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GENERAL DECAY OF SOLUTIONS FOR A NONLINEAR VISCOELASTIC WAVE EQUATION WITH NONLOCAL BOUNDARY DAMPING

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Abstract. In this work we show that under weaker assumptions on the memory kernel g, exponential and polynomial decay rates of the solution energy in Li and Zhao [8] are only special cases. Our result improves earlier results in the literature.

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1. INTRODUCTION

In this article, we investigate the following initial value problem

$$\begin{cases} u_{tt} - k_0 \Delta u + \displaystyle\int_0^t g(t-s)\mathrm{div}\big[a(x)\nabla u(s)\big]ds + b(x)h(u_t) = 0, \\ \qquad\qquad (x,t) \in \Omega \times (0,\infty), \\ -k_0 \dfrac{\partial u}{\partial \nu} + \displaystyle\int_0^t g(t-s)\big(a(x)\nabla u(s)\big)\,.\,\nu ds = f(u), \quad (x,t) \in \Gamma_1 \times (0,\infty), \\ u(x,t) = 0, \quad (x,t) \in \Gamma_0 \times (0,\infty), \\ u(x,0) = u_0(x) \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \end{cases} \tag{1.1}$$

where $k_0 > 0$, Ω is a bounded domain in $R^n (n \geq 1)$ with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1, \overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \varnothing$, Γ_0 and Γ_1 are closed with positive measures, ν is the unit outward normal to $\partial\Omega$, g denotes the memory kernel and a, b, h and f are real valued functions which satisfy appropriate conditions.

This problem arises in the study of motion of viscoelastic materials. We refer to [7,18] for mathematical analysis on the motions of materials with memory. The above problem with dirichlet boundary conditions has been considered by many authors. In this regard we recall the pioneer works by Cavalcanti et al. [1, 3], Santos [19] and Rivera et al. [15, 16].

In [8], Li and Zhao studied the problem (1.1) and proved exponential and polynomial decay results under weaker assumptions on g which improved [17]. In fact, in [17], the authors studied problem (1.1) with nonlinear boundary damping when $f(u) = |u|^{\gamma}u$ and $b(x) = 1$ on Ω. Assuming that the kernel g in the memory term decays exponentially, they showed exponential energy decay by using the perturbed energy method provided that $\|g\|_{L^1[0,\infty)}$ is small enough. In [9] Li et al. considered a related problem with nonlinear boundary dissipation. Under suitable conditions on the initial data and relaxation function, they established existence and uniqueness of global solutions by means of Galerkin method and showed that the energy decays exponentially if the decay rate of the memory kernel is also exponentially. These results have been recently improved by Wu and Chen [21] where the authors considered a nonlinear wave equation with boundary dissipation in presence of a local damping term, $b(x)u_t$, in Ω. The authors used Lyapunov functions to establish general decay rate of solution energy which is not necessarily of exponential or polynomial type. However, in order to prove main results, they supposed that the function $a(x)$ satisfies

$$|\nabla a(x)|^2 \leq \alpha_1^2 |a(x)|. \tag{1.2}$$

For more related results about the boundary stabilization we refer to Cavalcanti et al.[2], Liu and Yu [10], Lu et al. [11], Messaoudi and Soufyane [13]. We can also recall some other pioneer papers in connecting with the viscoelasticity such as Sobrinho and Rivera [20], Fabrizio and Polidro [6], Rivera et al. [14] and Dafermos [4].

In this work, we study problem (1.1). We show that for a certain class of relaxation functions, the decay rate of the energy is similar to that of g. Therefore, our result improves earlier results in [8, 9] where only the exponential and polynomial decay rates are obtained. The main point of the contribution is based on an inequality given by Martinez [12]. In this way, we are allowed to weaken some technical assumptions for the kernel g or even for the function $a(x)$ (such as inequality (1.2) which has been considered in [21]).

2. Preliminaries

In this section we present some materials that will be needed throughout the paper. We begin by presenting the precise hypotheses on the problem (1.1).

(H_1) $a, b : \Omega \to R$ are positive functions so that $a, b \in L^{\infty}(\Omega)$ and

$$b(x) \geq b_0 > 0. \tag{2.1}$$

(H_2) $f : R \to R$ satisfies

$$f(s)s \geq 2F(s) \geq 0, \quad \forall s \in R, \tag{2.2}$$

where

$$F(z) = \int_0^z f(s)ds. \tag{2.3}$$

(H_3) $h : R \to R$ is a nondecreasing function, such that for some positive constants α and β, satisfies

$$h(s)s \geq \alpha|s|^2, \qquad |h(s)| \leq \beta|s|, \qquad \forall s \in R. \tag{2.4}$$

(H_4) $g : [0,\infty) \to [0,\infty)$ is a non-increasing C^1 function such that

$$g(0) > 0, \qquad k_0 - \|a\|_\infty \int_0^{+\infty} g(s)ds = l > 0, \tag{2.5}$$

and there exists a non-increasing positive differentiable function ξ such that

$$g'(t) \leq -\xi(t)g(t), \qquad \forall t \geq 0, \qquad \int_0^{+\infty} \xi(s)ds = \infty. \tag{2.6}$$

We will also consider the Hilbert space

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) \ : \ u = 0 \text{ on } \Gamma_0\}.$$

Lemma 1 (Poincaré inequality). *There exists a positive constant B such that*

$$\|u\|_{L^2(\Omega)} \leq B\|\nabla u\|_{L^2(\Omega)},$$

for all $u \in H^1_{\Gamma_0}(\Omega)$.

Referring to [5, 17], we state the following existence and uniqueness theorem.

Theorem 1. *If $(u_0,u_1) \in \big(H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\big) \times H^1_{\Gamma_0}(\Omega)$, then the problem* (1.1) *has a unique solution satisfying*

$$u \in L^\infty_{loc}\big(0,\infty; H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega)\big), \quad u_t \in L^\infty_{loc}\big(0,\infty; H^1_{\Gamma_0}(\Omega)\big),$$

$$u_{tt} \in L^\infty_{loc}\big(0,\infty; L^2(\Omega)\big).$$

Moreover

$$u \in C\big([0,\infty); H^1_0(\Omega)\big), \qquad u_t \in C\big([0,\infty); L^2(\Omega)\big).$$

Finally, we present the following lemma by Martinez [12] which plays important role in our proof.

Lemma 2. *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonincreasing function and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a C^2 increasing function such that $\psi(0) = 0$ and $\lim_{t\to+\infty}\psi(t) = +\infty$. Assume that there exists $c > 0$ for which*

$$\int_t^{+\infty} \psi'(s)E(s)ds \leq cE(t), \qquad \forall t \geq 0, \tag{2.8}$$

then

$$E(t) \leq \lambda E(0)e^{-\omega\psi(t)}, \tag{2.9}$$

for some positive constants ω and λ.

3. ENERGY DECAY

In this section we state and prove our main result. First, we define the energy related to problem (1.1) as in [8]

$$E(t)=\frac{1}{2}\int_{\Omega}|u_t(t)|^2dx+\frac{1}{2}\int_{\Omega}\Big(k_0-a(x)\int_0^t g(s)ds\Big)|\nabla u(t)|^2dx$$
$$+\frac{1}{2}(g\circ\nabla u)(t)+\int_{\Gamma_1}F(u)d\Gamma, \tag{3.1}$$

where

$$(g\circ\nabla u)(t)=\int_{\Omega}\int_0^t g(t-s)a(x)|\nabla u(t)-\nabla u(s)|^2dsdx.$$

We also observe that

$$\int_{\Omega}\Big(k_0-a(x)\int_0^t g(s)ds\Big)|\nabla u(t)|^2dx\ge$$
$$\Big(k_0-\|a\|_{\infty}\int_0^t g(t)dt\Big)\|\nabla u(t)\|_2^2\ge l\|\nabla u(t)\|_2^2.$$

Lemma 3 (Lemma 2.3 of [8]). *The energy function $E(t)$ satisfies $E(t)\ge 0$ and*

$$E'(t)=\frac{1}{2}(g'\circ\nabla u)(t)-\frac{1}{2}g(t)\int_{\Omega}a(x)|\nabla u(t)|^2dx-\int_{\Omega}b(x)h(u_t)u_tdx\le 0. \tag{3.2}$$

Theorem 2. *Assume that $(u_0,u_1)\in\big(H^2(\Omega)\cap H^1_{\Gamma_0}(\Omega)\big)\times H^1_{\Gamma_0}(\Omega)$, $(H_1)-(H_3)$ and (2.5) hold, and*

$$a(x)\ge a_0>0. \tag{3.3}$$

(i) If

$$g'(t)\le -cg(t),\quad c>0, \tag{3.4}$$

then, the energy $E(t)$ of problem (1.1) satisfies the decay rate

$$E(t)\le 4E(0)e^{-Ct},\qquad \forall t\ge T.$$

(ii) If

$$g'(t)\le -cg^{1+\frac{1}{p}}(t),\quad p>2,\ \ c>0, \tag{3.5}$$

then, the energy $E(t)$ of problem (1.1) satisfies the decay rate

$$E(t)\le\frac{C}{(1+t)^p},\qquad \forall t\ge T,$$

for some $C,T>0$.

Proof. See [8], Theorems 3.1 and 3.2. □

In the next theorem, we state our main result. We extend the above rates of decay to a more general case which is similar to that of g. In fact, we use the assumption (2.6) which is weaker than (3.4) and (3.5) .

Theorem 3. *Assume that* $(\mathrm{H}_1)-(\mathrm{H}_4)$ *hold. If the initial data* $(u_0,u_1)\in\big(H^2(\Omega)\cap H^1_{\Gamma_0}(\Omega)\big)\times H^1_{\Gamma_0}(\Omega)$, *then the solution of problem* (1.1) *satisfies*

$$E(t)\le KE(0)e^{-\kappa\int_0^t\xi(s)ds}, \tag{3.6}$$

for some $K,\kappa>0$.

Proof. Multiplying $(1.1)_1$ by $\xi(t)u$ and integrating over $\Omega\times[t_1,t_2]$ $(0\le t_1\le t_2)$, we get

$$\begin{aligned}&\int_{t_1}^{t_2}\xi(t)\int_\Omega uu_{tt}dxdt+k_0\int_{t_1}^{t_2}\xi(t)\int_\Omega|\nabla u(t)|^2dxdt\\&+\int_{t_1}^{t_2}\xi(t)\int_\Omega b(x)uh(u_t)dxdt\qquad+\int_{t_1}^{t_2}\xi(t)\int_{\Gamma_1}uf(u)d\Gamma dt\\&-\int_{t_1}^{t_2}\xi(t)\int_\Omega\nabla u(t)\,.\int_0^t g(t-s)\big(a(x)\nabla u(s)\big)dsdxdt=0.\end{aligned} \tag{3.7}$$

For the last term in the right hand side of (3.7) we have

$$\begin{aligned}&\int_\Omega\nabla u(t)\,.\int_0^t g(t-s)\big(a(x)\nabla u(s)\big)dsdx\\&=\int_\Omega\nabla u(t)\,.\int_0^t g(t-s)a(x)\big(\nabla u(s)-\nabla u(t)\big)dsdx\\&\quad+\int_0^t g(s)ds\int_\Omega a(x)|\nabla u(t)|^2dx.\end{aligned} \tag{3.8}$$

Substituting (3.8) in (3.7) and using (2.2) , (2.3) and (3.1) we obtain

$$\begin{aligned}&2\int_{t_1}^{t_2}\xi(t)E(t)dt\\&\le-\int_{t_1}^{t_2}\xi(t)\int_\Omega uu_{tt}dxdt+\int_{t_1}^{t_2}\xi(t)\|u_t\|_2^2dt-\int_{t_1}^{t_2}\xi(t)\int_\Omega b(x)uh(u_t)dxdt\\&\quad+\int_{t_1}^{t_2}\xi(t)\int_\Omega\nabla u(t)\,.\int_0^t g(t-s)a(x)\big(\nabla u(s)-\nabla u(t)\big)dsdxdt\\&\quad+\int_{t_1}^{t_2}\xi(t)(g\circ\nabla u)(t)dt.\end{aligned} \tag{3.9}$$

For the first term in the right hand side of (3.9) we have

$$\begin{aligned}&-\int_{t_1}^{t_2}\xi(t)\int_\Omega uu_{tt}dxdt\\&=-\int_\Omega\xi(t)uu_tdx\Big|_{t_1}^{t_2}+\int_{t_1}^{t_2}\xi'(t)\int_\Omega uu_tdxdt+\int_{t_1}^{t_2}\xi(t)\|u_t\|_2^2dt.\end{aligned} \tag{3.10}$$

By Lemma 1 and (3.1) we have

$$\left|-\int_\Omega \xi(t)uu_t dx\Big|_{t_1}^{t_2}\right| \le \sum_{i=1}^{2}\left|\xi(t)\int_\Omega uu_t dx\right|_{t=t_i}$$
$$\le \sum_{i=1}^{2}\left[\xi(t)\left(\frac{B^2}{2}\|\nabla u\|_2^2+\frac{1}{2}\|u_t\|_2^2\right)\right]_{t=t_i}$$
$$\le \sum_{i=1}^{2}\left[\left(\frac{B^2}{l}+1\right)\xi(t)E(t)\right]_{t=t_i} \le 2\left(\frac{B^2+l}{l}\right)\xi(t_1)E(t_1). \quad (3.11)$$

Similarly,

$$\left|\int_{t_1}^{t_2}\xi'(t)\int_\Omega uu_t dxdt\right| \le \int_{t_1}^{t_2}|\xi'(t)|\left(\frac{B^2}{2}\|\nabla u\|_2^2+\frac{1}{2}\|u_t\|_2^2\right)$$
$$\le -\left(\frac{B^2+l}{l}\right)\int_{t_1}^{t_2}\xi'(t)E(t)dt \le \left(\frac{B^2+l}{l}\right)\xi(t_1)E(t_1). \quad (3.12)$$

To estimate the last term in the right-hand side of (3.10), we use (3.2), (2.1) and (2.4) to obtain

$$E'(t) \le -\int_\Omega b(x)h(u_t)u_t dx \le -b_0\int_\Omega h(u_t)u_t dx \le -b_0\alpha\|u_t\|_2^2.$$

Therefore,

$$\int_{t_1}^{t_2}\xi(t)\|u_t\|_2^2 dt \le -\frac{1}{b_0\alpha}\int_{t_1}^{t_2}\xi(t)E'(t)dt \le \frac{1}{b_0\alpha}\xi(t_1)E(t_1). \quad (3.13)$$

From (3.10)-(3.13) we get

$$\left|-\int_{t_1}^{t_2}\xi(t)\int_\Omega uu_{tt}dxdt\right| + \int_{t_1}^{t_2}\xi(t)\|u_t\|_2^2 dt \le \left(3l^{-1}(B^2+l)+\frac{2}{b_0\alpha}\right)\xi(t_1)E(t_1). \quad (3.14)$$

By (2.4), Young's inequality, (3.1) and (3.2) we arrive at

$$\left|-\int_{t_1}^{t_2}\xi(t)\int_\Omega b(x)uh(u_t)dxdt\right| \le \beta\int_{t_1}^{t_2}\xi(t)\int_\Omega b(x)\left(\frac{\delta B^2}{2}|\nabla u|^2+\frac{1}{2\delta}|u_t|^2\right)dxdt$$
$$\le \frac{\delta}{2}\beta B^2\|b\|_\infty\int_{t_1}^{t_2}\xi(t)\|\nabla u\|_2^2 dt - \frac{\beta}{2\delta\alpha}\int_{t_1}^{t_2}\xi(t)E'(t)dt$$
$$\le \frac{\delta}{l}\beta B^2\|b\|_\infty\int_{t_1}^{t_2}\xi(t)E(t)dt + \frac{\beta}{2\delta\alpha}\xi(t_1)E(t_1). \quad (3.15)$$

Also, we have

$$\int_\Omega a(x)\nabla u(t)\,.\int_0^t g(t-s)\big(\nabla u(s)-\nabla u(t)\big)dsdx$$

$$\le \delta \int_\Omega a(x)|\nabla u(t)|^2 dx + \frac{1}{4\delta}\int_\Omega a(x)\Big(\int_0^t g(t-s)\big(\nabla u(s)-\nabla u(t)\big)ds\Big)^2 dx$$

$$\le \delta \|a\|_\infty \|\nabla u\|_2^2 + \frac{1}{4\delta}\int_0^t g(s)ds \int_\Omega \int_0^t g(t-s)a(x)|\nabla u(s)-\nabla u(t)|^2 ds dx$$

$$\le \frac{2\delta}{l}\|a\|_\infty E(t) + \frac{k_0 - l}{4\delta\|a\|_\infty}(g\circ\nabla u)(t). \tag{3.16}$$

By (2.6) we have

$$\xi(t)(g\circ\nabla u)(t) \le -(g'\circ\nabla u)(t) \le -2E'(t). \tag{3.17}$$

Finally, using (3.14)-(3.17), the estimate (3.9) takes the form

$$\Big[2-\frac{\delta}{l}\Big(\beta B^2\|b\|_\infty + 2\|a\|_\infty\Big)\Big]\int_{t_1}^{t_2}\xi(t)E(t)dt$$

$$\le \Big(3l^{-1}(B^2+l)+\frac{2}{b_0\alpha}+\frac{\beta}{2\delta\alpha}\Big)\xi(t_1)E(t_1) - \Big(\frac{k_0-l}{2\delta\|a\|_\infty}+2\Big)\int_{t_1}^{t_2}E'(t)dt$$

$$\le \Big[\Big(3l^{-1}(B^2+l)+\frac{2}{b_0\alpha}+\frac{\beta}{2\delta\alpha}\Big)\xi(0) + \frac{k_0-l}{2\delta\|a\|_\infty}+2\Big]E(t_1) \tag{3.18}$$

Choosing δ small enough and letting t_2 go to infinity, we rewrite (3.18) as

$$\int_t^{+\infty}\xi(t)E(t)dt \le \lambda E(t), \qquad \forall t \ge 0, \tag{3.19}$$

for some $\lambda > 0$. Then, the assumptions of Lemma 2 satisfied with $\psi(t) = \int_0^t \xi(s)ds$. Therefore (3.6) is established and the proof of Theorem 3 is now complete. □

Remark 1. We note that similar rates of decay were given in [21]. However, we did not use (3.3) which has been supposed in [8]-H_1 and [21]-A_3.

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