Critical case for singularly perturbed linear boundary-value problems of ordinary differential equations

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CRITICAL CASE FOR SINGULARLY PERTURBED LINEAR BOUNDARY-VALUE PROBLEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. The conditions under which a unique asymptotic representation of the solution of boundary-value problems exists for singularly perturbed systems of ordinary differential equations are shown in the work. The solution is obtained with the help of boundary functions and pseudo-inverse matrices.

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1. Statement of the Problem

We consider the singularly perturbed differential system

\[ \varepsilon \dot{x} = Ax + \varepsilon A_1(t)x + \varphi(t), \quad t \in [a, b], \quad 0 < \varepsilon \ll 1, \quad (1.1) \]

\[ lx(\cdot) = h, \quad h \in \mathbb{R}^m, \quad (1.2) \]

where the coefficients of system (1.1) and equation (1.2) satisfy the conditions:

(H1) \( A \) is a constant \((n \times n)\) matrix. If \( \lambda_i \) are eigenvalues of \( A \), then \( \lambda_i = 0, \ i = 1, k, k < n, \ \text{Re} \lambda_i < 0, \ i = k + 1, n, \) as \( p, \ p < k, \) linear independent eigenvectors of matrix \( A \) correspond to the zero eigenvalue;

(H2) \( A_1(t) \) is an \((n \times n)\) matrix, \( A_1(t) \in C^\infty[a, b], \ \varphi(t) \) is an \( n \)-dimensional vector-function \( \varphi(t) \in C^\infty[a, b]; \)

(H3) \( l : C[a, b] \to \mathbb{R}^m \) is an \( m \)-dimensional linear bounded vector-functional, \( l = \text{col}(l_1, \ldots, l_m); \)

(H4) The degenerate (\( \varepsilon = 0 \)) system (1.1), \( Ax_0 + \varphi(t) = 0, \) is solvable with respect to \( x_0. \)

We look for an \( n \)-dimensional vector-function \( x(t, \varepsilon): x(\cdot, \varepsilon) \in C^1[a, b], \ x(t, \cdot) \in C(0, \varepsilon_0), \) satisfying (1.1), (1.2) and following relation \( \lim_{\varepsilon \to 0} x(t, \varepsilon) = x_0(t), \ t \in (a, b). \)
We shall consider the case $m \neq n$ and $p < k$. We use an asymptotic method of the boundary functions and construct an asymptotic series for the boundary-value problem (1.1), (1.2) with $\det A = 0$ (the critical case [10]).

In the case $m = n$ and $p = k$ an asymptotic solution of the Cauchy problem and two point boundary-value problem for linear and quasilinear systems is studied in [10] on the basis of the method of boundary functions. In the non-critical case $m \neq n$ and $\det A \neq 0$ the system is studied in [8]. When $m \neq n$ and $p = k$, the problem (1.1), (1.2) is considered in [8].

The construction of an asymptotic solution of (1.1), (1.2) in this work $m \neq n$, $p < k$ is represented on the basis of generalized inverse matrices and projectors [1, 4, 7] and central canonical form [2, 3].

We denote by $n_1, n_2, \ldots, n_p$ ($\sum_{i=1}^{p} n_i = k$) the lengths of the Jordan cells. We will consider the case where $n_1 > \cdots > n_s$, $n_{s+1} = \cdots = n_{p-1} = n_p = 1$, i.e., the matrix $A$ has a block diagonal representation

$$A = \text{diag}(\bar{A}, J_1, J_2, \ldots, J_s, \Theta_{p-s}),$$

(1.3)

where $\bar{A}$ is a $((n - k) \times (n - k))$ matrix and has eigenvalues with negative real parts, $J_i, i = 1, s$, are $(n_i \times n_i)$ Jordan cells, and $\Theta_{p-s}$ is the $((p - s) \times (p - s))$ zero matrix.

By $A^\dagger$, we denote the unique Moore–Penrose pseudo-inverse $(n \times n)$ matrix of the matrix $A$ [4, 7]. Denote by $P_A$ and $P_{A^*}$ orthoprojectors $P_A : \mathbb{R}^n \to \ker A$, $P_{A^*} : \mathbb{R}^n \to \ker A^*, A^* = A^T$. According to (H1) we find $\text{rank } A = n - p$ and $\text{rank } P_A = \text{rank } P_{A^*} = n - (n - p) = p$. Let $P_{A_p}$ be a $(n \times n)$ matrix with $P$ linear independent columns from the matrix $P_A$, and let $P_{A_p}$ be a $(p \times n)$ matrix with $p$ linear independent rows of the matrix $P_{A_p}$.

Let $C = P_{A_p}P_{A_p}$ be an $(m \times n)$-constant matrix.

**Lemma 1.** $\text{rank } C = p - s$.

**Proof.** The proof is based on the equalities $J_iJ_i^\dagger = \text{diag} (1, 1, \ldots, 1, 0)$ and $J_i^\dagger J_i = \text{diag} (0, 1, \ldots, 1, 1)$. Keeping in mind the representation

$$A^\dagger = \text{diag}(\tilde{A}^{-1}, J_1^\dagger, J_2^\dagger, \ldots, J_s^\dagger, \Theta_{p-s})$$

and the equalities $P_A = E_n - \tilde{A}A$, $P_{A^*} = E_n - AA^\dagger$, we get that $C = P_{A_p}P_{A_p} = \text{diag} (0, E_{p-s})$, i.e., $\text{rank } C = p - s$. \qed

We consider the degenerate differential system

$$C \frac{d}{dt} z(t) = B(t)z(t) + l(t), \quad t \in [a, b],$$

(1.4)

where $C$ is the matrix from Lemma 1.1, $B(t) = P_{A_p}A_1(t)P_{A_p}$ is $(p \times p)$ matrix, and $l$ is a $p$-dimensional vector-function, $l(t) \in C^\infty[a, b]$. 
Let the matrix $B(t)$ have the block representation
\[
\begin{pmatrix}
B_{11}(t) & B_{12}(t) \\
B_{21}(t) & B_{22}(t)
\end{pmatrix},
\]
where the matrices $B_{11}$, $B_{12}$, $B_{21}$, and $B_{22}$ have dimensions $((p-s) \times (p-s))$, $((p-s) \times s)$, $(s \times (p-s))$, and $s \times s$, respectively.

**Lemma 2.** System (1.4) takes the central canonical form if and only if $\det B_{11} \neq 0$ \(\forall t \in [a, b]\).

**Proof.** The proof of Lemma 2 is based on Lemma 1 and the work [3]. \(\square\)

In accordance with Lemma 1 under $p \neq s$ and Lemma 2 ($p \times p$), matrices $P(t)$ and $Q(t)$ exist such that substituting $z(t) = Q(t)y(t)$ and multiplying by $P(t) \frac{d}{dt}$ on the left, the system (1.4) takes central canonical form
\[
\begin{pmatrix}
E_{p-s} & 0 \\
0 & \Theta_s
\end{pmatrix} \frac{dy(t)}{dt} = \begin{pmatrix}
L(t) & 0 \\
0 & E_s
\end{pmatrix} y(t) + \begin{pmatrix}
\mu(t) \\
\nu(t)
\end{pmatrix},
\]
where $\Theta_s$ is the $(s \times s)$ zero matrix, $L(t)$ is a $((p-s) \times (p-s))$ matrix, $E_{p-s}$ and $E_s$ are $((p-s) \times (p-s))$ and $(s \times s)$ unit matrices, respectively, and $\mu(t)$ and $\nu(t)$ are $(p-s)$ and $s$-dimensional vector-functions such that
\[
P(t)y(t) = \begin{pmatrix}
\mu(t) \\
\nu(t)
\end{pmatrix}.
\]

Let the $(p-s)$-dimensional vector-function $u(t)$ and $s$-dimensional vector-function $v(t)$ are such that $y(t) = \begin{pmatrix}
u(t) \\
v(t)
\end{pmatrix}$. Then the system (1.5) takes the form
\[
\begin{align*}
\dot{u}(t) &= L(t)u(t) + \mu(t), \\
0 &= v(t) + \nu(t).
\end{align*}
\]
(1.7)

We denote by $\Phi(t)$ a normal fundamental matrix of the solutions of the system $\dot{x} = L(t)x$. Then system (1.7) has a generalized solution
\[
\begin{align*}
u(t) &= \Phi(t)\Phi^{-1}(t)\eta + \bar{u}(t), \quad \eta \in \mathbb{R}^{p-s}, \\
\nu(t) &= -v(t),
\end{align*}
\]
(1.8)

where $\bar{u}(t) = \Phi(t)\int_a^t \Phi^{-1}(s)\mu(s)ds$.

Let the matrix $Q(t)$ be reduced to the block form $Q(t) = [Q_1(t), Q_2(t)]$, where $Q_1(t)$ is a $(p \times (p-s))$ matrix and $Q_2(t)$ is a $(p \times s)$ matrix. Keeping in mind the substitution $z(t) = Q(t)y(t)$, where $y(t) = [u(t), v(t)]^T$, we obtain
\[
z(t) = Q_1(t)u(t) + Q_2(t)v(t).
\]

In the last equality we substitute solution (1.8). Thus,
\[
z(t) = \Phi(t,a)\eta + \bar{z}(t), \quad t \in [a, b], \eta \in \mathbb{R}^{p-s},
\]
(1.9)
where
\[ \tilde{\Phi}(t, a) = Q_1(t)\Phi(t)\Phi^{-1}(a) \] is a \((p \times (p - s))\) matrix and
\[ \tilde{z}(t) = Q_1(t)\tilde{u}(t) - Q_2(t)\nu(t) \] (1.10)
The following lemma is needed.

**Lemma 3.** Let the matrix \(A\) satisfy condition (H1), and let the vector-function \(f(\tau) \in C[0, +\infty)\) and satisfy the inequality \(\|f(\tau)\| < c_1e^{-\alpha_1\tau}\), where \(\tau \geq 0, c_1 > 0,\) and \(\alpha_1 > 0\). Then there exist positive constants \(c\) and \(\gamma\) such that the system \(dx/d\tau = Ax + f(\tau)\) has a particular solution of the form
\[ x(\tau) = \int_{0}^{\infty} K(\tau, s)f(s)ds, \]
satisfying the inequality \(\|x(\tau)\| \leq c \exp(-\gamma\tau), \ \tau \geq 0,\) where
\[ K(\tau, s) = \begin{cases} X(\tau)PX^{-1}(s) & \text{for } 0 \leq s \leq \tau < \infty, \\ -X(\tau)(I - P)X^{-1}(s) & \text{for } 0 < \tau < s \leq \infty, \end{cases} \]
and \(P\) is the spectral projector of the matrix \(A\) to the left semi-plane.

The lemma is proved analogously to a similar lemma in [5].

2. **Formally asymptotic expansion**

We shall seek for a formally asymptotic expansion of the solution of problem (1.1), (1.2) in the form of the regular and singular series
\[ x(t, \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i(x_i(t) + \Pi_i(\tau), \quad \tau = \frac{t - a}{\varepsilon}, \] (2.1)
where \(x_i(t)\) and \(\Pi_i(\tau)\) are unknown \(n\) vector functions. By \(\Pi_i(\tau)\) (see [10]) we denote the boundary function in a neighbourhood of the point \(t = a\). They will be constructed so that when \(0 < \varepsilon \leq \varepsilon_0\), the inequalities
\[ \|\Pi_i(\tau)\| \leq \gamma_i \exp(-\alpha_i\tau), \] (2.2)
where \(\gamma_i\) and \(\alpha_i\) are positive constants for \(i = 0, 1, 2, \ldots\) and \(\tau \geq 0\), hold in \([a, b]\).

Formally, by substituting (2.1) in (1.1), (1.2), for \(x_i(t)\) we obtain the systems
\[ Ax_i(t) = f_i(t), \quad t \in [a, b], \quad i = 0, 1, \ldots, \] (2.3)
where
\[ f_i(t) = \begin{cases} -\varphi(t) & \text{for } i = 0, \\ L_1(x_{i-1}(t)) & \text{for } i = 1, 2, \ldots, \end{cases} \]
and \( L_1 \) is the differential operator \( L_1(x(t)) = \frac{dx(t)}{dt} - A_1(t)x \). The boundary functions \( \Pi_i(\tau) \) are solutions of the boundary problems

\[
\frac{d}{d\tau} \Pi_i(\tau) = A\Pi_i(\tau) + \psi_i(\tau), \quad \tau \in [0, \tau_b], \quad \tau_b = \frac{b - a}{\varepsilon},
\]

\[
l(x_0(\cdot)) + l\left( \Pi_i \left( \frac{t \cdot a - b}{\varepsilon} \right) \right) = \begin{cases} h & \text{for } i = 0, \\ 0 & \text{for } i = 1, 2, \ldots, \end{cases}
\]

where

\[
\psi_i(\tau) = \begin{cases} 0 & \text{for } i = 0, \\ \sum_{q=1}^{i} \frac{1}{q!} \tau^q A_1^{(q)}(a) \Pi_{i-1-q}(\tau) & \text{for } i = 1, 2, \ldots, \end{cases}
\]

We denote the normal fundamental matrix of the solutions of the homogeneous system \( \frac{dx}{dt} = Ax \), \( \tau \in [0, \tau_b] \), by \( X(\tau) = \exp(\tau A) \). Let \( X_{n-k}(\tau) \) be an \((n \times (n - k))\) matrix with \((n - k)\) columns from the matrix \( X(\tau) \), consisting of exponentially small functions (see [8]).

### 2.1. Obtaining the coefficients \( x_0(t) \) and \( \Pi_0(\tau) \)

Consider systems (2.3)–(2.6) for \( i = 0 \). Then the degenerate system

\[
Ax_0(t) + \varphi(t) = 0
\]

is solvable with respect to \( x_0(t) \) (according to (H4)) if and only if \( P_{A^\top} \varphi(t) = 0 \) for all \( t \in [a, b] \), and it has a solution

\[
x_0(t) = P_{A^\top} \alpha_0(t) - A^\top \varphi(t),
\]

where \( \alpha_0(t) \) is an arbitrary \( p \)-dimensional vector-function.

The general solution of system (2.4) has the form

\[
\Pi_0(\tau) = X_{n-k}(\tau)c_0, \quad c_0 \in \mathbb{R}^{n-k}.
\]

We define the vector-function \( \alpha_0(t) \) by obtaining of \( x_1(t) \). Consider the system \( Ax_1(t) = f_1(t) \), where \( f_1(t) = L_1(x_0(t)) \). The latter system has a solution

\[
x_1(t) = P_{A^\top} \alpha_1(t) + A^\top L_1(x_0(t))
\]

if and only if \( P_{A^\top} L_1(x_0(t)) = 0 \) for all \( t \in [a, b] \). Keeping in mind the representation \( x_0(t) \) from (2.8) and \( L_1 \), we obtain the differential system for \( \alpha_0(t) \),

\[
C \frac{d}{dt} \alpha_0(t) = B(t)\alpha_0(t) + g_0(t), \quad t \in [a, b],
\]

where \( g_0(t) = -P_{A^\top} L_1(A^\top \varphi(t)) \). System (2.11) coincides with system (1.4) at \( h(t) \equiv g_0(t), t \in [a, b] \). Then, according to Lemma 2 and the equality (1.9), we obtain

\[
\alpha_0(t) = \Phi(t, a)\eta_0 + \tilde{\alpha}_0(t), \quad t \in [a, b], \quad \eta_0 \in \mathbb{R}^{p-n},
\]

where \( \tilde{\alpha}_0(t) \) is an arbitrary \((n - k)\)-vector.
where $\tilde{\alpha}(t) = Q_1(t)\tilde{u}_0(t) - Q_2(t)v_0(t)$,

$$\tilde{u}_0(t) = \Phi(t) \int_0^t \Phi^{-1}(s)\mu(s)_0 ds,$$

$$P(t)g_0(t) = \left( \begin{array}{c} \mu_0(t) \\ v_0(t) \end{array} \right),$$

and $\Phi(t, a)$, $Q(t) = [Q_1(t), Q_2(t)]$, and $P(t)$ are the matrices from Section 1. The vector-functions $u_0(t)$ and $v_0(t)$ are solutions of the following system (see (1.7)):

$$\dot{u}_0(t) = L(t)u_0(t) + \mu_0(t),$$

$$0 = v_0(t) + v_0(t).$$

We substitute (2.12) into equality (2.8), and for $x_0(t)$ we obtain

$$x_0(t) = P_{Ap} \Phi(t, a)\eta_0 + P_{Ap} \tilde{\alpha}_0(t) - A^1\varphi(t).$$ (2.13)

Finally, for obtaining the functions $x_0(t)$ and $\Pi_0(t)$ it is sufficient to determine the vectors $\eta_0 \in \mathbb{R}^{p-s}$ and $c_0 \in \mathbb{R}^{n-k}$. In this connection, we use the boundary condition (2.5) for $i = 0$, where we substitute (2.13) and (2.9). We obtain the vectors $\eta_0$ and $c_0$ by the system

$$D_0(\varepsilon)c_0 + S_0\eta_0 = h_0,$$ (2.14)

where $D(\varepsilon) = lX_{n-k}(\cdot)$ is an $(m \times (n-k))$ matrix, $S_0 = lAp\Phi(\cdot, a)$ is an $(n \times (p-s))$ matrix, $h_0 = h - l(A_p\tilde{\alpha}_0(\cdot)) - l(A^1\varphi(\cdot))$ is an $m$-dimensional vector.

Keeping in mind the expression of the matrix $X_{n-k}(\varepsilon)$ and the form of the functional $l(\cdot)$, we assume that $D_0(\varepsilon) = D_0 + O(\varepsilon e^\alpha \exp(-\alpha/\varepsilon))$, where $\alpha > 0$, $s \in \mathbb{N}$, $D_0$ is a $(m \times (n-k))$-constant matrix, and $O(\varepsilon e^\alpha \exp(-\alpha/\varepsilon))$ we denote a matrix consisting of elements infinitely small with respect to $\varepsilon$. Because the elements of the matrix $D_0$ are continuous for all $\varepsilon \in (0, \varepsilon_0]$ and $\lim_{\varepsilon \to 0} D_0(\varepsilon) = D_0$, then we determine the matrix $D_0(\varepsilon)$ for $\varepsilon = 0$, putting $D_0(0) = D_0$. We neglect the exponentially small elements in the matrix $D_0(\varepsilon)$ and system (2.14) takes the form

$$M \left( \begin{array}{c} c_0 \\ \eta_0 \end{array} \right) = h_0,$$ (2.15)

where $M = [D_0, S_0]$ is a $(m \times (n + p - k - s))$ constant matrix.

Let the following condition hold:

(H5) rank $M = m = n - k + p - s$.

Then $\det M \neq 0$ and system (2.15) is always solvable and

$$c_0 = [M^{-1}]_{n-k}h_0,$$

$$\eta_0 = [M^{-1}]_{p-s}h_0,$$ (2.16)

where $[M^{-1}]_{n-k}$ and $[M^{-1}]_{p-s}$ are the first $(n-k)$ and last $(p-s)$ rows of the matrix $M^{-1}$. We should note that in this case $n - m = k - p + s > 0$, i. e., $n > m$. 

We substitute (2.16) into (2.13) and (2.9) and get
\[
\begin{align*}
\dot{x}_0(t) &= P_{A_p} \Phi(t,a)[M^{-1}]_{p-1} h_0 + \tilde{x}_0(t), \\
\Pi_0(t) &= X_{n-k}(t)[M^{-1}]_{n-k} h_0,
\end{align*}
\]  
(2.17)
where \(\tilde{x}_0(t) = P_{A_p} \tilde{\alpha}_0(t) - A^t \tilde{\varphi}(t)\).

2.2. Obtaining the coefficients \(x_1(t)\) and \(\Pi_1(t)\). To obtain the coefficient \(x_1(t)\) from (2.10), it is sufficient to determine the function \(\alpha_1(t)\). This will be realized in terms of the coefficient \(x_2(t)\). System (2.3) under \(i = 2\) has a solution \(x_2(t) = P_{A_p} \alpha_2(t) + A^t L_1(x_1(t))\) if and only if
\[
P_{A_p}^2 L_1(x_1(t)) = 0
\]
for all \(t \in [a,b]\). In the last equation we substitute \(x_1(t)\) from (2.10). Keeping in mind the form of the operator \(L_1\), for determining the function \(\alpha_1(t)\), we obtain the degenerate differential system
\[
C A \frac{d}{dt} \alpha_1(t) + g_1(t) = B(t) \alpha_1(t), \quad t \in [a,b],
\]
(2.18)
where \(g_1(t) = -P_{A_p}^2 L_1(A^t L_1 x_0(t))\).

System (2.18) coincides with system (2.3), (1.4) at \(l(t) \equiv g_1(t), t \in [a,b]\) and in accordance with Lemma 1.2 and equation (1.9), we obtain
\[
\alpha_1(t) = \Phi(t,a) \eta_1 + \tilde{\alpha}_1(t), \quad t \in [a,b], \quad \eta_1 \in \mathbb{R}^{p-s},
\]
(2.19)
where \(\tilde{\alpha}_1(t) = Q_1(t) \tilde{u}_1(t) - Q_2(t) v_1(t)\),
\[
\tilde{u}_1(t) = \Phi(t) \int_a^t \Phi^{-1}(s) u_1(s) ds,
\]
and \(P(t) g_1(t) = \begin{pmatrix} \nu(t) \\ \nu'(t) \end{pmatrix}\). The vector-functions \(u_1(t)\) and \(v_1(t)\) are solutions of system (1.7), where \(u(t) = u_1(t), v(t) = v_1(t)\).

We substitute (2.19) into \(x_1(t)\) from (2.10) and obtain
\[
x_1(t) = P_{A_p} \Phi(t,a) \eta_1 + P_{A_p} \tilde{\alpha}_1(t) + A^t L_1(x_0(t)). \quad (2.20)
\]
In accordance with Lemma 3, the general solution of the system (2.4) at \(i = 1\) is
\[
\Pi_1(t) = X_{n-k}(t) c_1 + \int_0^t K(t,s) \psi_1(s) ds, \quad c_1 \in \mathbb{R}^{n-k}. \quad (2.21)
\]
We substitute (2.20) and (2.21) into (2.5) at \(i = 1\). The constant vectors \(\eta_1\) and \(c_1\) are obtained by the system
\[
D_0(\varepsilon) c_1 + S_0 \eta_1 = h_1(\varepsilon), \quad (2.22)
\]
where
\[
h_1(\varepsilon) = -l \left( \int_0^{\infty} K \left( \frac{t}{\varepsilon}, s \right) \psi_1(s) ds \right) - l \left( P_{A_p} \tilde{\alpha}_1(\cdot) \right) - l \left( A^t L_1(x_0(\cdot)) \right).
\]
Obviously,
\[ h_1(\varepsilon) = h_{10} + O(\varepsilon^{-1} \exp(-\alpha/\varepsilon)), \]
i.e., \( h_1(\varepsilon) \) is with continuous elements for all \( \varepsilon \in (\varepsilon_0] \) and \( \lim_{\varepsilon \to 0} h_1(\varepsilon) = h_{10} \). Then we determine \( h_1(\varepsilon) \) for \( \varepsilon = 0 \), putting \( h_1(0) = h_{10} \). Since \( D_0(0) = D_0 \), in system (2.22) we neglect the exponentially small elements and obtain
\[ M^{(c_1)}_{(\eta_1)} = h_{10}, \]
where \( M \) is the matrix from Section 2.1.

In accordance with condition (H5), the solution of the system (2.23) we substitute into (2.20) and (2.21). Consequently, the coefficients \( x_1(t) \) and \( \Pi_1(\tau) \) have the form
\[ x_1(t) = P_{A_p} \Phi(t, a)[M^{-1}]_{p^{-1}} h_{10} + \bar{x}_1(t), \]
\[ \Pi_1(\tau) = X_{n-k}(\tau)[M^{-1}]_{n-k} h_{10} + \bar{\Pi}_1(\tau), \]
where \( \bar{x}_1(t) = P_{A_p} \bar{x}_1(t) + A^T L_1(x_0(t)) \) and \( \bar{\Pi}_1(\tau) = \int_0^{+\infty} K(\tau, s) \psi_1(s)ds \).

2.3. Determining the coefficients \( x_q(t) \) and \( \Pi_q(\tau) \), \( q > 1 \). The inductive approach shows that the coefficients \( x_q(t) \) and \( \Pi_q(\tau) \) (\( q > 1 \)) have the form
\[ x_q(t) = P_{A_p} \Phi(t, a)[M^{-1}]_{p^{-1}} h_{q0} + \bar{x}_q(t), \]
\[ \Pi_q(\tau) = X_{n-k}(\tau)[M^{-1}]_{n-k} h_{q0} + \bar{\Pi}_q(\tau), \]
where
\[ h_{q0} = \lim_{\varepsilon \to 0} h_q(\varepsilon), \]
\[ h_q(\varepsilon) = -i \left( \int_0^{+\infty} K \left( \frac{\cdot - a}{\varepsilon}, s \right) \psi_q(s)ds \right) - l(P_{A_p} \bar{x}_q(\cdot)) = l(A^T L_1(x_{q-1}(\cdot))), \]
\[ \bar{x}_q(t) = P_{A_p} \bar{x}_q(t) + A^T L_1(x_{q-1}(t)), \]
\[ \bar{\Pi}_q(\tau) = \int_0^{+\infty} K(\tau, s) \psi_q(s)ds. \]

Assume that the coefficients \( x_i(t) \) and \( \Pi_i(\tau) \) \( i = \frac{1}{q-1} \) are determined. System (2.3) for \( i = q \) a solution
\[ x_q(t) = P_{A_p} a_q(t) + A^T L_1(x_{q-1}(t)) \]
if and only if \( P_{A_q^*} L_1(x_q-1(t)) = 0 \) for all \( t \in [a, b] \). However, this equality is fulfilled because it is used in obtaining the function \( a_{q-1}(t) \). This solution \( a_{q-1}(t) \) participates in \( x_{q-1}(t) \), which with respect to the induction hypothesis is determined completely.
The function $\alpha_q(t)$ is obtained from the solvability condition $P_{\bar{\alpha}}^p L_1(x_q(t)) = 0$, \(\forall t \in [a, b]\) of system (2.3) for \(i = q + 1\) $A x_{q+1}(t) = L_1(x_q(t))$. Thus, we obtain the following differential system (see (1.4)):

$$\frac{d}{dt} \alpha_q(t) = B(t) \alpha_q(t) + g_q(t), \quad t \in [a, b],$$

where $g_q(t) = -P_{\bar{\alpha}}^p L_1(A^\dagger L_1(x_q(t)))$.

By Lemma 2 and equation (1.9) we find

$$\alpha_q(t) = \Phi(t, a) \eta_q + \bar{\alpha}_q(t), \quad t \in [a, b], \quad \eta_q \in \mathbb{R}^{p-s},$$

where $\bar{\alpha}_q(t) = Q_1(t) \bar{\alpha}_q(t) - Q_2(t) v_q(t)$,

$$\bar{\alpha}_q(t) = \Phi(t) \int_a^t \Phi^{-1} (s) \mu_q(s) d s,$$

and $P(t) g_q(t) = \left( \frac{\mu_q(t)}{v_q(t)} \right)$.

The vector-functions $u_q(t)$ and $v_q(t)$ are solutions of system (1.7), where $u(t) = u_q(t)$, $v(t) = v_q(t)$.

We substitute (2.28) into (2.27) and obtain

$$x_q(t) = P_{\bar{\alpha}}^p \Phi(t, a) \eta_q + P_{\bar{\alpha}}^p \bar{\alpha}_q(t) + A^\dagger L_1(x_{q-1}(t)), \quad \eta_q \in \mathbb{R}^{p-s}.$$  

The general solution of system (2.4) for $i = q$ is

$$\Pi_q(\tau) = X_{n-k}(\tau) c_q + \int_0^{\tau} K(\tau, s) \psi_q(s) d s, \quad c_q \in \mathbb{R}^{n-k}.$$  

We substitute (2.29) and (2.30) in the boundary condition (2.5) for $i = q$ and get the system

$$D_0(\varepsilon) c_q + S_0 h_q = h_q(\varepsilon),$$

where

$$h_q(\varepsilon) = - \int_0^{\infty} K\left(\frac{a}{\varepsilon}, s\right) \psi_q(s) d s - l(P_{\bar{\alpha}}^p \bar{\alpha}_q(\cdot)) - l(A^\dagger L_1(x_{q-1}(\cdot))).$$

Since

$$D_0(\varepsilon) = D_0 + O(\varepsilon^\delta \exp (-\alpha/\varepsilon)), \quad D_0 = \lim_{\varepsilon \to 0} D_0(\varepsilon), \quad h_q(\varepsilon) = h_{q0} + O(\varepsilon^\delta \exp (-\alpha/\varepsilon)),$$

and $h_{q0} = \lim_{\varepsilon \to 0} h_q(\varepsilon)$, after ignoring the exponentially small elements, the last system takes the form

$$M\begin{pmatrix} c_q \\ \eta_q \end{pmatrix} = h_{q0},$$

with the solution (see (H5))

$$c_q = [M^{-1}]_{n-k} h_{q0}, \quad \eta_q = [M^{-1}]_{p-s} h_{q0}.$$  

We substitute (2.31) in (2.29) and (2.30) and obtain the equations (2.25), (2.26).
All the boundary functions $\Pi_i(\tau)$ satisfy inequalities (2.2). This follows from Lemma 3 and the inequality
$$||X_{n-k}(\tau)|| \leq c_1 \exp(-\beta_1 \tau),$$
where $c_1 > 0$, $\beta_1 > 0$, and $\tau > 0$. After sequential analysis we get
$$||\Pi_0(\tau)|| \leq ||X_{n-k}(\tau)|| ||M^{-1}_{\Pi_0}|| ||h_0|| \leq c_1 \exp(-\beta_1 \tau)c_2c_3 = \gamma_0 \exp(-\alpha_0 \tau),$$
where $||M^{-1}_{\Pi_0}|| \leq c_2$, $||h_0|| \leq c_3$, $\gamma_0 = c_1c_2c_3$, $\alpha_0 = \beta_1$, and
$$||\Pi_1(\tau)|| \leq ||X_{n-k}(\tau)|| ||M^{-1}_{\Pi_0}|| ||h_{10}|| + ||\Pi_1(\tau)|| \leq c_1 \exp(-\beta_1 \tau)c_2c_3 + c_1 \exp(-\beta_1 \tau) \leq (c_1c_2c_3 + c_1) \exp(-\alpha_1 \tau) = \gamma_1 \exp(-\alpha_1 \tau),$$
where $|h_{10}| \leq c_3$, $||\Pi_1(\tau)|| \leq c_1 \exp(-\beta_1 \tau)$, and $\alpha_1 = \max(\beta_1, \beta_1)$. Finally,
$$||\Pi_q(\tau)|| \leq ||X_{n-k}(\tau)|| ||M^{-1}_{\Pi_0}|| ||h_{q0}|| + ||\Pi_q(\tau)|| \leq c_1 \exp(-\beta_1 \tau)c_2c_3 + c_1 \exp(-\beta_q \tau) \leq (c_1c_2c_3 + c_1) \exp(-\alpha_q \tau) = \gamma_q \exp(-\alpha_q \tau),$$
where $|h_{q0}| \leq c_3$, $||\Pi_q(\tau)|| \leq c_1 \exp(-\beta_q \tau)$, and $\alpha_q = \max(\beta_1, \beta_q)$. Thus, the following theorem is true.

**Theorem 1.** Let conditions (H1)–(H5) hold and let $\det B_{11}(t) \neq 0$. Then the boundary-value problems (1.1), (1.2) have a formally asymptotic solution of form (2.1). The coefficients of the regular and singular series have representations (2.17) and (2.25) for $q = 1, 2, \ldots$. For the boundary functions, the following estimate holds:
$$||\Pi_q(\tau)|| \leq \gamma_q \exp(-\alpha_q \tau), \quad q = 0, 1, 2, \ldots,$$
where $\gamma_q$ and $\alpha_q$ are positive constants. Moreover, the equality
$$\lim_{\varepsilon \to 0} x(t, \varepsilon) = x_0(t)$$
holds for $t \in (a, b]$.

**Remark 1.** The case where $\text{rank } M = n_1 < \min(m, n - k + p - s)$ and $p = s$ is of independent interest.

**3. A bound of the remainder term of the asymptotic series**

The solution of the boundary-value problem (1.1), (1.2) we seek in the form
$$x(t, \varepsilon) = X_n(t, \varepsilon) + u_n(t, \varepsilon),$$
where $X_n(t, \varepsilon) = \sum_{i=0}^{n} e^i(x_i(t) + \Pi_i(\tau))$, $\tau = \frac{t-a}{b-a}$, $t \in [a, b]$.

We shall prove that, for $t \in [a, b]$ and $\varepsilon \in (0, \varepsilon_0]$, the function $u_n(t, \varepsilon)$ satisfies the inequality $||u_n(t, \varepsilon)|| \leq K\varepsilon^{n+1}$, where $K > 0$ and $\lim_{\varepsilon \to 0} x(t, \varepsilon) = x_0(t)$. 
Let the smoothness degree of the elements of the matrix $A_1(t)$ and the function $\varphi(t)$ is $n + 2$.

If $u_n(t, \varepsilon) = \varepsilon^{n+1}(x_{n+1}(t) + \Pi_{n+1}) + u_{n+1}(t, \varepsilon)$ and we should prove that $\|u_{n+1}(t, \varepsilon)\| \leq \bar{K}\varepsilon^{n+1}$, $\bar{K} > 0$, then there would exist a positive constant $K$ such that $\|u_n(t, \varepsilon)\| \leq K\varepsilon^{n+1}$.

Substituting $x(t, \varepsilon) = \chi_{n+1}(t, \varepsilon) + u_{n+1}(t, \varepsilon)$ in problem (1.1), (1.2), for the determination of $u_{n+1}(t, \varepsilon)$, we get the boundary-value problem

$$
\varepsilon \frac{d u_{n+1}(t, \varepsilon)}{d t} = A u_{n+1}(t, \varepsilon) + G(t, u_{n+1}, \varepsilon),
$$

(3.2)

$$
l(u_{n+1}, \varepsilon)) = 0.
$$

(3.3)

The function $G(t, u_{n+1}, \varepsilon)$ has the form

$$
G(t, u_{n+1}(t, \varepsilon), \varepsilon) = A X_{n+1}(t, \varepsilon) + \varepsilon A_1(t, \varepsilon)[X_{n+1}(t, \varepsilon) + u_{n+1}(t, \varepsilon)] +
$$

$$
+ \varphi(t) - \varepsilon \frac{d X_{n+1}(t, \varepsilon)}{d t}
$$

and satisfies the following conditions:

I. $\|G(t, 0, \varepsilon)\| \leq \xi \varepsilon^{n+2}$, where $\xi > 0$;

II. For all $\eta > 0$, exists $\delta = \delta(\eta)$ and $\varepsilon_0 = \varepsilon_0(\eta)$ such that if $\|u_{n+1}'\| \leq \delta$ and $\|u_{n+1}''\| \leq \delta$, then

$$
\|G(t, u_{n+1}', \varepsilon) - G(t, u_{n+1}''', \varepsilon)\| \leq \eta \|u_{n+1}' - u_{n+1}''\|
$$

for $t \in [a, b]$ and $0 < \varepsilon \leq \varepsilon_0$.

Let $A = \text{diag}(\bar{A}, \bar{A})$, $\bar{A} = \text{diag}(J, \Theta_{p-s})$ is a $(k \times k)$ matrix, $J = \text{diag}(J_1, \ldots, J_s)$ is a $((k - p + s) \times (k - p + s))$ matrix. Then we represent $u_{n+1}$ in the form

$$
u_{n+1}(t, \varepsilon) = (\omega_1(t, \varepsilon), \omega_2(t, \varepsilon), \omega_3(t, \varepsilon))^T,
$$

where $\omega_1(t, \varepsilon)$ is a $(n - k)$-dimensional vector, $\omega_2(t, \varepsilon)$ is a $(k - p + s)$-dimensional vector, and $\omega_3(t, \varepsilon)$ is a $(p - s)$-dimensional vector.

We introduce the following notation:

$$
A_1(t, \varepsilon) = \begin{pmatrix} A_{111}(t, \varepsilon) & A_{112}(t, \varepsilon) \\ A_{121}(t, \varepsilon) & A_{122}(t, \varepsilon) \end{pmatrix},
$$

where $A_{111}(t, \varepsilon)$ is a $(n - k) \times (n - k)$ matrix, $A_{112}(t, \varepsilon)$ is a $(n - k) \times k$ matrix, $A_{121}(t, \varepsilon)$ is a $(k \times (n - k))$ matrix, $A_{122}(t, \varepsilon)$ is a $(k \times k)$ matrix;

$$
A_{112}(t) = \begin{pmatrix} B_1(t) & B_2(t) \end{pmatrix}, \quad A_{121}(t) = \begin{pmatrix} C_1(t) \\ C_2(t) \end{pmatrix}, \quad A_{122}(t) = \begin{pmatrix} D_{11}(t) & D_{12}(t) \\ D_{21}(t) & D_{22}(t) \end{pmatrix},
$$

where $B_1(t)$ is a $(n - k) \times (k - p + s)$ matrix, $B_2(t)$ is a $(n - k) \times (p - s)$ matrix, $C_1(t)$ is a $(k - p + s) \times (n - k)$ matrix, $C_2(t)$ is a $(p - s) \times (n - k)$ matrix, $D_{11}(t)$ is
a \((k - p + s) \times (k - p + s)\) matrix, \(D_{12}(t)\) is a \((k - p + s) \times (p - s)\) matrix, \(D_{21}(t)\) is a \((p - s) \times (k - p + s)\) matrix, \(D_{22}(t)\) is a \((p - s) \times (p - s)\) matrix;

\[
G(t, 0, 0, \varepsilon) = \begin{pmatrix} G_1(t, 0, 0, 0, \varepsilon) \\ G_2(t, 0, 0, 0, \varepsilon) \\ G_3(t, 0, 0, 0, \varepsilon) \end{pmatrix},
\]

where \(G_1(t, 0, 0, 0, \varepsilon)\) is a \((n - k)\)-dimensional vector, \(G_2(t, 0, 0, 0, \varepsilon)\) is a \((k - p + s)\)-dimensional vector, \(G_3(t, 0, 0, 0, \varepsilon)\) is a \((p - s)\)-dimensional vector.

System (3.2) takes the form

\[
\begin{align*}
\varepsilon \frac{d\omega_1}{dt} &= \dot{\lambda}\omega_1 + \varepsilon A_{111}(t)\omega_1 + \varepsilon B_1(t)\omega_2 + \varepsilon B_2(t)\omega_3 + G_1(t, 0, 0, 0, \varepsilon), \\
\varepsilon \frac{d\omega_2}{dt} &= (J + \varepsilon D_{11}(t))\omega_2 + \varepsilon D_{12}(t)\omega_3 + \varepsilon C_1(t)\omega_1 + G_2(t, 0, 0, 0, \varepsilon), \\
\varepsilon \frac{d\omega_3}{dt} &= \varepsilon D_{21}(t)\omega_2 + \varepsilon D_{22}(t)\omega_3 + \varepsilon C_2(t)\omega_1 + G_3(t, 0, 0, 0, \varepsilon).
\end{align*}
\] (3.4)

(3.5)

(3.6)

Obviously, the inequalities \(||G_i(t, 0, 0, 0, \varepsilon)|| \leq c_{1i}e^{\sigma_1 t}, c_{1i} > 0, i = 1, 2, 3\) hold on \([a, b]\).

Let \(W(t, s, \varepsilon)\) and \(V(t, s)\) be the fundamental matrices for the homogeneous systems \(\varepsilon \ddot{x} = \tilde{A}\dot{x}\) and \(\dot{x} = \tilde{D}_{22}x\). Here, \(W(s, s, \varepsilon) = E_{n-k}\) and \(V(s, s) = \tilde{E}_{p-s}\) are the unit matrices.

Let the Cauchy problem for the homogeneous system \(\varepsilon \ddot{x} = (J + \varepsilon D_{11}(t))\dot{x}\) have only a trivial solution, and system (3.4) has the particular solution

\[
\omega_2(t, \varepsilon) = \int_a^b K(t, s, \varepsilon) [\varepsilon D_{12}(s)\omega_3 + \varepsilon C_1(s)\omega_1 + G_2(s, 0, 0, 0, \varepsilon)] ds, \quad t \in [a, b],
\]

where

\[
K(t, s, \varepsilon) = \begin{cases} \frac{1}{\varepsilon} \tilde{X}(t, \varepsilon) \tilde{X}^{-1}(s, \varepsilon), & \tau_{i-1} \leq s \leq t, \\
0, & \tau_{i-1} \leq t \leq s,
\end{cases}
\]

if the eigenvalues of the matrix \(J + \varepsilon D_{11}(t)\) are purely imaginary and

\[
K(t, s, \varepsilon) = \begin{cases} \frac{1}{\varepsilon} \tilde{X}(t, \varepsilon) P \tilde{X}^{-1}(s, \varepsilon), & \tau_{i-1} \leq s \leq t, \\
-\frac{1}{\varepsilon} \tilde{X}(t, \varepsilon) (I - P) \tilde{X}^{-1}(s, \varepsilon), & \tau_{i-1} \leq t \leq s,
\end{cases}
\]

if the eigenvalues are with a positive or negative real part. The matrix \(P\) is a spectral projector of the matrix \(J + \varepsilon D_{11}(t)\) on the left half-plane, and \(\tilde{X}(t, \varepsilon)\) is a normal fundamental matrix for the system \(\varepsilon \ddot{x} = (J + \varepsilon D_{11}(t))\dot{x}\).

Obviously, \(\int_a^b ||K(t, s, \varepsilon)|| ds \leq \xi_1, \xi_1 > 0, \) for \(t \in [a, b], \varepsilon \in (0, \varepsilon_0]\).
Lemma 4 ([6, 10]). For the matrix $W(t, s, \varepsilon)$, when $a < s \leq t \leq b$, $0 < \varepsilon \leq \varepsilon_0$, the exponential estimate

$$
\|W(t, s, \varepsilon)\| \leq \beta \exp\left(-\alpha \left(\frac{t-s}{\varepsilon}\right)\right), \quad a \leq s \leq t \leq b,
$$

is fulfilled, where $\alpha > 0$, $\beta > 0$.

It is clear that $\|V(t, s, \varepsilon)\| \leq \beta_1$, where $a \leq s \leq t \leq b$, $\beta_1 > 0$.

Lemma 5. Any continuous solution of system (3.4)–(3.6) is a solution of the system of integral equations

$$
\omega_1(t, \varepsilon) = W(t, a, \varepsilon)\omega_1(a, \varepsilon) + \int_a^t \frac{1}{\varepsilon} [\varepsilon A_{11}(s)\omega_1(s, \varepsilon) + \varepsilon B_1(s)\omega_2(s, \varepsilon) + G_1(s, 0, 0, \varepsilon)] ds,
$$

(3.7)

$$
\omega_2(t, \varepsilon) = \int_a^b K(t, s, \varepsilon) [\varepsilon D_{12}(s)\omega_3(s, \varepsilon) + \varepsilon C_1(s)\omega_1(s, \varepsilon)]
+ G_2(s, 0, 0, \varepsilon)] ds,
$$

(3.8)

$$
\omega_3(t, \varepsilon) = V(t, a)\omega_3(a, \varepsilon) + \int_a^t V(t, s) \frac{1}{\varepsilon} [\varepsilon D_{21}(t)\omega_2(s, \varepsilon) + \varepsilon D_{22}(s)\omega_3(s, \varepsilon) + G_3(s, 0, 0, \varepsilon)] ds.
$$

(3.9)

We substitute $u_{n+1}(t, \varepsilon) = (\omega_1(t, \varepsilon), \omega_2(t, \varepsilon), \omega_3(t, \varepsilon))^T$ into the boundary condition (3.3) and obtain

$$
\bar{I}_1\omega_1(\cdot, \varepsilon) + \bar{I}_2\omega_2(\cdot, \varepsilon) + \bar{I}_3\omega_3(\cdot, \varepsilon) = 0,
$$

where $\bar{I}_i$, $i = 1, 2, 3$ are linear $m$-dimensional bounded functionals. After transformations using (3.7)–(3.9), we obtain

$$
\omega_1(t, \varepsilon) = W_i(t, a, \varepsilon)\omega_1(a, \varepsilon) + V_i(t, a, \varepsilon)\omega_3(a, \varepsilon) + S_i(t, \omega_1, \omega_3, a, \varepsilon),
$$

(3.10)

$i = 1, 2, 3$, where $W_i(t, a, \varepsilon) = W(t, a, \varepsilon)$, and $W_i$, $V_i$, $S_i$, $i = 1, 2, 3$, are functions such that, for all $t \in [a, b]$ and $\varepsilon \in (0, \varepsilon_0]$,

$$
\|W_i(t, a, \varepsilon)\| \leq k_i \varepsilon, \quad k_i > 0, \quad i = 1, 2,
$$

$$
\|V_i(t, a, \varepsilon)\| \leq d_i \varepsilon, \quad d_i > 0, \quad i = 1, 2,
$$

$$
\|S_i(t, 0, 0, 0, a, \varepsilon)\| \leq \beta_2 + \varepsilon d_3, \quad \beta_2 > 0, \quad d_3 > 0,
$$

(3.11)

and

$$
\|S_i(t, \omega_1^2, \omega_3^2, a, \varepsilon) - S_i(t, \omega_1^1, \omega_3^1, a, \varepsilon)\| \leq \varepsilon r_1 \max_{s \in [a, b]} \left(\|\omega_1^2(t, \varepsilon) - \omega_1^1(t, \varepsilon)\| + \|\omega_3^2(t, \varepsilon) - \omega_3^1(t, \varepsilon)\|\right),
$$

(3.12)
where \( r_i > 0, \ i = 1, 2, 3 \). It follows from relation (3.10) that the vector \( \omega(a, \varepsilon) = (\omega_1(a, \varepsilon), \omega_3(a, \varepsilon))^T \) is determined by the equation

\[
R(\varepsilon)\omega(a, \varepsilon) = q(a, \omega_1, \omega_3),
\]

(3.13)

where \( R(\varepsilon) = [R_1(\varepsilon) \ R_2(\varepsilon)] \) is an \((m \times (n + p - k - s))\) matrix, \( R_1(\varepsilon) = \bar{I}_1 W_1(\cdot, a, \varepsilon) + \bar{I}_2 W_2(\cdot, a, \varepsilon) + \bar{I}_3 W_3(\cdot, a, \varepsilon) \) is an \((m \times (n-k))\) matrix, \( R_2(\varepsilon) = \bar{I}_1 V_1(\cdot, a, \varepsilon) + \bar{I}_2 V_2(\cdot, a, \varepsilon) + \bar{I}_3 V_3(\cdot, a, \varepsilon) \) is an \((m \times (p-s))\) matrix, and

\[
q(\varepsilon, \omega_1, \omega_3) = -\bar{I}_1 S_1 (\cdot, \omega_1, \omega_3, a, \varepsilon) - \bar{I}_2 S_2 (\cdot, \omega_1, \omega_3, a, \varepsilon) - \bar{I}_3 S_3 (\cdot, \omega_1, \omega_3, a, \varepsilon)
\]

is an \( m \)-dimensional vector. Also, one has \( ||q(\varepsilon, 0, 0)|| \leq c_4 \varepsilon^{n+1} \), \( c_4 > 0 \), and

\[
||q(\varepsilon, \omega^1_1, \omega^2_3) - q(\varepsilon, \omega^3_1, \omega^3_3)|| \leq \varepsilon r_4 \max_{i} \left( ||\omega^1_i - \omega^3_i|| + ||\omega^2_i - \omega^3_i|| \right),
\]

where \( r_4 > 0 \). Since

\[
R(\varepsilon) = R_0 + O\left(\exp\left(-\frac{\alpha}{\varepsilon}\right)\right),
\]

where \( R_0 \) is a constant matrix, then the following condition is fulfilled:

(H6) \( m = n + p - k - s; \ \det R(\varepsilon) \neq 0 \ \forall \varepsilon \in [0, \varepsilon_0] \).

System (3.13) is always solvable and

\[
\omega_1(a, \varepsilon) = [R^{-1}]_{n-k} q(a, \omega_1, \omega_3),
\]

\[
\omega_3(a, \varepsilon) = [R^{-1}]_{p-s} q(a, \omega_1, \omega_3),
\]

(3.14)

We shall substitute (3.14) into (3.7)–(3.9) and obtain a system which will be solved by the method of successive approximations. Let

\[
\omega^0_i(t, \varepsilon) = 0,
\]

\[
\omega^{i+1}_i(t, \varepsilon) = W_i(t, a, \varepsilon)[R^{-1}]_{n-k} q(a, \omega^1_i, \omega^3_i) + V_i(t, a, \varepsilon)[R^{-1}]_{p-s} q(a, \omega^1_i, \omega^3_i) + S_i(t, \omega^1_i, \omega^3_i, a, \varepsilon), \quad i = 1, 2, 3,
\]

(3.15)

be the Picard successive approximations.

**Theorem 2.** Let the conditions of Theorem 1 and assumption (H6) be fulfilled. If \( ||R^{-1}|| \leq c_R \), then there exists a positive constant \( K \) such that the asymptotic solution of the boundary-value problem (1.1), (1.2) has representation (3.1), where \( u_0(t, \varepsilon) \) satisfies the inequality

\[
||u_0(t, \varepsilon)|| \leq K \varepsilon^{n+1}.
\]

Moreover, \( x(t, \varepsilon) \) approaches the generating system when \( \varepsilon \to 0 \) and \( t \in (a, b) \).

**Proof.** By virtue of (3.10), (3.11), and (3.12), for the first approximation, we have

\[
\max_{\varepsilon \in [a, b]} ||\omega^1_i(t, \varepsilon) - \omega^0_i(t, \varepsilon)|| \leq K_{i1}, \quad K_{i1} > 0,
\]

where the constant \( K_{i1} \varepsilon^{n+1} \) is determined by the constants \( c_R, k_i, d_i, e_i, \) and \( r_i \).
Let $K^1 = \max_i (K_{i1})$ and $K^1 \varepsilon^{n+1} = \delta$. For the last approximation we have

$$\max_{t \in [a, b]} \| \omega^2_i(t, \varepsilon) - \omega^1_i(t, \varepsilon) \| \leq \varepsilon K_{i2}, \quad K_{i2} > 0, \ i = 1, 2, 3.$$  

Let $\varepsilon_0 = \frac{1}{2} \min_i (1/K_{i2})$. Then

$$\max_{t \in [a, b]} \| \omega^2_i(t, \varepsilon) - \omega^1_i(t, \varepsilon) \| \leq \frac{1}{2} \delta = \frac{1}{2^2} 2\delta.$$  

Inductively we obtain

$$\max_{t \in [a, b]} \| \omega^{k+1}_i(t, \varepsilon) - \omega^k_i(t, \varepsilon) \| \leq \frac{1}{2^{k+1}} 2\delta.$$  

This reveals that in the segment $[a, b]$, when $\varepsilon$ is sufficiently small, the successive approximations (3.15) are absolutely and uniformly convergent. In addition, we have

$$\| \omega^{k+1}_i(t, \varepsilon) \| \leq \sum_{j=1}^{k+1} \| \omega^j_i(t, \varepsilon) - \omega^{j-1}_i(t, \varepsilon) \| \leq \left( 1 + \frac{1}{2} + \cdots + \frac{1}{2^k} \right) \delta \leq \left( 1 + \frac{1}{2} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}} + \cdots \right) \delta = 2\delta.$$  

Let

$$\lim_{k \to \infty} \omega^k_i(t, \varepsilon) = \omega_i(t, \varepsilon)$$

satisfy (3.10) identically. Then, on the interval $[a, b]$, for $\varepsilon \to 0$, the inequality

$$\| \omega_i(t, \varepsilon) \| \leq 2\delta$$

is fulfilled. Consequently, system (3.10) has an unique continuous solution, which does not escape from the domain $\{ (t, \omega) | a \leq t \leq b, \| \omega \| \leq 2\delta \}$. Then, for all $t \in [a, b]$ and $\varepsilon \in (0, \varepsilon_0]$,

$$\| u_{n+1}(t, \varepsilon) \| \leq \sum_{i=1}^{3} \| \omega_i(t, \varepsilon) \| \leq 6\delta = 6K^1 \varepsilon^{n+1},$$

i. e., there exists a positive constant $K$ such that the inequality

$$\| u_n(t, \varepsilon) \| \leq K \varepsilon^{n+1}$$

is fulfilled and

$$\lim_{\varepsilon \to 0} x(t, \varepsilon) = x_0(t)$$

for all $t \in [a, b]$. □
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