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SQUARES OF CONGRUENCE SUBGROUPS OF THE EXTENDED MODULAR GROUP

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Abstract. In this paper, we generalize some results related to the congruence subgroups of modular group Γ, given in [7] and [6] by Kiming, Schütt, and Verrill, to the extended modular group Π.

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1. INTRODUCTION

The modular group $\Gamma = PSL(2,\mathbb{Z})$ is the discrete subgroup of $PSL(2,\mathbb{R})$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \qquad \text{and} \qquad S(z) = -\frac{1}{z+1}.$$

Then modular group Γ has a presentation

$$\Gamma = < T, S \mid T^2 = S^3 = I > \cong \mathbb{Z}_2 * \mathbb{Z}_3.$$

The extended modular group $\Pi = PGL(2,\mathbb{Z})$ has been defined by adding the reflection $R(z) = 1/\overline{z}$ to the generators of the modular group Γ. The extended modular group Π has a presentation, see [5],

$$\Pi = < T, S, R \mid T^2 = S^3 = R^2 = (RT)^2 = (RS)^2 = I > \cong D_2 *_{\mathbb{Z}_2} D_3.$$

Here T, S and R have matrix representations

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

respectively (in this work, we identify each matrix A in $GL(2,\mathbb{Z})$ with $-A$, so that they represent the same element of $PGL(2,\mathbb{Z})$). Thus the modular group $\Gamma = PSL(2,\mathbb{Z})$ is a subgroup of index 2 in the extended modular group Π.

Let us define Π^m as the subgroup generated by the m^{th} powers of all elements of Π, for some positive integer m. The subgroup Π^m is called the m^{th} *power subgroup* of Π. As fully invariant subgroups, they are normal in Π.

Then, power subgroups of the extended modular group Π were examined by Sahin, Ikikardes and Koruoglu in [10]. The authors showed that

$$\left|\Pi : \Pi^2\right| = 4,\ \Pi^2 = \Gamma^2.$$
$$\Pi^2 = < S, TST \mid (S)^3 = (TST)^3 = I > \cong \mathbb{Z}_3 * \mathbb{Z}_3,$$

Also, from [5], we have the following. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represent a general element of Π. For each integer $N \geq 1$, we define

$$\Pi(N) = \{A \in \Pi \mid a \equiv d \equiv \pm 1 \text{ and } b \equiv c \equiv 0 \pmod N\},$$
$$\Gamma(N) = \Pi(N) \cap \Gamma.$$

These are normal subgroups of finite index in Π, and they are called as the *principal congruence subgroups.* If $N > 2$ then $\Pi(N) = \Gamma(N)$ and if $N = 2$ then $\Pi(2) \geq \Gamma(2) \geq \Pi(4) = \Gamma(4)$. A subgroup K of Π contains some $\Pi(N)$ if and only if it contains some $\Gamma(N)$. Such a subgroup K is called a *congruence subgroup*, and the *level* of K is the least n such that $\Pi(N) \leq K$. Any other subgroup of finite index in Π is called a *non-congruence subgroup*.

The most important of the congruence subgroups of Π are

$$\Pi_0(N) = \{A \in \Pi \mid c \equiv 0 \pmod N\}$$

and

$$\Pi^1(N) = \{A \in \Pi \mid a \equiv d \equiv \pm 1 \text{ and } c \equiv 0 \pmod N\}.$$

From [9], it is known that

$$\Pi_0(N) = \Gamma_0(N) \cup TR.\Gamma_0(N) \quad \text{and} \quad \Pi^1(N) = \Gamma^1(N) \cup TR.\Gamma^1(N)\ .$$

Also, it is clear that $\Pi^1(N) \lhd \Pi_0(N)$ and for $N > 2$, $\left|\Pi_0(N) : \Pi^1(N)\right| = \varphi(N)/2$ where φ is the Euler Phi function (for the index $\Gamma^1(N)$ in $\Gamma_0(N)$, see [4]).

On the other hand, in [7] and [6], Kiming Schütt, and Verrill studied lifts of projective congruence subgroups. Now, we recall the following information from [7]. For a subgroup Λ of $SL(2, \mathbb{Z})$ denote by $\overline{\Lambda}$ the image of Λ in $PSL(2,\mathbb{Z})$. A lift of $\overline{\Lambda}$ is a subgroup of $SL(2, \mathbb{Z})$ that projects to $\overline{\Lambda}$ in $PSL(2,\mathbb{Z})$. A lift is called a congruence lift if it is a congruence subgroup.

In [7] and [6], the authors gave some consequences of their main results for the groups generated by squares of elements in congruence subgroups. These results are

$a)$ $\Gamma(N)^2$ is a congruence if and only if $N \leq 2$.

$b)$ All lifts of $\Gamma_0(N) \leq PSL(2,\mathbb{Z})$ are congruence subgroups of $SL(2,\mathbb{Z})$ if and only if either $N \in \{3,4,8\}$ or if $4 \nmid N$ and all odd prime divisors of N are congruent to 1 modulo 4.

c) All lifts of $\Gamma^1(N) \leq PSL(2,\mathbb{Z})$ are congruence subgroups of $SL(2,\mathbb{Z})$ if and only if $N \leq 4$.

The congruence and principal congruence subgroups (especially, $\Pi(2)$, $\Gamma(2)$, $\Gamma_0(N)$ and $\Gamma^1(N)$) of Γ and Π have been studied from various aspects in the literature, for example, number theory, modular forms, modular curves, Belyi's theory, graph theory, (please see [1], [2], [3] and [8]).

In this paper, we generalize the above results related with congruence subgroups of Γ, given in [7] and [6], to the extended modular group Π.

2. Squares of Congruence Subgroups of Π

From [5], if $N > 2$ then $\Pi(N) = \Gamma(N)$ and so $\Pi(N)^2 = \Gamma(N)^2$. Thus, if $N > 2$ then $\Pi^2(N)$ is not a congruence. Also, from [10] and [5], $\Pi^2(1) = \Pi'$ and $\Pi(6) \leq \Pi^2(1)$ and so $\Pi^2(1)$ is a congruence subgroup. Therefore we need the following theorem.

Theorem 1. $\Pi(2)^2 = \Pi(4)$.

Proof. We know that the group structure of $\Pi(2)$ is

$$\begin{aligned} \Pi(2) &= < TR, RSTS, RS^2TS^2 \mid (TR)^2 = (RSTS)^2 = (RS^2TS^2)^2 = I > \\ &\cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2. \end{aligned}$$

Let $a = TR$, $b = RSTS$, $c = RS^2TS^2$. Then the quotient group $\Pi(2)/\Pi(2)^2$ is the group obtained by adding the relation $X^2 = I$ for all $X \in \Pi(2)$ to the relations of $\Pi(2)$. Thus we have

$$\Pi(2)/\Pi(2)^2 \cong < a,\ b,\ c \mid a^2 = b^2 = c^2 = (ab)^2 = (ac)^2 = (bc)^2 = ... = I > .$$

As $a^2 = b^2 = c^2 = I$, we obtain

$$\Pi(2)/\Pi(2)^2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Therefore, we obtain $\left|\Pi(2) : \Pi(2)^2\right| = 8$.

Thus we use the Reidemeister-Schreier process to find the presentation of the subgroup $\Pi(2)^2$. Now we choose $\Sigma = \{I, a, b, c, ab, ac, bc, abc\}$ as a Schreier transversal for $\Pi(2)^2$. According to the Reidemeister-Schreier method, we can form all possible products :

$$\begin{array}{lll}
I.a.(a)^{-1} = I, & I.b.(b)^{-1} = I, & I.c.(c)^{-1} = I, \\
a.a.(I)^{-1} = I, & a.b.(ab)^{-1} = I, & a.c.(ac)^{-1} = I, \\
b.a.(ab)^{-1} = baba, & b.b.(I)^{-1} = I, & b.c.(bc)^{-1} = I, \\
c.a.(ac)^{-1} = caca, & c.b.(bc)^{-1} = cbcb, & c.c.(I)^{-1} = I, \\
ab.a.(b)^{-1} = abab, & ab.b.(a)^{-1} = I, & ab.c.(abc)^{-1} = I, \\
ac.a.(c)^{-1} = acac, & ac.b.(abc)^{-1} = acbcba, & ac.c.(a)^{-1} = I, \\
bc.a.(abc)^{-1} = bcacba, & bc.b.(c)^{-1} = bcbc, & bc.c.(b)^{-1} = I, \\
abc.a.(bc)^{-1} = abcacb, & abc.b.(ac)^{-1} = abcbca, & abc.c.(ab)^{-1} = I,
\end{array}$$

as $a^{-1} = a$, $b^{-1} = b$, and $c^{-1} = c$. Also, since $(baba)^{-1} = abab$, $(caca)^{-1} = acac$, $(cbcb)^{-1} = bcbc$, $(bcacba)^{-1} = abcacb$ and $(acbcba)^{-1} = abcbca$, the generators of $\Pi(2)^2$ are $abab = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$, $acac = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$,
$bcbc = \begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix}$, $abcacb = \begin{pmatrix} -7 & -12 \\ -4 & -7 \end{pmatrix}$ and $abcbca = \begin{pmatrix} 5 & -4 \\ 4 & -3 \end{pmatrix}$.

From [7, Lemma 32], $\Pi(2)^2 = \Gamma(4)$. As $\Gamma(4) = \Pi(4)$, we obtain $\Pi(2)^2 = \Pi(4)$. □

Using the above results, we have the following.

Proposition 1. *$\Pi(N)^2$ is a congruence if and only if $N \leq 2$.*

Now we present some results related with the congruence subgroups $\Pi_0(N)$ and $\Pi^1(N)$ of Π. To do this, we suppose that

$$A = \begin{pmatrix} x & * \\ 0 & x^{-1} \end{pmatrix} \pmod N$$

is an element of $\Gamma_0(N)$. Then

$$TR.A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & * \\ 0 & x^{-1} \end{pmatrix} = \begin{pmatrix} -x & * \\ 0 & x^{-1} \end{pmatrix} \pmod N$$

is an element of $\Pi_0(N)$. Therefore

$$(TRA)^2 = \begin{pmatrix} -x & * \\ 0 & x^{-1} \end{pmatrix} \begin{pmatrix} -x & * \\ 0 & x^{-1} \end{pmatrix} = \begin{pmatrix} x^2 & * \\ 0 & x^{-2} \end{pmatrix} \pmod N$$

is an element of $\Gamma_0(N)^2$. Thus, we get $\Pi_0(N)^2 = \Gamma_0(N)^2$.

Similarly to the case $\Pi_0(N)$, if

$$B = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N$$

is an element of $\Gamma^1(N)$, then

$$TR.B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & * \\ 0 & 1 \end{pmatrix} \pmod N$$

is an element of $\Pi^1(N)$. Therefore

$$(TRB)^2 = \begin{pmatrix} -1 & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & * \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N$$

is an element of $\Gamma^1(N)^2$ and so we obtain $\Pi^1(N)^2 = \Gamma^1(N)^2$.

On the other hand, if $\Pi_0(N)$ and $\Pi^1(N)$ are not congruence, then $\Pi_0(N)^2$ and $\Pi^1(N)^2$ are not congruence, since any lift of $\Pi_0(N)$ (or $\Pi^1(N)$) necessarily contains $\Pi_0(N)^2$ (or $\Pi^1(N)^2$), from [7, Lemma 5]. Consequently, we have the following.

Corollary 1. *a) $\Pi_0(N)^2$ is not congruence if and only if either $N \notin \{3,4,8\}$ or if $4 \mid N$ and all odd prime divisors of N are congruent to 3 modulo 4.*
b) $\Pi^1(N)^2$ is not congruence if and only if $N > 4$.

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