α-co-closed if for each $(x, y) \notin G(F)$, there exist α-open set U and clopen set V containing x and y, respectively, such that $(U \times V) \cap G(F) = \emptyset$.

Definition 9. A space X is said to be co-Hausdorff if for each pair of distinct points x and y in X, there exist disjoint clopen sets U and V in X such that $x \in U$ and $y \in V$.

Theorem 18. *If a multifunction $F : X \to Y$ is an upper slightly α-continuous multifunction such that $F(x)$ is mildly compact relative to Y for each $x \in X$ and Y is co-Hausdorff space, then the multigraph $G(F)$ of F is α-co-closed in $X \times Y$.*

PROOF. $(x, y) \notin G(F)$. That is $y \notin F(x)$. Since Y is co-Hausdorff, for each $z \in F(x)$, there exist disjoint clopen sets $V(z)$ and $U(z)$ of Y such that $z \in U(z)$ and $y \in V(y)$.

Then $\{U(z) : z \in F(x)\}$ is clopen cover of $F(x)$ and since $F(x)$ is mildly compact, there exists a finite number of points, say, $z_1, z_2, z_3, \dots, z_n$ in $F(x)$ such that

$$F(x) \subset \bigcup\{U(z_i) : i = 1, 2, 3, \dots, n\}.$$

Put

$$U = \bigcup\{U(z_i) : i = 1, 2, 3, \dots, n\} \text{ and } V = \bigcap\{V(y_i) : i = 1, 2, 3, \dots, n\}.$$

Then U and V are clopen in Y such that $F(x) \subset U$, $y \in V$ and $U \cap V = \emptyset$. Since F is an upper slightly α-continuous multifunction, there exists α-open W containing x such that $F(W) \subset U$. We have $(x, y) \in W \times V \subset (X \times Y) \setminus G(F)$. We obtain that $(W \times V) \cap G(F) = \emptyset$ and hence $G(F)$ is α-co-closed in $X \times Y$. □

Theorem 19. *Let $F : X \to Y$ be a multifunction having α-co-closed multigraph $G(F)$. If B is a mildly compact subset relative to Y, then $F^-(B)$ is α-closed in $X \times Y$.*

Proof. Let $x \in X \setminus F^-(B)$. For each $y \in B$, $(x, y) \notin G(F)$ and there exist an α-open set $U(y) \subset X$ and a clopen set $V(y) \subset Y$ containing x and y, respectively, such that $F(U(y)) \cap V(y) = \emptyset$. That is $U(y) \cap F^-(V(y)) = \emptyset$. Then $\{V(y) : y \in B\}$ is clopen cover of B and since B is mildly compact relative to Y, there exists a finite subset B_0 of B such that $B \subset \bigcup\{V(y) : y \in B_0\}$. Put

$$U = \bigcap\{U(y) : y \in B_0\}.$$

Then U is α-open in X, $x \in U$ and $U \cap F^-(B) = \emptyset$; i. e., $x \in U \subset X \setminus F^-(B)$. This shows that $F^-(B)$ is α-closed in X. □

Recall that a multifunction $F : X \to Y$ is said to be punctually connected if, for each $x \in X$, $F(x)$ is connected.

Definition 10. A space X is called α-connected provided that X is not the union of two disjoint nonempty α-open sets.

Theorem 20. *Let F be a multifunction from an α-connected topological space X onto a topological space Y such that F is punctually connected. If F is an upper slightly α-continuous multifunction, then Y is a connected space.*

Proof. Let $F : X \to Y$ be a an upper slightly α-continuous multifunction from a α-connected topological space X onto a topological space Y. Suppose that Y is not connected and let $Y = H \cup K$ be a partition of Y. Then both H and K are open and closed subsets of Y. Since F is an upper slightly α-continuous multifunction, $F^+(H)$ and $F^+(K)$ are α-open subsets of X. In view of the fact that $F^+(H)$, $F^+(K)$ are disjoint and F is punctually connected, $X = F^+(H) \cup F^+(K)$ is a partition of X. This is contrary to the α-connectedness of X. Hence, it is obtained that Y is a connected space. □

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