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GROUPOIDS CORRESPONDING TO RELATIONAL SYSTEMS

IVAN CHAJDA AND HELMUT LÄNGER

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Abstract. Groupoids corresponding to certain relational systems, i. e. sets with one binary relation, are considered. To each directed relational system there exists at least one such groupoid. It is shown how properties of the relational system can be characterized by properties of a corresponding groupoid. Further, for a given groupoid a sufficient condition is provided that guarantees the existence of a relational system to which this groupoid corresponds. This approach enables us to introduce the concept of congruence on a relational system and hence to produce quotient relational systems.

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It was aleady shown in [3] that to every directed relational system with one binary relation there can be assigned a groupoid reflecting its relational properties. However, the definition given in [3] does not fit well to certain relational systems, in particular to quasiordered or even ordered sets in a way similar to semilattices which are treated (under the name directoids) e. g. in [2] and [6]. Hence, we modify this definition here in order to be in accordance with that of a directoid in a directed ordered set. It turns out that we obtain a different characterization of relational properties by means of properties of the corresponding groupoid. In particular, several important properties (as quasiordered or ordered set) can be expressed by identities in the corresponding groupoid.

Our next task is to apply the method of a corresponding groupoid in order to introduce the concept of a congruence of a relational system which enables us to produce quotient relational systems having the same relational properties as the original ones.

In the following we agree to write ab instead of $a \cdot b$.

We start with the definition of a (directed) relational system.

Definition 1. By a *relational system* we mean an ordered pair $\mathcal{A} = (A, R)$ where A is a set and R is a binary relation on A. If $a, b \in A$ then we define the *upper cone*

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$U_R(a,b)$ of a and b *with respect to* R by $U_R(a,b) := \{x \in A \mid (a,x),(b,x) \in R\}$. The relational system $\mathcal{A}$ is called *directed* if $U_R(a,b) \neq \emptyset$ for all $a,b \in A$.

Now we define a groupoid corresponding to a relational system in a way different from that in [3].

Definition 2. A groupoid $\mathcal{G} = (A,\cdot)$ is said to *correspond* to a relational system $\mathcal{A} = (A,R)$ if for all $x,y \in A$ the following hold:

(i) If $(x,y) \in R$ then $xy = y$.
(ii) If $(x,y) \notin R$ and $(y,x) \in R$ then $xy = x$.
(iii) If $(x,y),(y,x) \notin R$ then $xy = yx \in U_R(x,y)$.

Remark 1. In the case $(x,y),(y,x) \notin R$ in (iii) above, the element $xy = yx$ is picked from $U_R(x,y)$ arbitrarily. Of course, if R is e. g. a directed ordering on A such that $\sup(x,y)$ exists for each $x,y \in A$, we can take $xy = \sup(x,y)$ in order to obtain the corresponding groupoid as a join semilattice. However, other choices can also be possible.

Remark 2. If $\mathcal{A}$ is directed then there exists at least one groupoid corresponding to $\mathcal{A}$. The converse is not true as can be seen from Example 1. The groupoid corresponding to a relational system need not be unique as can be seen from Example 2.

Example 1. If $A = \{a,b\}$, $R = \{(a,b),(b,a)\}$ and $\cdot$ denotes the binary operation on A defined by $ax := b$ and $bx := a$ for all $x \in A$ then $\mathcal{G} = (A,\cdot)$ corresponds to $\mathcal{A} = (A,R)$ though $\mathcal{A}$ is not directed.

Example 2. If $A = \{a,b,c\}$ and $R = \{(a,a),(a,b),(b,a),(b,c),(c,a),(c,c)\}$ then $\mathcal{G} = (A,\cdot)$ corresponds to $\mathcal{A} = (A,R)$ if and only if there exists an $x \in \{a,c\}$ such that the operation table of $\cdot$ looks as follows:

$\cdot$	a	b	c
a	a	b	a
b	a	x	c
c	a	c	c

Now we prove some lemmata which will be used later on.

Lemma 1. *Let $\mathcal{A} = (A,R)$ be a relational system, $\mathcal{G} = (A,\cdot)$ a groupoid corresponding to $\mathcal{A}$ and $a,b \in A$. Then $(a,b) \in R$ if and only if $ab = b$.*

Proof. If $(a,b) \in R$ then $ab = b$. If $(a,b) \notin R$ and $(b,a) \in R$ then $ab = a \neq b$. If, finally, $(a,b),(b,a) \notin R$ then $ab \neq b$ since $(a,ab) \in R$ and $(a,b) \notin R$. □

Lemma 2. *Let $\mathcal{A} = (A,R)$ be a relational system, $\mathcal{G} = (A,\cdot)$ a groupoid corresponding to $\mathcal{A}$ and $a,b \in A$. Then the following hold:*

(i) *If $(a,b),(b,a) \in R$ then $ab = b$ and $ba = a$.*
(ii) *If $(a,b) \in R$ and $(b,a) \notin R$ then $ab = ba = b$.*

(iii) *If* $(a,b) \notin R$ *and* $(b,a) \in R$ *then* $ab = ba = a$.

Proof. Clear. □

To every groupoid we now assign a relational system.

Definition 3. For every groupoid $\mathcal{G} = (A,\cdot)$ let $\mathcal{A}(\mathcal{G})$ denote the relational system $(A, R(\mathcal{G}))$, where $R(\mathcal{G}) := \{(x,xy) \mid x,y \in A\}$.

Lemma 3. *Let* $\mathcal{A} = (A,R)$ *be a relational system and* $\mathcal{G} = (A,\cdot)$ *a groupoid corresponding to* $\mathcal{A}$*. Then* $R \subseteq R(\mathcal{G})$*. If* R *is reflexive then* $\mathcal{A} = \mathcal{A}(\mathcal{G})$*.*

Proof. Let $a,b \in A$. If $(a,b) \in R$ then $(a,b) = (a,ab) \in R(\mathcal{G})$. Now assume R to be reflexive. If $(a,b) \in R$ then $(a,ab) = (a,b) \in R$. If $(a,b) \notin R$ and $(b,a) \in R$ then $(a,ab) = (a,a) \in R$. If, finally, $(a,b),(b,a) \notin R$ then $ab \in U_R(a,b)$, and hence, $(a,ab) \in R$. This shows $R(\mathcal{G}) \subseteq R$ completing the proof of the lemma. □

Remark 3. The example below shows that R need not to coincide with $R(\mathcal{G})$.

Example 3. Let $A = \{a,b,c\}$, $R := \{(a,a),(a,b),(a,c),(b,c),(c,c)\}$ and $\mathcal{A} = (A,R)$ and define a binary operation $\cdot$ on A by

$\cdot$	a	b	c
a	a	b	c
b	b	c	c
c	c	c	c

Then $\mathcal{G} = (A,\cdot)$ corresponds to $\mathcal{A}$, but

$$R(\mathcal{G}) = \{(a,a),(a,b),(a,c),(b,b),(b,c),(c,c)\} \neq R.$$

Definition 4. A binary relation R on a set A is called *strictly connex* if for all $x,y \in A$ either $(x,y) \in R$ or $(y,x) \in R$.

Remark 4. If R is strictly connex then it is reflexive. If R is strictly connex and symmetric then $R = A^2$.

Now we show how properties of a relational system are reflected by properties of a corresponding groupoid.

Theorem 1. *Let* $\mathcal{A} = (A,R)$ *be a relational system and* $\mathcal{G} = (A,\cdot)$ *a groupoid corresponding to* $\mathcal{A}$*. Then the following hold:*

(i) R *is reflexive if and only if* $xx = x$ *for all* $x \in A$*.*
(ii) R *is symmetric if and only if* $(xy)x = x$ *for all* $x,y \in A$*.*
(iii) *If* R *is symmetric then* $x(xy) = xy$ *for all* $x,y \in A$*.*
(iv) R *is antisymmetric if and only if* $xy = yx$ *for all* $x,y \in A$*.*
(v) *If* $(xy)z = x(yz)$ *for all* $x,y,z \in A$ *then* R *is transitive.*
(vi) *If* $x((xy)z) = (xy)z$ *for all* $x,y,z \in A$ *then* R *is transitive.*

(vii) *R is a quasiorder if and only if $xx = x$ and $x((xy)z) = (xy)z$ for all $x, y, z \in A$.*
(viii) *R is a tolerance if and only if $(xy)x = x$ and $x((xy)z) = (xy)z$ for all $x, y, z \in A$.*
(ix) *R is a partial order if and only if $xx = x$, $xy = yx$ and $x((xy)z) = x(yz)$ for all $x, y, z \in A$.*
(x) *R is an equivalence relation if and only if $xx = (xy)x = x$ and $x((xy)z) = (xy)z$ for all $x, y, z \in A$.*
(xi) *R is asymmetric if and only if $(xy)x \neq x$ for all $x, y \in A$.*
(xii) *If for all $x, y \in A$ either $xy \neq yx$ or $x(xy) \neq xy$ or $y(xy) \neq xy$ then R is strictly connex.*

Proof.

(i) follows from Lemma 1.

Let $a, b \in A$.

(ii) First assume R to be symmetric. If $(a,b) \in R$ then $(b,a) \in R$ and hence $(ab,a) = (b,a) \in R$. If $(a,b) \notin R$ then $(b,a) \notin R$ and therefore $ab \in U_R(a,b)$, whence $(a,ab) \in R$ which shows $(ab,a) \in R$. Hence in any case $(ab,a) \in R$, i. e. $(ab)a = a$. Conversely, assume $(xy)x = x$ for all $x, y \in A$. If $(a,b) \in R$ then $ab = b$, whence $ba = (ab)a = a$, i. e. $(b,a) \in R$.
(iii) If $(a,b) \in R$ then $(a,ab) = (a,b) \in R$, i. e. $a(ab) = ab$. Otherwise $(a,b),(b,a) \notin R$ and hence $ab \in U_R(a,b)$, i. e. $(a,ab) \in R$ which means $a(ab) = ab$.
(iv) First assume R to be antisymmetric. If $(a,b),(b,a) \in R$ then $ab = b = a = ba$. In all the other cases $ab = ba$ according to Lemma 2. Conversely, assume $xy = yx$ for all $x, y \in A$. If $(a,b),(b,a) \in R$ then $a = ba = ab = b$.

Let $c \in A$.

(v) If $(a,b),(b,c) \in R$ then $ab = b$ and $bc = c$ and hence

$$ac = a(bc) = (ab)c = bc = c,$$

i. e. $(a,c) \in R$.
(vi) If $(a,b),(b,c) \in R$ then $ab = b$ and $bc = c$ and hence

$$ac = a(bc) = a((ab)c) = (ab)c = bc = c,$$

i. e. $(a,c) \in R$.
(vii) If R is reflexive and transitive then $(a,ab),(ab,(ab)c) \in R$ according to Lemma 3 and hence $(a,(ab)c) \in R$, i. e. $a((ab)c) = (ab)c$. The converse implication follows from (i) and (vi).
(viii) First assume R to be symmetric and transitive. Then $(ab)a = a$ according to (ii). Because of (iii) we have $(a,ab),(ab,(ab)c) \in R$ and hence $(a,(ab)c) \in R$, i. e. $a((ab)c) = (ab)c$. The converse implication follows from (ii) and (vi).
(ix) follows from (iv) and (vii).

(x) follows from (i) and (viii) or from (ii) and (vii).

(xi) First, assume R to be asymmetric. If $(a,b) \in R$ then $(ab,a) = (b,a) \notin R$. If $(a,b) \notin R$ and $(b,a) \in R$ then $(ab,a) = (a,a) \notin R$. If, finally, $(a,b),(b,a) \notin R$ then $(ab,a) \notin R$ since otherwise we would have $(a,ab) \notin R$ which is a contradiction. Therefore also in this case $(ab,a) \notin R$. Hence in all the cases $(ab,a) \notin R$ which yields $(ab)a \neq a$. Conversely, assume $(xy)x \neq x$ for all $x,y \in A$. If $(a,b),(b,a) \in R$ then we would have $a \neq (ab)a = ba = a$, a contradiction. Hence R is asymmetric.

(xii) This follows from Definition 2.

□

Remark 5. Example 2 shows that in (iii) of Theorem 1 the converse implication does not hold. The following example shows that in (v) of Theorem 1 the converse implication does not hold:

Example 4. Let $A = \{a,b,c\}$, $R := \{(a,b),(a,c),(b,c),(c,c)\}$ and $\mathcal{A} = (A,R)$ and define a binary operation $\cdot$ on A by

$\cdot$	a	b	c
a	c	b	c
b	b	c	c
c	c	c	c

Then $\mathcal{G} = (A,\cdot)$ corresponds to $\mathcal{A}$, but

$$(aa)b = cb = c \neq b = ab = a(ab).$$

Example 3 shows that in (vi) of Theorem 1 the converse implication does not hold, since $\mathcal{G} = (A,\cdot)$ corresponds to $\mathcal{A}$ and R is transitive, but

$$b((ba)a) = b(ba) = bb = c \neq b = ba = (ba)a.$$

The following example shows that in (xii) of Theorem 1 the converse implication does not hold:

Example 5. Put $A = \{a,b,c\}$, $R := \{(a,a),(a,b),(b,b),(b,c),(c,a),(c,c)\}$ and $\mathcal{A} = (A,R)$ and define a binary operation $\cdot$ on A by

$\cdot$	a	b	c
a	a	b	a
b	b	b	c
c	a	c	c

Then $\mathcal{G} = (A,\cdot)$ corresponds to $\mathcal{A}$, but

$$ab = ba, a(ab) = ab \text{ and } b(ab) = bb = b = ab.$$

Now, we provide sufficient conditions for a groupoid in order to correspond to a relational system:

Theorem 2. *Let $\mathcal{G} = (A, \cdot)$ be a groupoid. Then the following are equivalent:*

(i) *$R(\mathcal{G})$ is reflexive, $\mathcal{A}(\mathcal{G})$ is directed and $\mathcal{G}$ corresponds to $\mathcal{A}(\mathcal{G})$.*

(ii) *$R(\mathcal{G})$ is reflexive and $\mathcal{G}$ corresponds to $\mathcal{A}(\mathcal{G})$.*

(iii) *For all $x, y \in A$ the following hold:*

$xx = x$

$x(xy) = xy$

either $xy = y$ *or* $xy = yx$.

Proof.
(i)⇒(ii):
This is trivial.
Let $a, b \in A$.
(ii)⇒(iii):
Since $\mathcal{G}$ corresponds to $\mathcal{A}(\mathcal{G})$ and $R(\mathcal{G})$ is reflexive we have $aa = a$. If $(a,b) \in R(\mathcal{G})$ then $a(ab) = ab$. If $(a,b) \notin R(\mathcal{G})$ and $(b,a) \in R(\mathcal{G})$ then $a(ab) = aa = a = ab$. If, finally, $(a,b),(b,a) \notin R(\mathcal{G})$ then $a(ab) = ab$ since $ab \in U_{R(\mathcal{G})}(a,b)$. If $ab \neq b$ then $(a,b) \notin R(\mathcal{G})$ and hence $ab = ba$, according to Lemma 2.
(iii)⇒(i):
We have $(a,a) = (a,aa) \in R(\mathcal{G})$. This shows that $R(\mathcal{G})$ is reflexive. By definition of $R(\mathcal{G})$ we have $(a,ab) \in R(\mathcal{G})$. If $ab = b$ then $(b,ab) = (b,b) \in R(\mathcal{G})$. If $ab \neq b$ then $(b,ab) = (b,ba) \in R(\mathcal{G})$. This shows that $\mathcal{A}(\mathcal{G})$ is directed. If $(a,b) \in R(\mathcal{G})$ then there exists an element c of A with $b = ac$ and hence $ab = a(ac) = ac = b$. If $(a,b) \notin R(\mathcal{G})$ then $ab \neq b$ and hence $ab = ba$. If $(a,b) \notin R(\mathcal{G})$ and $(b,a) \in R(\mathcal{G})$ then $ab = ba = a$. If, finally, $(a,b),(b,a) \notin R(\mathcal{G})$ then $ab = ba \in U_{R(\mathcal{G})}(a,b)$. This shows that $\mathcal{G}$ corresponds to $\mathcal{A}(\mathcal{G})$. □

The concept of a groupoid corresponding to a relational system enables us to construct quotient relational systems. The *quotient system* of a relational system is defined as follows:

Definition 5 (cf. [9])**.** For a relational system $\mathcal{A} = (A, R)$ and an equivalence relation Θ on A define $\mathcal{A}/\Theta := (A/\Theta, R/\Theta)$ where

$$R/\Theta := \{([x]\Theta, [y]\Theta) \mid (x,y) \in R\}.$$

We are now able to prove that a quotient of a groupoid corresponding to a relational system corresponds to the corresponding quotient of the relational system.

Lemma 4. *Let $\mathcal{A} = (A, R)$ be a relational system, $\mathcal{G} = (A, \cdot)$ a corresponding groupoid and $\Theta \in \mathrm{Con}\mathcal{G}$. Then $\mathcal{G}/\Theta$ corresponds to $\mathcal{A}/\Theta$.*

Proof. Let $a, b \in A/\Theta$. If $(a,b) \in R/\Theta$ then there exists $(c,d) \in R$ with $(a,b) = ([c]\Theta, [d]\Theta)$ and hence,

$$ab = [c]\Theta \cdot [d]\Theta = [cd]\Theta = [d]\Theta = b.$$

If $(a,b) \notin R/\Theta$ and $(b,a) \in R/\Theta$ then there exists $(e,f) \in R$ with $(b,a) = ([e]\Theta, [f]\Theta)$. Now $(f,e) \in R$ would imply $(a,b) = ([f]\Theta,[e]\Theta) \in R/\Theta$, a contradiction. Therefore $(f,e) \notin R$ and hence

$$ab = [f]\Theta \cdot [e]\Theta = [fe]\Theta = [f]\Theta = a.$$

Finally, assume $(a,b),(b,a) \notin R/\Theta$. Since $a,b \in A/\Theta$ there exist $g,h \in A$ with $(a,b) = ([g]\Theta,[h]\Theta)$. Because of $(a,b),(b,a) \notin R/\Theta$ we have $(g,h),(h,g) \notin R$. Since $\mathcal{G}$ corresponds to $\mathcal{A}$ we have $gh = hg \in U_R(g,h)$. Now

$$ab = [g]\Theta \cdot [h]\Theta = [gh]\Theta = [hg]\Theta = [h]\Theta \cdot [g]\Theta = ba,$$
$$(a,ab) = ([g]\Theta,[gh]\Theta) \in R/\Theta \text{ since } (g,gh) \in R \text{ and}$$
$$(b,ab) = ([h]\Theta,[gh]\Theta) \in R/\Theta \text{ since } (h,gh) \in R.$$

This shows $ab = ba \in U_{R/\Theta}(a,b)$ completing the proof of the lemma. □

It was shown e. g. in [1] that the quotient of a poset $(A,\leq)$ by an equivalence relation Θ as given in Definition 5 need not be a poset since transitivity need not be satisfied by the relation $\leq/\Theta$. (For literature concerning congruences on posets cf. e. g. [4], [5], [7] and [8]). However, as shown in Theorem 1, if R is a quasiorder or even a partial order, this can be expressed in a corresponding groupoid by certain identities. If we factor a groupoid by a congruence, the identities of the groupoid are preserved. This motivates us to introduce the following concept:

Definition 6. Let $\mathcal{A} = (A,R)$ be a relational system and Θ an equivalence relation on A. The relation Θ is called a *congruence* on $\mathcal{A}$ if there exists a groupoid $\mathcal{G} = (A,\cdot)$ corresponding to $\mathcal{A}$ such that $\Theta \in \operatorname{Con}\mathcal{G}$. Let $\operatorname{Con}\mathcal{A}$ denote the set of all congruences on $\mathcal{A}$.

Now we can prove that under a certain assumption certain properties of relational systems remain valid when these systems are factorized.

Theorem 3. *Let $\mathcal{A} = (A,R)$ be a relational system satisfying one of the following properties:*

(i) *R is reflexive.*
(ii) *R is symmetric.*
(iii) *R is antisymmetric.*
(iv) *R is a quasiorder.*
(v) *R is a tolerance.*
(vi) *R is a partial order.*
(vii) *R is an equivalence relation.*

If $\Theta \in \operatorname{Con}\mathcal{A}$ then $\mathcal{A}/\Theta$ has the same property.

Proof. Assume $\Theta \in \operatorname{Con}\mathcal{A}$. Then there exists a groupoid $\mathcal{G} = (A,\cdot)$ corresponding to $\mathcal{A}$ such that $\Theta \in \operatorname{Con}\mathcal{G}$. According to Lemma 4, $\mathcal{G}/\Theta$ corresponds to $\mathcal{A}/\Theta$. Since each one of the properties (i) – (vii) for $\mathcal{A}$ can be characterized by identities holding in $\mathcal{G}$, we have that these identities also hold in $\mathcal{G}/\Theta$. This means that $\mathcal{A}/\Theta$ has the corresponding property. □

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Authors' addresses

Ivan Chajda
Palacký University Olomouc, Faculty of Science, Department of Algebra and Geometry, 17. listopadu 12, 77146 Olomouc, Czech Republic
E-mail address: ivan.chajda@upol.cz

Helmut Länger
TU Wien, Faculty of Mathematics and Geoinformation, Institute of Discrete Mathematics and Geometry, Wiedner Hauptstraße 8-10, 1040 Vienna, Austria
E-mail address: helmut.laenger@tuwien.ac.at