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ON Λ-STATISTICAL CONVERGENCE OF ORDER α IN RANDOM 2-NORMED SPACE

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Abstract. Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ with $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. In the present paper, we introduce the notion of Λ-statistical convergence of order α, Λ-statistical Cauchy sequences of order α in random 2-normed spaces and obtain some results. We display examples which show that our method of convergence is more general in random 2-normed space.

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1. Introduction and background

The idea of the statistical convergence was given by Zygmund [25] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Fast [5] and Steinhaus [23] and then reintroduced by Schoenberg [20] independently. Over the years, statistical convergence has been developed in [3, 6, 7, 12, 16, 17, 24] and turned out very useful to resolve many convergence problems arising in Analysis.

Definition 1 ([5]). A sequence $x = (x_k)$ of numbers is said to be statistically convergent to a number L if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - L| > \epsilon\}| = 0,$$

where vertical bars denotes the cardinality of enclosed set. In this case, we write $S - \lim_{k\to\infty} x_k = L$.

In literature, several interesting generalizations of statistical convergence have been appeared. One among these is λ-statistical convergence given by Mursaleen [14] with the help of a non-decreasing sequence $\lambda = (\lambda_n)$. Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ with

$$\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1.$$

The idea of λ-statistical convergence can be connected to the generalized de la Vallée-Poussin mean. It is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k, \quad \text{where } I_n = [n - \lambda_n + 1, n] \text{ for } n = 1, 2, \ldots$$

Definition 2 ([14])**.** A sequence $x = (x_k)$ of numbers is said to be λ-statistically convergent to a number L provided that for every $\epsilon > 0$,

$$\lim_{n \to \infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - L| \geq \epsilon\}| = 0.$$

In this case, the number L is called λ-statistical limit of the sequence $x = (x_k)$ and we write $S_\lambda - \lim_{k \to \infty} x_k = L$.

Recently, for $\alpha \in (0, 1]$ Çolak and Bektaş[2] generalized Definition 2 in terms of λ-statistical convergence of order α and obtained some analogous results.

Definition 3 ([2])**.** Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers as defined above and $0 < \alpha \leq 1$ be given. A sequence $x = (x_k)$ of numbers is said to be λ-statistically convergent of order α if there is a number L such that

$$\lim_{n \to \infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : |x_k - L| \geq \epsilon\}| = 0.$$

In this case, we write $S_\lambda^\alpha - \lim_{k \to \infty} x_k = L$.

We next quote the following definition due to Mursaleen and Noman [19] on μ-convergent series.

Definition 4 ([19])**.** Let $\mu = (\mu_k)$ be a sequence of positive real numbers tending to infinity such that

$$0 < \mu_0 < \mu_1 < \mu_2 \ldots \text{ and } \mu_k \to \infty \text{ as } k \to \infty.$$

Then a sequence $x = (x_k)$ of numbers is said to be μ-convergent to a number l if $\Lambda x_k \to l$ as $k \to \infty$, where

$$\Lambda x_k = \frac{1}{\mu_k} \sum_{i=0}^{k} (\mu_i - \mu_{i-1}) x_i$$

Esi and Braha [4] used Definition 4 to introduce a new notion called Λ-statistical convergence in random 2-normed spaces and studied some of its properties. Before we proceed further it would be better to recall the ideas of probabilistic and random 2-normed spaces which are of much interest in the study of random operator equations. The concept of probabilistic normed spaces was initially introduced by A. N. Sherstnev [22] in 1962. Menger [13] introduced the notion of probabilistic metric spaces in 1942. The idea of Menger [13] was to use distribution function instead of non-negative real numbers as values of the metric. In last few years these spaces are grown up rapidly and many detereministic results of linear normed spaces are obtained for probabilistic normed spaces. For a detailed study on probabilistic functional analysis, we refer [1, 10, 11, 18, 21]. In 2005, Goleţ [9] used the concept of 2-norm of Gähler [8] and presented generalized probabilistic normed space which he called random 2-normed space.

Let $\mathbb{R}$ denotes the set of reals and $\mathbb{R}_0^+ = [0,\infty)$. A function $f : \mathbb{R} \to \mathbb{R}_0^+$ is called a distribution function if it is non-decreasing and left-continuous with $\inf_{t\in\mathbb{R}} f(t) = 0$ and $\sup_{t\in\mathbb{R}} f(t) = 1$.

We will denote the set of all distribution functions by $\mathfrak{D}$.

Also, a distance distribution function is a non decreasing function F defined on $\mathbb{R}^+ = [0,\infty]$ that satisfies $F(0) = 0$ and $F(\infty) = 1$, and is left continuous on $(0,\infty)$. Let $\mathfrak{D}^+$ denotes the set of all distance distribution functions.

A triangular norm, briefly t-norm, is a binary operation $*$ on $[0,1]$ which is continuous, commutative, associative, non-decreasing and has 1 as neutral element, i.e., it is the continuous mapping $* : [0,1]\times[0,1] \to [0,1]$ such that for all $a,b,c \in [0,1]$:

(1) $a * 1 = a$,
(2) $a * b = b * a$,
(3) $c * d \geq a * b$ if $c \geq a$ and $d \geq b$,
(4) $(a * b) * c = a * (b * c)$.

The $*$ operations $a * b = max\{a+b-1,0\}$, $a * b = ab$, and $a * b = min\{a,b\}$ on $[0,1]$ are t-norms.

In following, we quote some needful definitions.

Definition 5 ([8])**.** Let X be a real vector space of dimension $d > 1$ (d may be infinite). A real valued function $||\cdot,\cdot|| : X^2 \to \mathbb{R}$ satisfying the following conditions:

(1) $||x_1,x_2|| = 0$, if and only if x_1,x_2 are linearly dependent.
(2) $||x_1,x_2|| = ||x_2,x_1||$ for all $x_1,x_2 \in X$,
(3) $||\alpha x_1,x_2|| = |\alpha|||x_1,x_2||$, for any $\alpha \in \mathbb{R}$ and
(4) $||x_1+x_2,x_3|| \leq ||x_1,x_3|| + ||x_2,x_3||$

is called a 2-norm and the pair $(X,||\cdot,\cdot||)$ is called a 2-normed space.

For example, if we take $X = \mathbb{R}^2$ with 2-norm $||x_1,x_2||$ = area of parallelogram spanned by the vectors x_1,x_2 which may be given explicitly by the formula

$$||x_1,x_2|| = |det(x_{ij})| = abs.\left(det\left(x_i,x_j\right)\right)$$

where $x_i = (x_{i1},x_{i2}) \in \mathbb{R}^2$ for each $i = 1,2$. Then $(X,||\cdot,\cdot||)$ is a 2-normed space.

Definition 6 ([9])**.** Let X be a real linear space of dimension $d > 1$ (d may be infinite), τ be a triangle function(a binary operation on $\mathfrak{D}^+$ which is associative, commutative, nondecreasing and ϵ_0 as a unit) and $\mathcal{F} : X\times X \to \mathfrak{D}^+$ (for $x,y \in X$, $\mathcal{F}(x,y;t)$ is the value of $\mathcal{F}(x,y)$ at $t \in \mathbb{R}$). Then $\mathcal{F}$ is called a probabilistic norm and $(X,\mathcal{F},\tau)$ a probabilistic 2-normed space if the following conditions are satisfied:

(1) $\mathcal{F}(x,y;t) = H_0(t)$ if x,y are linearly dependent, where $H_0(t) = 0$ if $t \leq 0$ and $H_0(t) = 1$ if $t > 0$.
(2) $\mathcal{F}(x,y;t) \neq H_0(t)$ if x,y are linearly independent,
(3) $\mathcal{F}(x,y;t) = \mathcal{F}(y,x;t)$ for all $x,y \in X$,
(4) $\mathcal{F}(\alpha x,y;t) = \mathcal{F}(x,y;\frac{t}{|\alpha|})$ for every $t > 0$, $\alpha \neq 0$ and $x,y \in X$,
(5) $\mathcal{F}(x+y,z;t) \geq \tau(\mathcal{F}(x,z;t),\mathcal{F}(y,z;t))$, where $x,y,z \in X$.

If (5) is replaced by $\mathcal{F}(x+y,z;t_1+t_2) \geq \mathcal{F}(x,z;t_1) * \mathcal{F}(y,z;t_2)$ for all $x,y,z \in X$ and $t_1,t_2 \in \mathbb{R}_0^+$ then $(X,\mathcal{F},*)$ is called a random 2-normed space.

Example 1. Every 2-normed space $(X,||\cdot,\cdot||)$ can be made a random 2-normed space by setting $\mathcal{F}(x,y;t) = H_0(t-||x,y||)$ where

$$H_a(t) = \begin{cases} 0, & \text{if } t \leq a, \\ 1, & \text{if } t > a \end{cases}$$

for all $x,y \in X$, $t > 0$ and $a * b = ab$; $a,b \in [0,1]$.

Example 2. Let $(X,||\cdot,\cdot||)$ be a 2-normed space with $||x,z|| = |x_1 z_2 - x_2 z_1|$; $x = (x_1,x_2)$, $z = (z_1,z_2)$ and $a * b = ab$ for all $a,b \in [0,1]$. For every $x,y \in X$ and $t > 0$ we define $\mathcal{F}(x,y;t) = \frac{t}{t+||x,y||}$, then $(X,\mathcal{F},*)$ is a random 2-normed space.

Definition 7 ([15])**.** Let $(X,\mathcal{F},*)$ be a random 2-normed space. Then a sequence $x = (x_k)$ is said to be convergent to $x_0 \in X$ with respect to the norm $\mathcal{F}$ if for every $\epsilon > 0$, $t \in (0,1)$ and $\theta \neq z \in X$, there exists a positive integer k_0 such that $\mathcal{F}(x_k - x_0, z; \epsilon) > 1-t$ whenever $k \geq k_0$. It is denoted by $\mathcal{F} - \lim x_k = x_0$.

Definition 8. p[15]] Let $(X,\mathcal{F},*)$ be a random 2-normed space. Then a sequence $x = (x_k)$ is said to be statistically convergent or S^{R2N} convergent to $x_0 \in X$ with respect to the norm $\mathcal{F}$ if for every $\epsilon > 0$, $t \in (0,1)$ and $\theta \neq z \in X$,

$$\delta(\{k \in \mathbb{N} : \mathcal{F}(x_k - x_0, z; \epsilon) \leq 1-t\}) = 0.$$

In this case, we write $S^{R2N} - \lim x_k = x_0$.

Definition 9 ([4])**.** Let $(X,\mathcal{F},*)$ be a random 2-normed space. Then a sequence $x = (x_k)$ is said to be Λ-statistically convergent with respect to the norm $\mathcal{F}$ provided that, for every $\epsilon > 0$, $t \in (0,1)$ and $\theta \neq z \in X$,

$$\delta_\Lambda(\{k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) \leq 1-t\}) = 0,$$

$i.e.$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) \leq 1-t\}| = 0.$$

In this case, we write $S_\Lambda^{R2N} - \lim x_k = x_0$.

Definition 10 ([4]). Let $(X, \mathcal{F}, *)$ be a random 2-normed space. Then a sequence $x = (x_k)$ is said to be Λ-statistically cauchy with respect to the norm $\mathcal{F}$ provided that, for every $\epsilon > 0$, $t \in (0,1)$ and $\theta \neq z \in X$, there exists a positive integer $k_0(\epsilon)$ such that for all $k, l \geq k_0$

$$\delta_\Lambda(\{k \in I_n : \mathcal{F}(\Lambda x_k - \Lambda x_l, z; \epsilon) \leq 1 - t\}) = 0.$$

In present paper, we study quite natural and new notion of Λ-statistical convergence of order α in random 2-normed spaces.

2. Main results

In this section, we begin with the following definitions of statistical and Λ-statistical convergence of order α in random 2-normed spaces.

Definition 11. A sequence $x = (x_k)$ in a random 2-normed space $(X, \mathcal{F}, *)$ is said to be statistically convergent of order α $(0 < \alpha \leq 1)$ to $x_0 \in X$ provided that, for every $\epsilon > 0$, $t \in (0,1)$ and $\theta \neq z \in X$,

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \in \mathbb{N} : \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) \leq 1 - t\}| = 0,$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{n^\alpha} |\{k \in \mathbb{N} : \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) > 1 - t\}| = 1.$$

In this case, we write $S^\alpha - \lim_{k\to\infty} x_k = x_0$.

Let $S^\alpha(X)$ denotes the set of all statistically convergent sequences of order α in a random 2-normed space $(X, \mathcal{F}, *)$.

Definition 12. Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ with $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. A sequence $x = (x_k)$ in a random 2-normed space $(X, \mathcal{F}, *)$ is said to be Λ-statistically convergent of order α $(0 < \alpha \leq 1)$ to $x_0 \in X$ provided that, for every $\epsilon > 0$, $t \in (0,1)$ and $\theta \neq z \in X$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) \leq 1 - t\}| = 0,$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) > 1 - t\}| = 1,$$

where λ_n^α denote the αth power of λ_n, *i.e.*, $(\lambda_n^\alpha) = (\lambda_1^\alpha, \lambda_2^\alpha, \lambda_3^\alpha, \cdots\cdots)$. In this case, we write $S_\Lambda^\alpha - \lim_{k\to\infty} x_k = x_0$.

Let $S_\Lambda^\alpha(X)$ denotes the set of all Λ-statistically convergent sequences of order α in a random 2-normed space $(X, \mathcal{F}, *)$.
For the particular choice $\alpha = 1$, Definition 12 coincides with the notion of Λ-statistical

convergence of [4]; For $\lambda_n = n$, Definition 12 coincides with the notion of statistical convergence of order α in random 2-normed space; For $\lambda_n = n$ and $\alpha = 1$, Definition 12 coincides with the notion of statistical convergence in random 2-normed space[15].
We next give an example that shows Definition 12 is well defined for $(0 < \alpha \leq 1)$ but not for $\alpha > 1$. In view of this, we need the following theorem with lemma.

Lemma 1. *Let $\lambda = (\lambda_n)$ be a non-decreasing sequence as defined above and $(X, \mathcal{F}, *)$ be a random 2-normed space. Let $0 < \alpha \leq 1$ and $x = (x_k)$ be a sequence in X. Then, for $\epsilon > 0$, $t \in (0,1)$ and $\theta \neq z \in X$, the following statements are equivalent:*

(1) $S_\Lambda^\alpha - \lim_{k\to\infty} x_k = x_0$,
(2) $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) \leq 1 - t\}| = 0$,
(3) $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) > 1 - t\}| = 1$,
(4) $S_\Lambda^\alpha - \lim_{k\to\infty} \mathcal{F}(\Lambda x_k - x_0, z; \epsilon) = 1$.

Theorem 1. *Let $(X, \mathcal{F}, *)$ be a random 2-normed space and $0 < \alpha \leq 1$ be given. If $S_\Lambda^\alpha - \lim_{k\to\infty} x_k = x_0$, then x_0 must be unique.*

Proof. Suppose $S_\Lambda^\alpha - \lim_{k\to\infty} x_k = y_0$ where $y_0 \neq x_0$. Given $\epsilon > 0$ and $t > 0$, choose $\eta > 0$ such that $(1-\eta) * (1-\eta) > 1 - \epsilon$. For $\theta \neq z \in X$, define

$$K_1(\eta, t) = \left\{ k \in I_n : \mathcal{F}\left(\Lambda x_k - x_0, z; \frac{t}{2}\right) \leq 1 - \eta \right\};$$
$$K_2(\eta, t) = \left\{ k \in I_n : \mathcal{F}\left(\Lambda x_k - y_0, z; \frac{t}{2}\right) \leq 1 - \eta \right\}.$$

Since $S_\Lambda^\alpha - \lim_{k\to\infty} x_k = x_0$ and $S_\Lambda^\alpha - \lim_{k\to\infty} x_k = y_0$, it follows for every $t > 0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K_1(\eta, t)| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K_2(\eta, t)| = 0$$

Let $K(\eta, t) = K_1(\eta, t) \cup K_2(\eta, t)$, then clearly $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K(\eta, t)| = 0$ which immediately implies $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K^c(\eta, t)| = 1$. Let $k \in K^c(\eta, t) = K_1^c(\eta, t) \cap K_2^c(\eta, t)$. Now one can write,

$$\mathcal{F}\left(x_0 - y_0, z; \frac{t}{2}\right) \geq \mathcal{F}\left(\Lambda x_k - x_0, z; \frac{t}{2}\right) * \mathcal{F}\left(\Lambda x_k - y_0, z; \frac{t}{2}\right)$$
$$> (1-\eta) * (1-\eta) > 1 - \epsilon.$$

Since ϵ is arbitrary, it follows that $\mathcal{F}\left(x_0 - y_0, z; \frac{t}{2}\right) = 1$, for $t > 0$ and $\theta \neq z \in X$. This shows that $x_0 = y_0$. □

Example 3. Let $X = \mathbb{R}^2$ with the 2-norm $||x, z|| = ||x_1 z_2 - x_2 z_1||$ where $x = (x_1, x_2)$, $z = (z_1, z_2)$ and $a * b = ab$ for all $a, b \in [0,1]$. Let $\mathcal{F}(x, z; t) = \frac{t}{t + ||x, z||}$

where $x \in X, t \in (0,1)$ and $\theta \neq z \in X$. Then $(\mathbb{R}^2, \mathcal{F}, *)$ is a random 2-normed space. We define a sequence $x = (x_k)$ as follows:

$$\Lambda x_k = \begin{cases} (1,0), & \text{if } k \text{ is even}, \\ (0,0), & \text{if } k \text{ is odd}. \end{cases}$$

For $\epsilon > 0$, $t \in (0,1)$, if we define

$$\begin{aligned} K(\epsilon,t) &= \{k \in I_n : \mathcal{F}(\Lambda x_k - \theta, z; t) \leq 1-\epsilon\}, \theta = (0,0) \\ &= \left\{k \in I_n : \frac{t}{t + ||\Lambda x_k - \theta, z||} \leq 1-\epsilon\right\} \\ &= \left\{k \in I_n : ||\Lambda x_k - \theta, z|| \geq \frac{\epsilon t}{1-\epsilon} > 0\right\} \\ &= \{k \in I_n : \Lambda x_k = (1,0)\} \\ &= \{k \in I_n : k \text{ is even}\}; \end{aligned}$$

then,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K(\epsilon,t)| = \lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : k \text{ is even}\}| \leq \lim_{n\to\infty} \frac{[\sqrt{\lambda_n}]+1}{2\lambda_n^\alpha} = 0$$

for $\alpha > 1$.
Similarly, for $\epsilon > 0$ and $t \in (0,1)$ if we define

$$B(\epsilon,t) = \{k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; t) \leq 1-\epsilon\}, x_0 = (1,0)$$

then

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |B(\epsilon,t)| = \lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : \text{k is odd }\}| \leq \lim_{n\to\infty} \frac{[\sqrt{\lambda_n}]+1}{2\lambda_n^\alpha} = 0$$

for $\alpha > 1$.

This shows that $S_\Lambda^\alpha - \lim_k x_k$ is not unique and we obtain a contradiction to Theorem 1.

Theorem 2. *Let $(X, \mathcal{F}, *)$ be a random 2-normed space and $0 < \alpha \leq 1$ be given. For a sequence $x = (x_k)$ in X if $\mathcal{F}_\Lambda - \lim_k x_k = x_0$, then $S_\Lambda^\alpha - \lim_k x_k = x_0$. However, the converse need not be true in general.*

Proof. Since $\mathcal{F}_\Lambda - \lim_k x_k = x_0$, so for $\epsilon > 0$, $t \in (0,1)$ and $\theta \neq z \in X$ there exists a positive integer n_0 such that $\mathcal{F}(\Lambda x_k - x_0, z; t) > 1-\epsilon$ for all $k \geq n_0$. Hence the set

$$K(\epsilon,t) = \{k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; t) \leq 1-\epsilon\} \subset \{1,2,3,\ldots,n_0-1\},$$

for which we have,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; t) \leq 1-\epsilon\}| = 0.$$

This shows that $S_{\Lambda}^{\alpha}-\lim_k x_k = x_0$. We next give the following example which shows that the converse need not be true.

Example 4. Consider the random 2-normed space as in Example 3. Define a sequence $x=(x_k)$ as follows:

$$\Lambda x_k = \begin{cases} (k,0), & \text{if } n-[\sqrt{\lambda_n}]+1 \le k \le n, \\ (0,0), & \text{otherwise.} \end{cases}$$

For $\epsilon > 0$ and $t > 0$ if we define $K(\epsilon,t) = \{k \in I_n : \mathcal{F}(\Lambda x_k, z; t) \le 1-\epsilon\}$, then one can write as in Example 3 $K(\epsilon,t) = \left\{k \in I_n : n-[\sqrt{\lambda_n}]+1 \le k \le n\right\}$

Thus,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} |K(\epsilon,t)| = \lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} \left|\left\{k \in I_n : n-[\sqrt{\lambda_n}]+1 \le k \le n\right\}\right|$$

$$\le \lim_{n\to\infty} \frac{[\sqrt{\lambda_n}]}{\lambda_n^{\alpha}} = 0$$

for $\frac{1}{2} < \alpha \le 1$. This shows that $S_{\Lambda}^{\alpha} - \lim_k x_k = 0$. But $\mathcal{F}_{\Lambda} - \lim_k x_k \neq 0$, since

$$\mathcal{F}(\Lambda x_k, z; t) = \frac{t}{t+||\Lambda x_k, z||} = \begin{cases} \frac{t}{t+kz_2}, & \text{if } n-[\sqrt{\lambda_n}]+1 \le k \le n, \\ 1, & \text{otherwise.} \end{cases}$$

which implies

$$\lim_{k\to\infty} \mathcal{F}(\Lambda x_k, z; t) = \begin{cases} 0, & \text{if } n-[\sqrt{\lambda_n}]+1 \le k \le n, \\ 1, & \text{otherwise.} \end{cases}$$

□

Theorem 3. *Let $(X,\mathcal{F},*)$ be a random 2-normed space and $0<\alpha\le 1$ be given. Let $x=(x_k)$ and $y=(y_k)$ be two sequences in X.*
(i) If $S_{\Lambda}^{\alpha} - \lim_{k\to\infty} x_k = x_0$ and $0 \neq c \in \mathbb{R}$, then $S_{\Lambda}^{\alpha} - \lim_{k\to\infty} c x_k = c x_0$.
(ii) If $S_{\Lambda}^{\alpha} - \lim_{k\to\infty} x_k = x_0$ and If $S_{\Lambda}^{\alpha} - \lim_{k\to\infty} y_k = y_0$, then $S_{\Lambda}^{\alpha} - \lim(x_k + y_k) = x_0 + y_0$.

Proof. The proof of the Theorem is not so hard so is omitted here. □

Theorem 4. *Let $(X,\mathcal{F},*)$ be a random 2-normed space and $0<\alpha\le\beta\le 1$ be given. Then $S_{\Lambda}^{\alpha}(X) \subset S_{\Lambda}^{\beta}(X)$ and the inclusion is strict for some α and β such that $\alpha<\beta$.*

Proof. If $0<\alpha\le\beta\le 1$, then for every $\epsilon>0$, $t>0$ and $\theta \neq z \in X$, we have

$$\frac{1}{\lambda_n^{\beta}} |\{k \in I_n : \mathcal{F}(\Lambda x_k - l, z; t) \le 1-\epsilon\}|$$
$$\le \frac{1}{\lambda_n^{\alpha}} |\{k \in I_n : \mathcal{F}(\Lambda x_k - l, z; t) \le 1-\epsilon\}|;$$

which immediately implies the inclusion $S_{\Lambda}^{\alpha}(X) \subset S_{\Lambda}^{\beta}(X)$. We next give an example that shows the inclusion in $S_{\Lambda}^{\alpha}(X) \subset S_{\Lambda}^{\beta}(X)$ is strict for some α and β with $\alpha < \beta$.

Example 5. Let $(\mathbb{R}^2, \mathcal{F}, *)$ be a random 2-normed space as defined above. We define a sequence $x = (x_k)$ as follows:

$$\Lambda x_k = \begin{cases} (1,0), & \text{if } n - [\sqrt{\lambda_n}] + 1 \leq k \leq n, \\ (0,0), & \text{otherwise.} \end{cases}$$

Then one can easily see $S_{\Lambda}^{\beta} - \lim_k x_k = 0$, *i.e.*, $x \in S_{\Lambda}^{\beta}(X)$ for $\frac{1}{2} < \beta \leq 1$ but $x \notin S_{\Lambda}^{\alpha}(X)$ for $0 < \alpha \leq \frac{1}{2}$. This shows that the inclusion in $S_{\Lambda}^{\alpha}(X) \subset S_{\Lambda}^{\beta}(X)$ is strict.

□

Theorem 5. *Let $(X, \mathcal{F}, *)$ be a random 2-normed space and $0 < \alpha \leq 1$ be given. If $x = (x_k)$ be a sequence in X, then $S_{\Lambda}^{\alpha} - \lim_k x_k = x_0$ if and only if there exists a subset $K = \{k_m : k_1 < k_2 < \dots\}$ of $\mathbb{N}$ such that $\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} |K| = 1$ and $\mathcal{F}_{\Lambda} - \lim_k x_k = x_0$.*

Proof. First suppose that $S_{\Lambda}^{\alpha} - \lim_k x_k = x_0$. For $t > 0$, $\theta \neq z \in X$ and $p \in \mathbb{N}$, if we define

$$K(p,t) = \left\{ k \in I_n : \mathcal{F}\left(\Lambda x_k - x_0, z; t\right) \leq 1 - \frac{1}{p} \right\}$$
$$M(p,t) = \left\{ k \in I_n : \mathcal{F}\left(\Lambda x_k - x_0, z; t\right) > 1 - \frac{1}{p} \right\};$$

then,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} |K(p,t)| = 0.$$

Also, for $p = 1, 2, 3, \dots$

$$M(1,t) \supset M(2,t) \supset \dots M(i,t) \supset M(i+1,t) \supset \dots \tag{2.1}$$

and

$$\lim_{n\to\infty} \frac{1}{\lambda_n^{\alpha}} |M(p,t)| = 1. \tag{2.2}$$

Now, to prove the result in one way, it is sufficient to prove that $\mathcal{F}_{\Lambda} - \lim_k x_k = x_0$ over $M(p,t)$. Suppose x_k is not convergent to x_0 over $M(p,t)$ with respect to the norm $\mathcal{F}_{\Lambda}$. Then, there exists some $\eta > 0$ such that

$$\{k \in \mathbb{N} : \mathcal{F}\left(\Lambda x_k - x_0, z; t\right) \leq 1 - \eta\}$$

for infinitely many terms x_k. Let

$$M(\eta,t) = \{k \in I_n : \mathcal{F}\left(\Lambda x_k - x_0, z; t\right) > 1 - \eta\}$$

and $\eta > \frac{1}{p}$ for $p = 1, 2, 3 \cdots$. This implies that $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |M(\eta,t)| = 0$. Also from (2.1), we have $K_1(p,t) \subset M(\eta,t)$ which gives that $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |M(p,t)| = 0$, this contradicts (2.2). Hence $\mathcal{F}_\Lambda - \lim_k x_k = x_0$.

Conversely, suppose that there exists a subset $K = \{k_m : k_1 < k_2 < \ldots\}$ of $\mathbb{N}$ such that $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K| = 1$ and $\mathcal{F}_\Lambda - \lim_k x_k = x_0$. Then for every $t > 0$, $\epsilon > 0$ and $\theta \neq z \in X$ there exists a positive integer k_0 such that

$$\{k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; t) > 1 - \epsilon\}$$

for all $k \geq k_0$. Since the set $\{k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; t) \leq 1 - \epsilon\}$ is contained in $\mathbb{N} - \{k_0 + 1, k_0 + 2, k_0 + 3, \cdots\}$ therefore,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |\{k \in I_n : \mathcal{F}(\Lambda x_k - x_0, z; t) \leq 1 - \epsilon\}| = 0.$$

Hence, $S_\Lambda^\alpha - \lim_k x_k = x_0$. □

Definition 13. Let $(X, \mathcal{F}, *)$ be a random 2-normed space. A sequence $x = (x_k)$ is said to be Λ-statistically Cauchy of order α $(0 < \alpha \leq 1)$ if for every $\epsilon > 0$, $t \in (0,1)$ and $\theta \neq z \in X$ there exists a positive integer k_0 such that for all $k, l \geq k_0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |k \in I_n : \mathcal{F}(\Lambda x_k - \Lambda x_l, z; \epsilon) \leq 1 - t\}| = 0,$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |k \in I_n : \mathcal{F}(\Lambda x_k - \Lambda x_l, z; \epsilon) > 1 - t\}| = 1.$$

Theorem 6. *Let $(X, \mathcal{F}, *)$ be a random 2-normed space and $0 < \alpha \leq 1$ be given. Then a sequence $x = (x_k)$ is said to be Λ-statistically convergent of order α iff it is Λ-statistically Cauchy of order α.*

Proof. Let $x = (x_k)$ be a Λ-statistically convergent sequence of order α. Suppose that $S_\Lambda^\alpha - \lim_k x_k = x_0$. Let $\epsilon > 0$. Choose $r > 0$ such that $(1 - r) * (1 - r) > 1 - \epsilon$. If we define

$$K(r,t) = \left\{ k \in I_n : \mathcal{F}\left(\Lambda x_k - x_0, z; \frac{t}{2}\right) \leq 1 - r \right\},$$

then

$$K^c(r,t) = \left\{ k \in I_n : \mathcal{F}\left(\Lambda x_k - x_0, z; \frac{t}{2}\right) > 1 - r \right\};$$

which gives by virtue of $S_\Lambda^\alpha - \lim_k x_k = x_0$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K(r,t)| = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K^c(r,t)| = 1.$$

Let $m \in K^c(r,t)$, then $\mathcal{F}\left(\Lambda x_m - x_0, z; \frac{t}{2}\right) > 1-r$. If we take

$$B(\epsilon,t) = \{k \in I_n : \mathcal{F}\left(\Lambda x_k - \Lambda x_m, z; t\right) \leq 1-\epsilon\},$$

then to prove the first part it is sufficient to prove that $B(\epsilon,t) \subset K(r,t)$. Let $k \in B(\epsilon,t)$, which gives $\mathcal{F}\left(\Lambda x_k - \Lambda x_m, z; t\right) \leq 1-\epsilon$. Suppose $k \notin K(r,t)$, then $\mathcal{F}\left(\Lambda x_k - x_0, z; \frac{t}{2}\right) > 1-r$. Now, we can observe that

$$\begin{aligned} 1-\epsilon \geq \mathcal{F}\left(\Lambda x_k - \Lambda x_m, z; t\right) &\geq \mathcal{F}\left(\Lambda x_k - x_0, z; \frac{t}{2}\right) * \mathcal{F}\left(\Lambda x_m - x_0, z; \frac{t}{2}\right) \\ &\geq (1-r)*(1-r) > 1-\epsilon. \end{aligned}$$

This contradiction shows that $B(\epsilon,t) \subset K(r,t)$ and therefore, one way of the theorem is proved.

Conversely, suppose that $x = (x_k)$ is Λ-statistically Cauchy sequence of order α but not Λ-statistically convergent of order α with respect to $\mathcal{F}$. Then for every $t > 0$, $\epsilon > 0$ and $\theta \neq z \in X$ there exists a positive integer m such that

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K(\epsilon,t)| = 0 \, \text{where} \, K(\epsilon,t) = \{k \in I_n : \mathcal{F}\left(\Lambda x_k - \Lambda x_m, z; t\right) \leq 1-\epsilon\}.$$

This implies that $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K^c(\epsilon,t)| = 1$. Choose $r > 0$ such that $(1-r)*(1-r) > 1-\epsilon$. Let

$$B(r,t) = \left\{k \in I_n : \mathcal{F}\left(\Lambda x_k - x_0, z; \frac{t}{2}\right) > 1-r\right\}.$$

Let $m \in B(r,t)$, then $\mathcal{F}\left(\Lambda x_m - x_0, z; \frac{t}{2}\right) > 1-r$.
Since

$$\begin{aligned} \mathcal{F}\left(\Lambda x_k - \Lambda x_m, z; t\right) &\geq \mathcal{F}\left(\Lambda x_k - x_0, z; \frac{t}{2}\right) * \mathcal{F}\left(\Lambda x_m - x_0, z; \frac{t}{2}\right) \\ &> (1-r)*(1-r) > 1-\epsilon; \end{aligned}$$

therefore,

$$\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} \left|\{k \in I_n : \mathcal{F}\left(\Lambda x_k - \Lambda x_m, z; t\right) > 1-\epsilon\}\right| = 0.$$

i.e. $\lim_{n\to\infty} \frac{1}{\lambda_n^\alpha} |K^c(\epsilon,t)| = 0$ which leads to a contradiction. Hence $x = (x_k)$ is Λ-statistically convergent of order α. □

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