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PARTITION PROPERTIES ON COUNTABLE BIPARTITE GRAPHS

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Abstract. We classify all countably infinite (on both sides) bipartite graphs G having a naturally defined version of the pigeonhole principle.

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1. INTRODUCTION

Definition 1. An $(\aleph_0, \aleph_0)$-bigraph is a structure $G = (X, Y, E)$, where $(X \cup Y, E)$ is a digraph such that $E \subseteq \{xy : x \in X, y \in Y\}$ and $|X| = |Y| = \aleph_0$. We call (X, Y) the *bipartition* of G, X the left side, and Y the right side.

The complement $\bar{G}$ of a bigraph $G = (X, Y, E)$ has the same set of vertices with the same bipartition, and $(X \times Y) \setminus E$ for the set of edges.

We say that a countable structure S has property $\mathcal{P}$ (also known as the pigeonhole principle) if for every partition of its domain into finitely many pieces at least one of the induced substructures is isomorphic to S. This property was investigated for various graph-theoretic structures. We list some such results.

Theorem 1 ([2], Proposition 3.4)**.** *The only countable graphs with the property $\mathcal{P}$ up to isomorphism are the empty graph, the complete graph and the Rado graph.*

Theorem 2 ([1], Theorem 1)**.** *The only countable tournaments with the property $\mathcal{P}$ up to isomorphism are the random tournament and tournaments ω^α and $(\omega^\alpha)^*$ for $0 < \alpha < \omega_1$.*

Theorem 3 ([3], Corollary 2.4)**.** *The only countable digraphs with the property $\mathcal{P}$ up to isomorphism are the empty digraph, the random tournament, tournaments ω^α and $(\omega^\alpha)^*$ for $0 < \alpha < \omega_1$, the random digraph, the random acyclic digraph and its inverse.*

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As we shall see, for bigraphs it makes sense only to consider certain altered property that we will call $\mathcal{P}'$. It is our goal to classify all countable bigraphs with property $\mathcal{P}'$.

Since random bigraphs will play an important role in our classification, we will now list some of their properties. More on these structures can be found in [5].

Definition 2. A $(\aleph_0,\aleph_0)$-bigraph (X,Y,E) is $(\aleph_0,\aleph_0)$-*random* if

$$\forall U,W \in [Y]^{<\aleph_0}\ (U\cap W=\varnothing \Rightarrow \exists x\in X(\forall u\in U\ xu\in E\ \wedge\ \forall w\in W\ xw\notin E)).$$

(X,Y,E) is $(\aleph_0,\aleph_0)$-*dense* if

$$\forall U,W \in [X]^{<\aleph_0}\ (U\cap W=\varnothing \Rightarrow \exists y\in Y(\forall u\in U\ uy\in E\ \wedge\ \forall w\in W\ wy\notin E)).$$

If G satisfies both these conditions we will call it $(\aleph_0,\aleph_0)$-random dense.

Proposition 1. *Let $G=(X,Y,E)$ be a $(\aleph_0,\aleph_0)$-random bigraph. Then every $y\in Y$ is of infinite degree in both G and $\bar{G}$.*

Clearly, the same results hold for vertices in X if G is (κ,λ,μ)-dense. In [4] it was shown that there is exactly one $(\aleph_0,\aleph_0)$-random dense bigraph up to isomorphism. This is not true for $(\aleph_0,\aleph_0)$-random bigraphs.

2. The classification

Now we turn to our main concern, a partition property resembling $\mathcal{P}$.

Definition 3. Let $G_1=(X_1,Y_1,E_1)$ and $G_2=(X_2,Y_2,E_2)$ be bigraphs. A bijection $f: X_1\cup Y_1\to X_2\cup Y_2$ is an isomorphism if for all $x\in X_1$, $y\in Y_1$: $xy\in E_1$ iff $f(x)f(y)\in E_2$.

Of course, for $(\aleph_0,\aleph_0)$-bigraphs it makes sense only to consider partitions into $(\aleph_0,\aleph_0)$-bigraphs (otherwise we could split the bigraph in two parts, each having one finite side). So we say that an $(\aleph_0,\aleph_0)$-bigraph G has property $\mathcal{P}'$ if:

$\mathcal{P}'$: for every partition of the set of vertices of G into finitely many pieces that each induce $(\aleph_0,\aleph_0)$-bigraphs at least one of the induced sub-bigraphs is isomorphic to G.

Example 1. $S=(X,Y,E)$ is defined by $X=\{x_n : n\in\omega\}$, $Y=\{y_n : n\in\omega\}$ and $E=\{x_ny_0 : n\in\omega\}$. S has the property $\mathcal{P}'$.

It is easy to see that if G has $\mathcal{P}$, then its complement $\bar{G}$ and the bigraph G^* obtained by reversing all the edges also have $\mathcal{P}'$.

Taking into account Propositions 1, 2 and 3 it would be natural to expect the $(\aleph_0,\aleph_0)$-random dense bigraph has the property $\mathcal{P}'$. However, this is not the case.

Lemma 1. *The* $(\aleph_0,\aleph_0)$*-random dense bigraph does not satisfy* $\mathcal{P}'$.

Proof. Let $y \in Y$ be arbitrary and let X_1 be the set of all its neighbors. By Proposition 1, both X_1 and $X \setminus X_1$ are infinite. Let $x \in X_1$ and let Y_0 be the set of all its neighbors. If we partition vertices of G into $(X \setminus X_1) \cup Y_0$ and $X_1 \cup (Y \setminus Y_0)$, in both induced sub-bigraphs we have an isolated vertex, so none of those parts can be isomorphic to G. □

In order to prove our main result, Theorem 4, we will consider several cases concerning the minimal degree of a vertex in G, and the number of vertices of minimal degree. $d_G(v)$ will denote the degree of a vertex v in G, and we will drop the G if it is clear from the context.

Lemma 2. *If all the vertices in both* G *and* $\bar{G}$ *are of infinite degree, then* G *does not satisfy* $\mathcal{P}'$.

Proof. Let $G = (X, Y, E)$ be an $(\aleph_0,\aleph_0)$-bigraph with property $\mathcal{P}'$ such that all the vertices in both G and $\bar{G}$ are of infinite degree. We prove that G must be $(\aleph_0,\aleph_0)$-random, and in a similar way it is proved that it must be $(\aleph_0,\aleph_0)$-dense. This will be a contradiction with Lemma 1.

Let $u_1, u_2, \ldots, u_m, w_1, w_2, \ldots, w_n \in Y$. Suppose there is no $x \in X$ connected to all $u_1, u_2 \ldots, u_m$, and not connected to any of $w_1, w_2 \ldots, w_n$. We define the partition $X = X_1 \cup \cdots \cup X_m \cup X'_1 \cup \cdots \cup X'_n$ as follows. First we enumerate the vertices in X: $X = \{x_k : k \in \omega\}$, and begin with $X_1, \ldots, X_m, X'_1, \ldots, X'_n$ empty. We proceed by recursion on k. If there is $i \in \{1, 2, \ldots, m\}$ such that $x_k u_i \notin E$, let r_k be such i that X_i is of the smallest cardinality (if there are several such i, choose any of them). On the other hand, if there is $j \in \{1, 2, \ldots, n\}$ such that $x_k w_j \in E$, let s_k be such j that X'_j is of the smallest cardinality. Now, if there is no j as above, put x_k in the set X_{r_k}; if there is no i as above then put x_k in the set X'_{s_k}. Otherwise, if $|X_{r_k}| \le |X'_{s_k}|$, put x_k in the set X_{r_k}, and otherwise in the set X'_{s_k}.

What we get in the end are disjoint sets $X_1, \ldots, X_m, X'_1, \ldots, X'_n$, each X_i containing no neighbors of u_i and each X'_i containing no non-neighbors of w_i. In addition, each X_i is infinite; suppose not and let, for example, X_i be the one with the smallest cardinality (the case when there are several such sets is considered similarly). Let $k_0 \in \omega$ be big enough such that by the k_0-th step of the recursion all the elements of X_i and at least $|X_i| + 1$ elements of other sets were added to them. Then for $k > k_0$ we would have no more elements x_k such that $x_k u_i \notin E$, which is impossible because u_i has infinitely many non-neighbors.

Now we partition Y into infinite pieces: $Y = Y_1 \cup \cdots \cup Y_m \cup Y'_1 \cup \cdots \cup Y'_n$ arbitrarily so that $u_i \in Y_i$ and $w_i \in Y'_i$. Finally, we split G into $m + n$ parts induced by $X_i \cup Y_i$ and $X'_i \cup Y'_i$. Clearly, none of the obtained sub-bigraphs can be isomorphic to G, since each of them either has an isolated vertex or a vertex connected to all the vertices from the opposite side, a contradiction. □

Let $G = (X, Y, E)$ be an $(\aleph_0, \aleph_0)$-bigraph. We denote $D_X = \{d(v) : v \in X\}$, $D_Y = \{d(v) : v \in Y\}$, $d_X = \min D_X$ and $d_Y = \min D_Y$.

Lemma 3. *If in an $(\aleph_0, \aleph_0)$-bigraph $G = (X, Y, E)$ that satisfies $\mathcal{P}'$ holds $d_X \in \omega \setminus \{0\}$ or $d_Y \in \omega \setminus \{0\}$, then G is isomorphic either to S or to S^*.*

Proof. Suppose $d_X \in \omega \setminus \{0\}$. We will consider the following cases:

Case 1° There is exactly one vertex in X of degree d_X, say x. Split G arbitrarily into sub-bigraphs G_1 and G_2. Let, for example, $x \in G_1$.

If G_1 is isomorphic to G, we move one of the neighbors of x (call it y) to G_2. If now $G_2 + y$ is isomorphic to G, it has a vertex x_2 of degree d_X, so we also move one of the neighbors of x_2 from G_1 to G_2.

If, on the other hand, G_2 is isomorphic to G then there is $x_2 \in X$ that has degree d_X in G_2. Now by moving one of the neighbors of x_2 (say y) from G_1 to G_2 we get a partition in which G_2 has no vertices of degree d_X. If now the other part $(G_1 - y)$ is isomorphic to G, we move one of the neighbors of x in the same direction.

Either way, we arrive at a partition into two parts, none of which is isomorphic to G.

Case 2° There are at least two (but finitely many) vertices in X of degree d_X, say $x_1, x_2, \dots, x_k$. Split G into two $(\aleph_0, \aleph_0)$-bigraphs G_1 and G_2 arbitrarily, but in such way that in each of the parts there is at least one of the vertices $x_1, x_2, \dots, x_k$. Let G_1 be isomorphic to G; then there are exactly k vertices of degree d_X on its left side. Let there be l vertices of degree d_X on the left side of G_2. If $l \geq k$ move one of the vertices $x_1, x_2, \dots, x_k$ from G_1 to G_2; otherwise move one of them in the other direction. In the partition obtained in this way none of the parts can be isomorphic to G.

Case 3° There are infinitely many vertices in X of degree d_X. If $d_X > 1$ then it is easy to split G so that there are two vertices in different parts such that each of them has in the induced sub-bigraph degree less than d_X. Now let $d_X = 1$. If there is no $y \in Y$ connected to all of the vertices of degree 1, we can split so that in both parts we get isolated vertices on the left side. So let $y \in Y$ be connected to all vertices of degree 1 in X. There may be no other vertices in X, because otherwise we could split into two parts G_1 and G_2 so that in G_1 we get isolated vertices and in G_2 having on the right side more that one vertex connected to vertices of degree 1. It follows that all the other vertices on the right side must be isolated, so the only remaining bigraph is S.

The corresponding cases when $d_Y \in \omega \setminus \{0\}$ are considered analogously and we get the bigraph S^* as another solution. □

This also takes care of the corresponding cases for $\bar{G}$ (i.e. when on one side of $\bar{G}$ there are vertices of finite degree, but no isolated vertices) and gives us two more solutions: $\bar{S}$ and $\bar{S}^*$. For the discussion in the upcoming theorem we will also need a property stronger than $\mathcal{P}'$:

$\mathcal{P}''$: for every partition of the set of vertices of G into finitely many pieces that each induce $(\aleph_0,\aleph_0)$-bigraphs all of the induced sub-bigraphs are isomorphic to G.

It is easy to see that, when investigating either of the properties $\mathcal{P}'$ and $\mathcal{P}''$, the condition is not weakened when "finitely many pieces" is replaced with "two pieces".

Lemma 4. *The only $(\aleph_0,\aleph_0)$-bigraphs with the property $\mathcal{P}''$ up to isomorphism are the empty $(\aleph_0,\aleph_0)$-bigraph and the complete $(\aleph_0,\aleph_0)$-bigraph.*

Proof. The cases $d_X \in \omega \setminus \{0\}$ and $d_Y \in \omega \setminus \{0\}$ are discussed in Lemma 3 and yield no solutions (S and S^* clearly do not satisfy $\mathcal{P}''$).

Let $d_X = 0$. The case when there are finitely many isolated vertices is easy: partition in an arbitrary way, and if both parts are isomorphic to G, then move one of the vertices that were isolated in G from one part to the other. If infinitely many vertices in X are isolated, we can split so that one of the parts is an empty bigraph, so G itself must be empty.

If in $\bar{G}$ there are vertices of finite degree, the proof is analogous, and gives us the complete bigraph. Finally, if both G and $\bar{G}$ have only vertices of infinite degree, by Lemma 2 we have no further solutions. □

Theorem 4. *The only $(\aleph_0,\aleph_0)$-bigraphs with the property $\mathcal{P}'$ up to isomorphism are the empty $(\aleph_0,\aleph_0)$-bigraph, the complete $(\aleph_0,\aleph_0)$-bigraph, the bigraphs S and S^* and their complements.*

Proof. Let $G = (X,Y,E)$ be an $(\aleph_0,\aleph_0)$-bigraph with property $\mathcal{P}'$. Since some of the cases were analyzed in Lemmas 2 and 3, we consider cases when one of the d_X and d_Y is 0, and the other is not in $\omega \setminus \{0\}$.

Case 1° Either X or Y has exactly one isolated vertex, say $x \in X$. We claim that $G - x$ (the bigraph obtained by removing x from G) has the stronger property $\mathcal{P}''$. To prove it, suppose $G - x$ is partitioned into two parts, thus inducing sub-bigraphs G_1 and G_2. If we add x as an isolated vertex to G_1, we get a partition of G into $G_1 + x$ and G_2. One of the parts must be isomorphic to G. Suppose it is G_2. Then, if we split G into G_1 and $G_2 + x$, G_1 must also be isomorphic to G. Hence each of the bigraphs G_1 and G_2 has exactly one isolated vertex, call them x_1 and x_2. But now, if we split G into $G_1 - x_1$ and $G_2 + x + x_1$, none of the two induced parts is isomorphic to G, a contradiction. So $G_1 + x$ must be isomorphic to G, and it follows that G_1 is isomorphic to $G - x$. In an analogous way we prove that G_2 is also isomorphic to $G - x$.

By Lemma 4 $G - x$ must be either empty or complete. The first possibility would mean that there were more than one isolated vertices in X, a contradiction. So G is isomorphic to $\bar{S}^*$.

If, instead of X, the right side Y has exactly one isolated vertex, we get $\bar{S}$ as a solution.

Case 2° There are at least two (but finitely many) isolated vertices in either X or Y; this is analogous to Case 2° of Lemma 3.

Case 3° There are infinitely many isolated vertices in both X and Y. Then we can split G into two empty sub-bigraphs. None of the parts is isomorphic to G, except if G itself is empty.

Case 4° There are infinitely many isolated vertices in X, and Y contains only vertices of infinite degree (or the other way around). Let $\langle x_k : k < \omega \rangle$ be an enumeration of the vertices from the left side. Again we will construct, by recursion on k, a partition that will be a counterexample for $\mathcal{P}'$. To begin with, let the sets X_0, X_1, Y_0 and Y_1 be empty. If x_k has "unused" neighbors from Y (not yet placed into Y_0 or Y_1), put x_k into X_1 and put one of its neighbors into Y_1; otherwise just place x_k in X_0. When we're finished, we add the rest of the vertices from Y to Y_0. Hence we obtained a partition of G into two parts, one of them containing no edges and the other without isolated vertices, which is a contradiction.

The corresponding cases for $\bar{G}$ (when $\bar{G}$ has isolated vertices) are discussed as above, so we also get the complete $(\aleph_0, \aleph_0)$-bigraph. □

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References

[1] A. Bonato, P. Cameron, and D. Delić, "Tournaments and orders with the pigeonhole property." *Canad. Math. Bull.*, vol. 43, pp. 397–405, 2000, doi: 10.4153/CMB-2000-047-6.

[2] P. Cameron, "The random graph," in *The Mathematics of Paul Erdos, Vol. 2*, R. L. Graham and J. Nešetřil, Eds. Berlin: Springer, 1997, ch. 7, pp. 333–351, doi: 10.1007/978-1-4614-7254-4.

[3] R. Diestel, I. Leader, A. Scott, and S. Thomassé, "Partitions and orientations of the Rado graph." *Trans. Am. Math. Soc.*, vol. 359, no. 5, pp. 2395–2405, 2007, doi: 10.1090/S0002-9947-06-04086-4.

[4] M. Goldstern, R. Grossberg, and M. Kojman, "Infinite homogeneous bipartite graphs with unequal sides." *Discrete Math.*, vol. 149, no. 1-3, pp. 69–82, 1996, doi: 10.1016/0012-365X(94)00339-K.

[5] B. Šobot, "Random bipartite graphs." *Novi Sad J. Math.*, vol. 44, no. 2, pp. 69–75, 2014.

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