Islands: from coding theory to enumerative combinatorics and to lattice theory - overview and open problems

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ISLANDS: FROM CODING THEORY TO ENUMERATIVE COMBINATORICS AND TO LATTICE THEORY – OVERVIEW AND OPEN PROBLEMS

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Abstract. As a tool to characterize instantaneous codes, Foldes and Singhi determined the maximum number of certain subsets of the set of the first *n* natural numbers in 2006. This motivated Czédli in 2009 to determine the maximum number of the analogous subsets of a rectangular grid. He called these subsets *islands*. The present paper summarizes recent results on this topic.

1. INTRODUCTION

The topic we are going to survey goes back to two papers. Firstly, to characterize instanteneous codes by equalities, Foldes and Singhi [9] considered certain families of subsets of $\{1, 2, \ldots, n\}$. They proved that the minimum number of eqations needed equals the maximum size of these families, see [9, Theorem 4].

Secondly, soon afterwards, Czédli [4] studied analogous families of certain subsets of the rectangular grid $\{1, \ldots, m\} \times \{1, 2, \ldots, n\}$. He coined the name (2-dimensional) *rectangular islands* for these subsets. In this terminology, the subsets of Foldes and Singhi are 1-dimensional islands. By the main result of [4], the maximum number of rectangular islands on a *m*-by-*n* grid is

$$f(m,n) = \left\lfloor \frac{mn + m + n - 1}{2} \right\rfloor. \tag{1.1}$$

To be more precise, f(m,n) is the maximum size of a family of rectangular islands on the *m*-by-*n* grid.

1.1. *Remark (added in July 30th, 2013).*

After the present paper was submitted, new lattice theoretical approach of the island topic came to existence, see Czédli [2]. This approach is related to [5] and to [14].

The aim of the present paper is to overview the results and directions appeared in several branches of mathematics after the above-mentioned two papers.

2. The definition of islands

First, we fix a finite grid, consisting of e.g. squares or equilateral triangles. Its cells constitute the so-called *board.* The cells have *neighbours.* In each particular cases, we have to define exactly, which cells are considered neighbours.

We put a real number into each cell, these are the so-called *heights.* Actually we have a real valued *height function* on the set of cells. We fix a shape (consisting of cells), e.g. rectangle or triangle. We call this rectangle/triangle an *island* if its cells have greater heights than the heights in the neighboring cells.

We call a rectangle/triangle on a given board an *island,* if for the cell t, if we denote its height by a_t, then for each cell $\hat{t}$ neighbouring with a cell of the rectange/triangle T, the inequality $a_{\hat{t}} < \min\{a_t : t \in T\}$ holds. For detailed didactidal introduction for the maximum number of rectangular islands see e.g. [10]. The collection of all islands for a given height function is called island system. Also, we will use the following terminology: system of rectangular islands, system of triangular islands, etc.
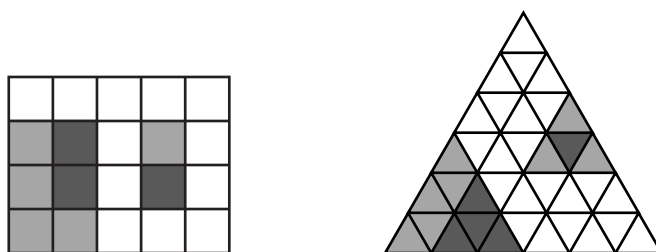

FIGURE 1. Rectangular and triangular islands

3. Proving results on the maximum number of islands

In this section, we present three proofs for formula (1.1) of Gábor Czédli [4] for rectangular islands. The presented methods are applicable for further cases, too.

3.1. *Lower bound*

We prove by induction on the number of cells that

$$f(m,n) \geq \left\lfloor \frac{mn+m+n-1}{2} \right\rfloor.$$

If $m = 1$, $n = 1$ or $m = n = 2$, then it is easy to check that the statement is true. Let $m,n > 2$. First, we put maximally many islands into the rectangles of sizes $(m-2) \times n$, and $1 \times n$. Between these rectangles, we put one row of cells with heights smaller then the minimum of the heights in the two rectangles, furthermore we put even smaller heights outside of our $m \times n$ rectangle. Then we apply the induction hypothesis, i.e. the inequation for smaller rectangles:

$$f(m,n) \geq f(m-2,n) + f(1,n) + 1 \geq$$

$$\geq \left\lfloor \frac{(m-2)n + (m-2) + n - 1}{2} \right\rfloor + \left\lfloor \frac{n+1+n-1}{2} \right\rfloor + 1 =$$

$$= \left\lfloor \frac{(m-2)n + (m-2) + n - 1 + 2n}{2} \right\rfloor + 1 = \left\lfloor \frac{mn+m+n-1}{2} \right\rfloor.$$

3.2. *Upper bound*

3.2.1. *Method A: lattice method*

The original method was based on lattice theory, using the result of [3] that any two weak bases of a finite distributive lattice have the same number of elements. This method produced the result first. This proof can be read in [4], and provided several research directions in lattice theory, e.g. [5,6,14].

A little bit later, two other methods appeared in [1]. In this paper we present only these proofs (method B and method C, as follows).

3.2.2. *Method B: induction*

If $m = n = 1$, then the statement is obviously true. Let $m > 1$ or $n > 1$. The induction hypothesis: if $u < m$ of $v < n$, then for the rectangle R of size $u \times v$, $f(R) = f(u,v) \leq \frac{1}{2}(u+1)(v+1) - 1$.

Let $\mathcal{J}^*$ denote such system of rectangular islands that contains maximally many rectangular islands. Denote by $\max \mathcal{J}^*$ the set of all maximal rectangular islands for $\mathcal{J}^*$, i.e. those set of rectangular islands that have only one bigger rectangular island. For rectangle of size $u \times v$ the number of grid points is $\|R\| = (u+1)(v+1)$. Now

$f(m,n) = 1 + \sum_{R \in \max \mathcal{J}^*} f(R) \leq 1 + \sum_{R \in \max \mathcal{J}^*} \left(\frac{1}{2}\|R\| - 1 \right) =$
$= 1 - |\max \mathcal{J}^*| + \frac{1}{2} \sum_{R \in \max \mathcal{J}^*} \|R\| \leq 1 - |\max \mathcal{J}^*| + \frac{1}{2}(m+1)(n+1).$
so we obtained $f(m,n) \leq 1 - |\max \mathcal{J}^*| + \frac{1}{2}(m+1)(n+1).$
If we have at least two maximal rectangular islands, then the proof is ready.

If we have only one maximal rectangular island, then one of the following inequalities are true:

$$f(m,n) \leq 1 - |\max \mathscr{I}^*| + \tfrac{1}{2}m(n+1) = 1 - 1 + \tfrac{1}{2}m(n+1) \leq \tfrac{1}{2}(m+1)(n+1) - 1.$$
$$f(m,n) \leq 1 - |\max \mathscr{I}^*| + \tfrac{1}{2}(m+1)n = 1 - 1 + \tfrac{1}{2}(m+1)n \leq \tfrac{1}{2}(m+1)(n+1) - 1.$$

If we have no maximal rectangular island, then we have only one rectangular island, so the proof is also ready.

3.2.3. *Method C: tree-graph method*

Our rectangular islands constitute tree graph by inclusion. In Figure 2, the islands are represented by grey rectangles, labelled at their bottom right corner. The heights of cells are indicated at their top left corners. The island R_i and, if exists, R_i' occur at water level i.
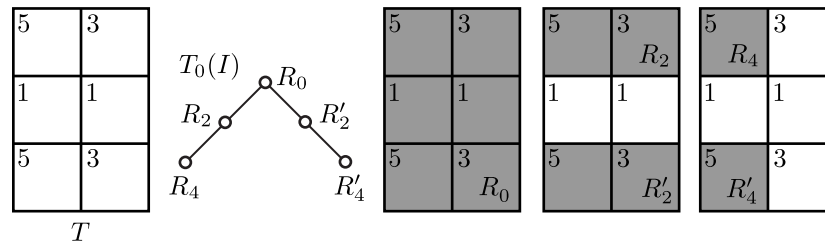

FIGURE 2. Islands and the corresponding tree

First, we need a Lemma.

Lemma 1. *Let T be a rooted tree such that any non-leaf node has at least 2 sons. Let ℓ be the number of leaves in T. Then $|V| \leq 2\ell - 1$.*

Proof. Let us direct the tree-graph from its root to the direction of its leaves. Then for the in-degrees and out-degrees the equation $\sum D^+ = \sum D^-$ holds. Now $\sum D^+ = |V| - 1$ because each nod has father except for the root. Moreover $\sum D^- \geq 2(|V| - \ell)$ because all non-leaf nod has at least two sons. So $\sum D^+ = |V| - 1 = \sum D^- \geq 2(|V| - \ell)$, i.e. we obtained $2\ell - 1 \geq |V|$.

$\square$

As we mentioned, our rectangular islands constitute tree-graphs by inclusion. For having at least binary tree-graph in all cases (condition of Lemma 1), we introduce the so-called "dummy island" in case the island shrinks when the water level increases. Figure 3 is derived from Figure 2. The dummy islands are depicted as thin, dark-grey rectangles.

We denote by s the number of the minimal rectangular islands, by d the number of dummy islands. By Figures 2 and 3, each minimal rectangular island covers at least four grid-points, each dummy island covers at least two grid-points.
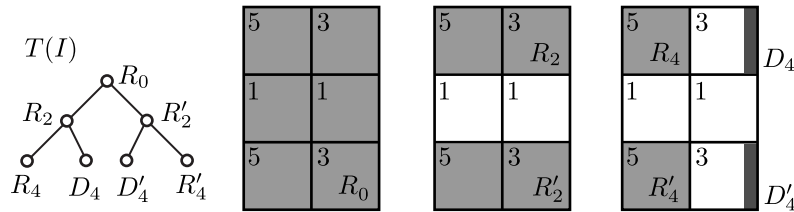
FIGURE 3. Islands and the corresponding tree with dummy islands D_4 and D_4'

This way we have covered not more then all the grid-points of the square grid, i. e.: $4s + 2d \leq (n+1)(m+1)$.

The number of leaves of is $\ell = s + d$. Hence by Lemma the number of islands is $|V| - d \leq (2\ell - 1) - d = 2s + d - 1 \leq \frac{1}{2}(n+1)(m+1) - 1$.

4. BRICK ISLANDS, HIGHER DIMENSIONS

In [4] a question was raised about the analogous problem in higher dimensions. Pluhár in [25] gave upper and lower bound for the maximum number of brick islands.

5. THE MAXIMUM NUMBER OF TRIANGULAR ISLANDS

Denote by $\mathrm{tr}(n)$ the maximum number of triangular islands in the equilateral triangle of sidelength n. The following lower and upper bounds are proved in [13]:

$$\frac{n^2 + 3n}{5} \leq \mathrm{tr}(n) \leq \frac{3n^2 + 9n + 2}{14}.$$

Furthermore, $\mathrm{tr}(n) = \frac{3n^2 + 9n + 2}{14}$ for infinitely many n. It is proved in [17] that

$$\lim_{n \to \infty} \frac{\mathrm{tr}(n)}{n^2} = \frac{3}{14}.$$

In [17] triangular islands were investigated not only on triangular, but also on trapezoid and paralelogram board, with triangular grid.

5.1. *Problem*

On triangular grid, not only triangular islands could be considered, but also other shapes, e.g. diamonds, paralelograms, trapezoids, hexagons.

6. THE MAXIMUM NUMBER OF SQUARE ISLANDS

It is proved in [12] that for the square grid of size $(m-1) \times (n-1)$ the maximum number of square islands $\mathrm{sq}(m,n)$:

$$\frac{1}{3}(mn - 2m - 2n) \leq \mathrm{sq}(m,n) \leq \frac{1}{3}(mn - 1).$$

This upper bound is sharp, whenever $m = n = 2^k$ for some $k \in \mathbb{N}$. Analogous upper and lower bounds are valid for the higher dimensional cases.

However, similarly to the triangular case, the lower and upper bounds are close, but exact formula for this case is also has not been proved.

7. EXACT RESULTS

The proofs of the following statements can be found in [1].

7.1. *Peninsulas (semi islands)*

Here we consider rectangular islands on the square grid that reaches at least one side of the rectangular board. The maximum number of such islands is:

$$p(m,n) = f(m,n) = \lfloor (mn + m + n - 1)/2 \rfloor.$$

7.2. *Cylindric board, rectangular islands*

We put square grid onto the surface of a cylinder. Denote by $h_1(m,n)$ the maxium number of rectangular islands: If $n \geq 2$, then $h_1(m,n) = \left\lfloor \frac{(m+1)n}{2} \right\rfloor$.

7.3. *Cylindric board, cylindric and rectangular islands*

It is possible to create cylindric islands on sylindric board. Denote by $h_2(m,n)$ the maxium number of rectangular or cylindric islands on the surface of a cylinder of height m. Of course, in this case, the maximum number of cilindric islands is more than in the previous case: If $n \geq 2$, then $h_2(m,n) = \left\lfloor \frac{(m+1)n}{2} \right\rfloor + \left\lfloor \frac{(m-1)}{2} \right\rfloor$.

7.4. *Torus board, rectangular islands*

We fold a torus from a rectangle of size $m \times n$. Denote by $t(m,n)$ the maximum number or rectangular islands on this board. If $m, n \geq 2$, then $t(m,n) = \left\lfloor \frac{mn}{2} \right\rfloor$.

7.5. *Islands in Boolean algebras*

Here the board consists of all vertices of a hypercube, i.e. the elements of a Boolean algebra $B = \{0,1\}^n$. We consider two cells neighbouring if their Hamming distance is 1. Our islands are Boolean algebras. We denote the maximum number of islands in the Boolean algebra $B = \{0,1\}^n$ by $b(n)$. Then, $b(n) = 1 + 2^{n-1}$.

8. THE MINIMUM SIZES OF MAXIMUM SYSTEMS OF ISLANDS

In general, a system of islands is maximal, if it cannot be extended to a larger system of islands.

Systems of rectangular islands on a $m \times n$ rectangle R constitute a partially ordered set I_R with respect to set inclusion. Let

$$g_r(m,n) = \min\{H : H \in \max(I_R)\}.$$

In [18] Lengvárszky proved that $g_r(m,n) = m+n-1$.

Also, he treated the triangular case in [17] and if I_T is the partially ordered set of triangular islands on the triangular board and if

$$g_t(m,n) = \min\{H : H \in \max(I_T)\},$$

then he obtained that $g_t y(n) = n$. In [17], the minimum number of maximal systems of triangular islands are also estimated on diamond, trapezoid and paralellogram, giving possibility for further research.

The minimum cardinality of maximal systems of square islands on a n^2 board is considered in [19]. Let

$$g_s(n) = \min\{H : H \in \max(I_S)\}.$$

In [19] Lengvárszky proved that $g_s(n) = n$.

In [7] Eccles investigated the higher dimensional generalizations, and he proved formula for the minimal size of cuboid islands $g_d(m_1,\ldots,m_d)$ in a cuboid of size $m_1 \times \cdots \times m_d$, namely: $g_d(m_1,\ldots,m_d) = \sum_{i=1}^{d} m_i - (d-1)$. Moreover, he proved that in a cube of size m^d the minimal size of cubic islands is given by $g'_d(m,\ldots,m) = m$.

9. CD-INDEPENDENCE AND CDW-INDEPENDENCE

It was observed that many subsets in island problems are CD-independent.

Let $\mathbb{P} = (P, \leq)$ be a partially ordered set and $a,b \in P$. The elements a and b are called *disjoint* and we write $a \perp b$ if

either $\mathbb{P}$ has least element $0 \in P$ and $\inf\{a,b\} = 0$,

or $\mathbb{P}$ is without 0 and the elements a and b have no common lowerbound.

Notice, that $a \perp b$ implies $x \perp y$ for all $x,y \in P$ with $x \leq a$ and $y \leq b$.

A nonempty set $X \subseteq P$ is called *CD-independent* if for any $x, y \in X$, $x \leq y$ or $y \leq x$, or $x \perp y$ holds. Maximal CD-independent sets (with respect to $\subseteq$) are called *CD-bases* in $\mathbb{P}$. In some papers, e.g. in [20, 21, 24] CD-independent sets are called *laminar* systems.

In [5] the authors showed that the CD-bases in a finite distributive lattice have the same number of elements, and conversely, if all finite lattices in a lattice variety have this property, then the variety must coincide with the variety of distributive lattices. From the proof of the main result of [5] it is clear that if we consider the maximum number of islands of arbitrary connected shape on the rectangular board, then it is bounded from above by the cardinality of a maximal chain of the powerset of the set of the cells, i.e. by $m \times n$.

In paper [14] it is shown that the CD bases of any poset $\mathbb{P}$ can be characterized as maximal chains in a related poset $\mathcal{D}(P)$. A nonempty set D of nonzero elements of P is called a *disjoint set* in $\mathbb{P}$, if $x \perp y$ holds for all $x,y \in D$, $x \neq y$; if the poset $\mathbb{P}$ contains 0-element, then $\{0\}$ is considered to be a disjoint set, too. Let $\mathcal{D}(P)$ denote

the set of all disjoint sets of $\mathbb{P}$. It is proved in [14] that if $\mathbb{P}$ is a complete lattice, then $\mathcal{D}(P)$ is also a lattice having a weak distributive property. If $\mathbb{P} = (P, \wedge)$ is a semilattice with 0, then for any $a, b \in P$ the relation $a \perp b$ means that $a \wedge b = 0$. Hence, a set $\{a_i \mid i \in I\}$ of nonzero elements is a disjoint system if and only if $a_i \wedge a_j = 0$, for all $i, j \in I$, $i \neq j$. A pair $a, b \in P$ with least upper bound $a \vee b$ in $\mathbb{P}$ is called a *distributive pair*, if $(c \wedge a) \vee (c \wedge b)$ exists in $\mathbb{P}$ for any $c \in P$, and $c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b)$. We say that $(P, \wedge)$ is *dp-distributive* (*distributive with respect to disjoint pairs*), if any $a, b \in P$ with $a \wedge b = 0$ is a distributive pair. Two known lattice classes are pointed out where the CD-bases in finite lattices have the mentioned property: The first class is the graded, dp-distributive lattices, and the second class is obtained by generalizing the properties of the so-called interval lattices (having their origin in graph theory).

A family $\mathcal{H} \subseteq P$ is *weakly independent* if

$$H \leq \bigvee_{i \in I} H_i \implies \exists i \in I : H \leq H_i$$

holds for all $H \in \mathcal{H}, H_i \in \mathcal{H} \, (i \in I)$. If $\mathcal{H}$ is both CD-independent and weakly independent, then we say that $\mathcal{H}$ is *CDW-independent*.

It is proved in [6] that any two CDW-bases of a finite distributive lattice have the same number of elements. Moreover, if a lattice variety contains a nondistributive lattice, then there exists a finite lattice in this variety that has two CDW-bases with different number of elements.

9.1. *Problem*

It is conjectured that the formula of Czédli for the maximum number of rectangular islands (and other cases) can be proved by using the main result of [5] or [6], but by our knowledge, up to now nobody has elaborated the proof. We encourage the reader to do it, as different proofs might lead to different research directions.

10. Height function with finite range

If we put only finitely many heights into the square cells of a rectangle, we obtain a much more complicated problem for rectangular islands, which is solved only for one dimension. Namely, assume that there are n cells in a single row and each cell has to be of height at least 0 and at most h, and let $I(n, h)$ denote the maximum number of islands in this case. It was proved in [11] that

$$I(n, h) = n - \left\lfloor \frac{n}{2^h} \right\rfloor$$

if we have n cells in one row. Notice that the whole board is not necessarily an island, only if we do not use the height 0. Now from this formula it is easy to see that if the height is at least 1 and at most h, then the maximum number of islands is:

$$I'(n,h) = n + 1 - \left\lfloor \frac{n}{2^{h-1}} \right\rfloor$$

as it appears in [22].

In [22] the cases $2 \times n$ and $3 \times n$ are solved if if the height is at least 1 and at most h. They obtained

$$I''(n,h) = \left\lfloor \frac{3n+1}{2} \right\rfloor + 1 + \left\lceil \frac{n}{2^{h-2}} \right\rceil$$

for the $2 \times n$ case and

$$I'''(n,h) = 2n + 2 + \left\lceil \frac{n}{2^{h-2}} \right\rceil$$

for the $3 \times n$ case.

10.1. *Problem*

The larger boards (m-by-n board with $m \geq 4$) were found problematic by [22].

11. ISLANDS AND CUTS OF LATTICE VALUED FUNCTIONS

A rectangular board equipped with a height fuction can be considered a fuzzy relation, see [16]. Fuzzy relations in many cases model real life better that crisp ones. Here we interpret the results of [16] with using the classical „height function" terminology only.

A *height function h* is a mapping from $\{1, 2, ..., m\} \times \{1, 2, ..., n\}$ to $\mathbb{N}$, $h : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$. Actually, a *fuzzy relation* in [16] is a mapping from $\{1, 2, ..., m\} \times \{1, 2, ..., n\}$ to $[0, 1]$, $h : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to [0, 1]$, but the results of [16] are valid if we consider co-domain $\mathbb{N}$.

The co-domain of the height function is the lattice $(\mathbb{N}, \leq)$, where $\mathbb{N}$ is the set of natural numbers under the usual ordering $\leq$ and suprema and infima are max and min, respectively.

For every $p \in \mathbb{N}$, the *cut of the height function*, p-*cut* of h is an ordinary relation h_p on $\{1, 2, ..., m\} \times \{1, 2, ..., n\}$ defined by

$$(x, y) \in h_p \text{ if and only if } h(x, y) \geq p.$$

We say that two rectangles $\{\alpha, ..., \beta\} \times \{\gamma, ..., \delta\}$ and $\{\alpha_1, ..., \beta_1\} \times \{\gamma_1, ..., \delta_1\}$ are *distant* if they are disjoint and for every two cells, namely (a, b) from the first rectangle and (c, d) from the second, we have $(a - c)^2 + (b - d)^2 \geq 4$.

The height function h is called *rectangular* if for every $p \in \mathbb{N}$, every nonempty p-cut of h is a union of distant rectangles.

We denote by $\mathcal{I}_{rect}(h)$ the system of islands defined by the height function h.

In [16] it is proved that for every height function

$$h : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \to \mathbb{N},$$

there is a rectangular height function

$$h^* : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \to \mathbb{N},$$

such that $\mathcal{I}_{rect}(h) = \mathcal{I}_{rect}(h^*)$.

Moreover, in [16] there is a representation theorem, namely:
let $h : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$ be a rectangular height function. Then there is a lattice L and an L-valued fuzzy relation Φ, such that the cuts of Φ are precisely all islands of h.

It is proved in [15] that for every rectangular height function

$$h^* : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \to \mathbb{N},$$

there is a rectangular height function

$$h^{**} : \{1, 2, ..., n\} \times \{1, 2, ..., m\} \to \mathbb{N},$$

such that $\mathcal{I}_{rect}(h^*) = \mathcal{I}_{rect}(h^{**})$ and in h^{**} every island appears exactly in one cut.

If a rectangular height function h^{**} has the property that each island appears exactly in one cut, then we call it *standard rectangular height function.*

We denote by $\Lambda_{max}(m, n)$ the maximum number of different nonempty p-cuts of a standard rectangular height function on the rectangular table of size $m \times n$. It is obtained in [15] that $\Lambda_{max}(m, n) = m + n - 1$. If $m, n \geq 3$ and the height function has maximally many rectangular islands, then the number of different nonempty cuts are strictly less than $\Lambda_{max}(m, n)$.

It is an interesting fact proved in [15] that if $m \geq 3$ and $n \geq 3$ and a height function $h : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$ has maximally many rectangular islands, then it has exactly two maximal islands.

We denote by $\Lambda_h^{cz}(m, n)$ the number of different nonempty cuts of a standard rectangular height function h in the case h has maximally many islands, i.e., when the number of islands is given by (1.1).

Let $h : \{1, 2, ..., m\} \times \{1, 2, ..., n\} \to \mathbb{N}$ be a standard rectangular height function having maximally many islands $f(m, n)$. Then,

$$\Lambda_h^{cz}(m, n) \geq \lceil log_2(m + 1) \rceil + \lceil log_2(n + 1) \rceil - 1,$$

this lower bound is sharp, moreover if $m, n \geq 2$, then $\Lambda_h^{cz}(m, n) \leq \lfloor \frac{m+n+3}{2} \rfloor$, in addition for $m, n \geq 3$ this bound is also sharp, all proved also in [15]. This result might lead us closer to the solution to the two-dimensional problem of height function with finite range, mentioned in the former section.

12. CONTINUOUS CASE

Up to now in this paper, only discrete boards were considered. In [21], Lengvárszky and Pach investigated continuous board; real-valued height function f is defined on a closed rectangle $R \subset \mathbb{R}^n$. A rectangle $S \subset R$ is a f-island, if there exists an open set $G \subset R$ containing S such that $f(x) < \inf_S f$ for every $x \in G \setminus S$. The set of all f-islands for a fixed f are called a *system of rectangular islands*. In [21] it is proved that the size of a maximal system of rectangular islands is either countable or continuum, both exist.

In paper [24] Pach et al. investigated the maximum number of continuous islands of arbitrary form. From an obtained more general condition they derived that the cardinality of maximal laminar i.e. CD-independent system of closed discs in $\mathbb{R}^n$ is either countable or continuum. They also proved that all island systems are laminar, but not every laminar system is a system of islands for some height function. This paper looks for necessary conditions for a laminar system in order to be a system of islands.

The paper [20] investigates island systems with continuous height functions. It shows that these system of islands are strongly laminar, i.e. laminar with the property that every two distinct sets in it have disjoint boundaries, however not vice versa. It is shown also in this paper that in the discrete case, i.e. for rectangular islands with integer cordinates for a maximal rectangular system of islands with continuous height function $|\mathcal{H}|$ on an $m \times n$ grid we have $\left\lceil \frac{\min(m,n)}{4} \right\rceil \leq |\mathcal{H}| \leq \left\lceil \frac{m}{2} \right\rceil \left\lceil \frac{n}{2} \right\rceil$. In the continous case this paper gives sufficient conditions for maximal strongly laminar systems to have cardinality countable or continuum.

13. GENERALIZATION

It is possible to generalize the notion of islands in such a way that formal concepts and prime implicants of Boolean functions are covered, the details can be read in [8].

An *island domain* is a pair $(\mathcal{C}, \mathcal{K})$, where $\mathcal{C} \subseteq \mathcal{K} \subseteq \mathcal{P}(U)$ for some nonempty finite set U such that $U \in \mathcal{C}$. By a *height function* we mean a map $h: U \to \mathbb{R}$.

We denote the cover relation of the poset $(\mathcal{K}, \subseteq)$ by $\prec$, and we write $K_1 \preceq K_2$ if $K_1 \prec K_2$ or $K_1 = K_2$.

We say that S is a *pre-island* with respect to the triple $(\mathcal{C}, \mathcal{K}, h)$, if every $K \in \mathcal{K}$ with $S \prec K$ satisfies

$$\min h(K) < \min h(S).$$

We say that S is an *island* with respect to the triple $(\mathcal{C}, \mathcal{K}, h)$, if every $K \in \mathcal{K}$ with $S \prec K$ satisfies

$$h(u) < \min h(S) \text{ for all } u \in K \setminus S.$$

The paper gives necessary condition for a system of pre-islands, called admissibility. This condition is not a sufficient one, hovever the paper proves that maximal pre-island systems and maximal admissible families are the same. An island domain

$(\mathcal{C}, \mathcal{K})$ is a *connective island domain* if

$$\forall A, B \in \mathcal{C} : (A \cap B \neq \varnothing \text{ and } B \nsubseteq A) \implies \exists K \in \mathcal{K} : A \subset K \subseteq A \cup B.$$

A result of [8] is that the following three conditions are equivalent for any island domain $(\mathcal{C}, \mathcal{K})$:

(i) $(\mathcal{C}, \mathcal{K})$ is a connective island domain.
(ii) Every system of pre-islands corresponding to $(\mathcal{C}, \mathcal{K})$ is CD-independent.
(iii) Every system of pre-islands corresponding to $(\mathcal{C}, \mathcal{K})$ is CDW-independent.

In connective island domains the systems of pre-islands are exactly the admissible systems. A binary relation $\delta \subseteq \mathcal{C} \times \mathcal{C}$ is defined that expresses the fact that a set $B \in \mathcal{C}$ is in some sense close to a set $A \in \mathcal{C}$:

$$A \delta B \Leftrightarrow \exists K \in \mathcal{K} : \ A \preceq K \text{ and } K \cap B \neq \varnothing.$$

We say that $A, B \in \mathcal{C}$ are *distant* if neither $A\delta B$ nor $B\delta A$ holds. The island domain $(\mathcal{C}, \mathcal{K})$ is called a *proximity domain*, if it is a connective island domain and the relation δ is symmetric for nonempty sets, that is

$$\forall A, B \in \mathcal{C} \setminus \{\varnothing\} : \ A\delta B \Leftrightarrow B\delta A.$$

It is proved in this paper that in proximity domains systems of islands are exactly the distant families.

14. Didactical aspects

It is worth to introduce this topic to young students because there are several elementary problems that they can treated themselves. Some of these approaches can be found in [10], [11], [22] or [23].

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