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CONGRUENCES IN TRANSITIVE RELATIONAL SYSTEMS

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ABSTRACT. A transitive relational system means a pair (A, R) where $A \neq \emptyset$ and R is a transitive binary relation on A. We define a congruence θ on (A, R) and a factor relation R/θ on the factor set A/θ such that the factor system $(A/\theta, R/\theta)$ is also a transitive relational system. We show that these congruences are in a one-to-one correspondence with the so-called LU-morphisms whenever the relation R is a quasiorder on A.

Mathematics Subject Classification: 08A02, 08A30

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THE CONCEPT OF A RELATIONAL SYSTEM was introduced by A. I. Maltsev [5, 6]. We will restrict our consideration to relational systems with only one binary relation. Hence, by a *relational system* we will mean a pair $\mathcal{A} = (A, R)$, where $A \neq \emptyset$ and $R \subseteq A \times A$, i. e., R is a binary relation on A. Relational systems play an important role both in mathematics and in applications since every formal description of a real system can be done by means of relations. For these considerations we often ask about a certain factorisation of a relational system $\mathcal{A} = (A, R)$ because it enables us to introduce the method of abstraction on $\mathcal{A}$. Hence, if θ is an equivalence relation on A, we ask about a 'factor relation' R/θ on the factor set A/θ such that the factor system $(A/\theta, R/\theta)$ shares some of 'good' properties of $\mathcal{A}$.

In this paper, we are mostly interested in relational systems $\mathcal{A} = (A, R)$ where R is *transitive*, i.e. $\langle a, b\rangle \in R$ and $\langle b, c\rangle \in R$ imply $\langle a, c\rangle \in R$. Then $\mathcal{A}$ is called a *transitive system*. A transitive relation formalises the concept of an “ordering” so that, in a set A, one can thus ask what elements of A go “before” or “after” a given element of A. Our topic is to define a congruence θ on $\mathcal{A}$ and a factor relation R/θ such that

(i) the system $(A/\theta, R/\theta)$ is also transitive, and if R is reflexive or symmetrical, then R/θ shares the same properties;
(ii) a possible common bound is preserved by our construction.

Let us note that a similar task for ordered sets was already solved in [4], and we will try to modify that construction for transitive relational systems.

A *quasiordered system* will mean a relational system $\mathcal{A} = (A, R)$ where R is a *quasiorder* on A, i. e., R is a reflexive and transitive relation. Quasiorders on a given set A form an algebraic lattice, which was studied, e. g., in [3]. Here, we are interested in quasiordered systems where elements may have common "lower" and/or "upper" bounds. The systems where every two elements of A have also suprema and infima with respect to the quasiorder R are very important in applications; they were investigated by the author in [1, 2]. However, the lower and upper bounds can be defined also for general relational systems as follows.

Notation 1. Let $\mathcal{A} = (A, R)$ be a relational system and $a, b \in A$. Introduce the following notation:

$$L_A(a, b) = \{x \in A; \langle x, a\rangle \in R \text{ and } \langle x, b\rangle \in R\},$$
$$U_A(a, b) = \{x \in A; \langle a, x\rangle \in R \text{ and } \langle b, x\rangle \in R\}.$$

If $a = b$, we will write $L_A(a)$ or $U_A(a)$ instead of $L_A(a, a)$ or $U_A(a, a)$, respectively. Clearly, if R is reflexive, then $a \in L_A(a)$ and $a \in U_A(a)$ for each $a \in A$. It is easy to prove that if R is transitive, then $\langle a, b\rangle \in R$ iff $L_A(a, b) = L_A(a)$ iff $U_A(a, b) = U_A(a)$.

Naturally, if R is transitive and $a, b \in R$, then $L_A(a, b)$ is the set of all lower bounds of a, b and $U_A(a, b)$ is the set of all upper bounds of a, b with respect to R.

If $f : A \to B$ is a mapping and $P \subseteq A$, we put $f(P) = \{f(z) : z \in P\}$.

Definition. Let $\mathcal{A} = (A, R), \mathcal{B} = (B, Q)$ be two relational systems. A surjective mapping $f : A \to B$ is called an *LU-morphism* if

$$f(L_A(x, y)) = L_B(f(x), f(y))$$

and

$$f(U_A(x, y)) = U_B(f(x), f(y)) \quad \text{for all } x, y \in A.$$

A mapping f is called a *homomorphism* of $\mathcal{A}$ into $\mathcal{B}$ if

$$\langle a, b\rangle \in R \Rightarrow \langle f(a), f(b)\rangle \in Q.$$

A homomorphism f is called *strong* if, for arbitrary $a, b \in A$, there exist $c, d \in A$ such that $f(c) = f(a)$, $f(d) = f(b)$ and $\langle f(a), f(b)\rangle \in Q \Rightarrow \langle c, d\rangle \in R$.

Lemma 1. *Let $\mathcal{A} = (A, R)$, $\mathcal{B} = (B, Q)$ be transitive relational systems and f be an LU-morphism of $\mathcal{A}$ onto $\mathcal{B}$. Then f is a homomorphism of $\mathcal{A}$ onto $\mathcal{B}$. If R is, moreover, reflexive, then f is a strong homomorphism.*

Proof. Suppose $\langle a, b\rangle \in R$. Since R is transitive, it implies $L_A(a, b) = L_A(a)$ and, therefore,

$$L_B(f(a), f(b)) = f(L_A(a, b)) = f(L_A(a)) = L_B(f(a)),$$

whence $\langle f(a), f(b)\rangle \in Q$; thus, f is a homomorphism. Suppose now that R is also reflexive. If $\langle f(a), f(b)\rangle \in Q$, then

$$f(L_A(a, b)) = L_B(f(a), f(b)) = L_B(f(a)) = f(L_A(a))$$

and, on account of reflexivity, we have $a \in L_A(a)$; thus, $f(a) \in f(L_A(a)) = f(L_A(a,b))$. Analogously, one can show that $f(b) \in f(U_A(a,b))$. Hence, there exist $c \in L_A(a,b)$ and $d \in U_A(a,b)$ such that $f(c) = f(a)$, $f(d) = f(b)$. The condition $c \in L_A(a,b)$ yields $\langle c,a\rangle \in R$ and $\langle c,b\rangle \in R$, and the condition $d \in U_A(a,b)$ implies that $\langle a,d\rangle \in R$ and $\langle b,d\rangle \in R$. Using the transitivity of R, we conclude that $\langle c,d\rangle \in R$. Hence, f is a strong homomorphism. □

If $f : A \to B$ is a mapping, we denote by θ_f the so-called *induced equivalence* on A, i. e., $\langle x,y\rangle \in \theta_f$ iff $f(x) = f(y)$.

We say that relational systems $\mathcal{A}$, $\mathcal{B}$ are *isomorphic*, in symbols $\mathcal{A} \cong \mathcal{B}$, if there exists a bijection $f : A \to B$ such that both f and f^{-1} are homomorphisms.

Theorem 1. *Let $\mathcal{A} = (A,R)$, $\mathcal{B} = (B,Q)$ be quasiordered relational systems and $f : A \to B$ a surjective mapping. The following statements are equivalent:*

(1) *f is an LU-morphism;*
(2) *f is a homomorphism and, for arbitrary $x,y \in A$ with $\langle f(x), f(y)\rangle \in Q$, there exist $u,v \in A$ such that $\langle v,x\rangle \in R$, $\langle x,u\rangle \in R$ and $\langle v,y\rangle \in R$, $\langle y,u\rangle \in R$ and $f(u) = f(y)$, $f(v) = f(x)$.*

Proof. The implication (1) ⇒ (2) follows directly by the same argument as in the proof of Lemma 1.

Let us prove the implication (2) ⇒ (1). Let f be a homomorphism of $\mathcal{A}$ onto $\mathcal{B}$. Then $f(U_A(x,y)) \subseteq U_B(f(x), f(y))$ and $f(L_A(x,y)) \subseteq L_B(f(x), f(y))$. Let us prove the converse inclusions. Suppose that $z \in U_B(f(x), f(y))$. Then $z = f(w)$ for some $w \in A$ with $\langle f(x), f(w)\rangle \in Q$, $\langle f(y), f(w)\rangle \in Q$. By (2), there exist $c,d \in A$ such that $\langle x,c\rangle \in R$, $\langle w,c\rangle \in R$ and $\langle y,d\rangle \in R$, $\langle w,d\rangle \in R$ and $f(c) = f(w) = f(d)$. Applying the reflexivity of Q, we obtain $\langle f(c), f(d)\rangle \in Q$ and, by (2), there exists $u \in A$ such that $\langle c,u\rangle \in R$, $\langle d,u\rangle \in R$ and $f(u) = f(c) = f(w) = z$. Since R is transitive, it follows that $\langle x,u\rangle \in R$, $\langle y,u\rangle \in R$, thus $u \in U_A(x,y)$, i. e., $z = f(u) \in f(U_A(x,y))$. Analogously, it can be shown that the inclusion $f(L_A(x,y)) \supseteq L_B(f(x), f(y))$ is true. □

Definition. Let $\mathcal{A} = (A,R)$ be a relational system and θ be an equivalence on A. Define a binary relation R/θ on the set A/θ as follows:

$$\langle [a]_\theta, [b]_\theta\rangle \in R/\theta \text{ iff there exist } x \in [a]_\theta \text{ and } y \in [b]_\theta \text{ with } \langle x,y\rangle \in R.$$

The system $\mathcal{A}/\theta = (A/\theta, R/\theta)$ will be called a *factor system* of $\mathcal{A}$ by θ.

The following statement is obvious.

Lemma 2. *Let $\mathcal{A} = (A,R)$ and θ be an equivalence on A. If R is reflexive or symmetrical, then R/θ also has this property.*

Definition. Let $\mathcal{A} = (A,R)$ be a relational system and θ be an equivalence on A. We say that θ is a *congruence on $\mathcal{A}$* if $\theta = R \times R$ or

(a) for arbitrary $x, y \in [a]_\theta$, there exists a $c \in [a]_\theta$ such that $\langle x, c\rangle \in R$ and $\langle y, c\rangle \in R$;

(b) if $\langle v, a\rangle \in R$, $\langle v, b\rangle \in R$, and $\langle v, a\rangle \in \theta$, then there exists a $t \in A$ such that $\langle a, t\rangle \in R$, $\langle b, t\rangle \in R$, and $\langle b, t\rangle \in \theta$

and the conditions (a) and (b) hold for R^{-1}.

Theorem 2. *Let $\mathcal{A} = (A, R)$ be a transitive relational system and θ be a congruence on $\mathcal{A}$. Then $\mathcal{A}/\theta = (A/\theta, R/\theta)$ is also a transitive relational system.*

Proof. Suppose $\langle [a]_\theta, [b]_\theta\rangle \in R/\theta$ and $\langle [b]_\theta, [c]_\theta\rangle \in R/\theta$. Then there exist $x \in [a]_\theta$, $y, y' \in [b]_\theta$, and $z \in [c]_\theta$ such that $\langle x, y\rangle \in R$ and $\langle y', z\rangle \in R$. By (a), there exists an $u \in [b]_\theta$ such that $\langle y, u\rangle \in R$ and $\langle y', u\rangle \in R$. Since R is transitive and $\langle x, y\rangle \in R$, we also have $\langle x, u\rangle \in R$. By (b), there exists a $v \in A$ such that $\langle u, v\rangle \in R$, $\langle z, v\rangle \in R$ and $\langle z, v\rangle \in \theta$, i. e., $v \in [c]_\theta$. However, $\langle x, u\rangle \in R$ and $\langle u, v\rangle \in R$ yield $\langle x, v\rangle \in R$; thus, $\langle [a]_\theta, [c]_\theta\rangle \in R/\theta$. □

Theorem 3. *Let $\mathcal{A} = (A, R)$, $\mathcal{B} = (B, Q)$ be quasiordered relational systems. Then:*

(1) *if $f : \mathcal{A} \to \mathcal{B}$ is an LU-morphism, then θ_f is a congruence on $\mathcal{A}$ and $\mathcal{A}/\theta_f \cong \mathcal{B}$;*

(2) *if θ is a congruence on $\mathcal{A}$, then the canonical mapping $h : \mathcal{A} \to \mathcal{A}/\theta$ (given by the relation $h(a) = [a]_\theta$) is an LU-morphism.*

Proof. (1) Suppose that $x, y \in [a]_{\theta_f}$. Then $f(x) = f(y)$ and, in view of the reflexivity of Q, we have $\langle f(x), f(y)\rangle \in Q$. By Theorem 1, there exists an $u \in A$ with $\langle x, u\rangle \in R$, $\langle y, u\rangle \in R$ and $f(x) = f(u) = f(y)$. Hence, $u \in [a]_{\theta_f}$. Analogously, one can show the existence of $v \in [a]_{\theta_f}$ with $\langle v, x\rangle \in R$, $\langle v, y\rangle \in R$, i. e., $[a]_{\theta_f}$ satisfies (a) and its dual (i. e., it is "directed").

Let us prove (b). Let $\langle v, a\rangle \in R$, $\langle v, b\rangle \in R$ and $\langle v, a\rangle \in \theta_f$. Then $f(v) = f(a)$ and, therefore, $f(U_A(a, b)) = U_B(f(a), f(b)) = U_B(f(v), f(b)) = U_B(f(b)) = f(U_A(b))$. Hence, there exists a $t \in A$ such that $t \in U_A(a, b)$ and $f(t) = f(b)$, whence $\langle b, t\rangle \in \theta_f$ and $\langle a, t\rangle \in R$, $\langle b, t\rangle \in R$. We have thus shown that (b) holds. Analogously, the dual of (b) can be obtained.

(2) Suppose that $a, b \in A$ and $\langle a, b\rangle \in R$. Since $a \in [a]_\theta$, $b \in [b]_\theta$, we have $\langle h(a), h(b)\rangle = \langle [a]_\theta, [b]_\theta\rangle \in R/\theta$, i. e., h (the canonical mapping) is a surjective homomorphism. Let $x, y \in A$ and $\langle h(x), h(y)\rangle \in Q$. Then $\langle [x]_\theta, [y]_\theta\rangle \in R/\theta$; thus, there exist $c \in [x]_\theta$, $d \in [y]_\theta$ with $\langle c, d\rangle \in R$. By (a), there exists a $v \in A$ with $\langle v, x\rangle \in R$, $\langle v, c\rangle \in R$ and $v \in [x]_\theta$, and there exists $t \in A$ with $\langle d, t\rangle \in R$, $\langle y, t\rangle \in R$ and $t \in [y]_\theta$. By (b), there is an $u \in A$ such that $\langle t, u\rangle \in R$, $\langle x, u\rangle \in R$ and $\langle u, t\rangle \in \theta$. On account of the transitivity of R, we also have $\langle x, u\rangle \in R$, $\langle y, u\rangle \in R$, and $u \in [y]_\theta$, i. e., $h(u) = h(y)$. Analogously, there is an $s \in A$ such that $\langle s, x\rangle \in R$, $\langle s, y\rangle \in R$, and $h(s) = h(x)$. By Theorem 1, h is an LU-morphism. □

Theorem 4. *Let $\mathcal{A} = (A, R)$ be a quasiordered system and θ be an equivalence on A. Then θ is a congruence on $\mathcal{A}$ if and only if the following assertion is true: for every $a \in A$, $[a]_\theta$ is directed and*

(i) $\langle a, b\rangle \in R$, $\langle a, a_1\rangle \in \theta \Rightarrow \exists b_1 \in A$ *with* $\langle a_1, b_1\rangle \in R$ *and* $\langle b_1, b\rangle \in \theta$;

(ii) $\langle a, b\rangle \in R$, $\langle b, b_1\rangle \in \theta \Rightarrow \exists a_1 \in A$ *with* $\langle a_1, b_1\rangle \in R$ *and* $\langle a_1, a\rangle \in \theta$.

Proof. (1) Suppose that $\langle a, b\rangle \in R$ and $\langle a, a_1\rangle \in \theta$ for some $a, a_1, b \in A$. By (a), there exists $d \in [a]_\theta$ with $\langle d, a_1\rangle \in R$, $\langle d, a\rangle \in R$ and, due to the transitivity, $\langle d, b\rangle \in R$. By (b), there exists $b_1 \in [b]_\theta$ such that $\langle a_1, b_1\rangle \in R$. We have obtained (i). Analogously, it can be shown that (ii) is true.

(2) Let θ be an equivalence on A satisfying (i) and (ii). Clearly, (i) + (ii) yields property (b). □

Corollary. *Let $\mathcal{A} = (A, R)$ be a quasiordered system and θ be an equivalence on A. Then θ is a congruence on $\mathcal{A}$ if and only if:*

(i) R/θ *is a quasiorder on* A/θ;

(ii) $[L_A(x, y)]_\theta = L_{A/\theta}([x]_\theta, [y]_\theta)$ *and* $[U_A(x, y)]_\theta = U_{A/\theta}([x]_\theta, [y]_\theta)$ *for arbitrary* $x, y \in A$.

Proof. If θ is a congruence on $\mathcal{A}$, then by Theorem 2 and Lemma 2, we obtain (i). Applying Theorem 3, we have (ii). Conversely, let θ be an equivalence on A satisfying (i) and (ii). Then the canonical mapping $h : A \to A/\theta$ is an LU-morphism and, due to Theorem 3, we have $\theta = \theta_h$ is a congruence on $\mathcal{A}$. □

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