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COEFFICIENT INEQUALITY FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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Abstract. Let $\mathcal{T}(s,\lambda,\mu,\beta)$ be the class of normalized functions, f defined in the open unit disk $\mathbb{U}$ by

$$\operatorname{Re}\left(\frac{\Theta_{\mu}^{\lambda,s+1} f(z)}{\Theta_{\mu}^{\lambda,s} g(z)}\right) > 0 \qquad (\mu \in \mathbb{N},\, \lambda, s \in \mathbb{N}_0,\, z \in \mathbb{U})$$

for some $g \in \mathcal{R}_{\beta}(s,\lambda,\mu,\beta)$. The authors in [15] introduced the operator $\Theta_{\mu}^{\lambda,s}$ defined by

$$\Theta_{\mu}^{\lambda,s} f(z) = z + \sum_{k=2}^{\infty} \frac{(k+\lambda-1)!(\mu-1)!}{\lambda!(k+\mu-2)!} k^s a_k z^k,$$

where $\mathcal{R}_{\beta}(s,\lambda,\mu,\beta)$ denotes the class of normalized functions g in the open unit disk $\mathbb{U}$ defined by

$$\left|\arg\left(\frac{\Theta_{\mu}^{\lambda,s+1} g(z)}{\Theta_{\mu}^{\lambda,s} g(z)}\right)\right| < \frac{\pi}{2}\beta \qquad (0 < \beta \leq 1)\,.$$

For $f \in \mathcal{T}(s,\lambda,\mu,\beta)$ and given by $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$, a sharp upper bound is obtained for $\left|a_3 - \tau a_2^2\right|$ when $\tau \geq 1$.

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1. Introduction and definitions

Let $\mathcal{A}$ denote the family of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. Further, let $\mathcal{S}$ denote the class of functions which are univalent in $\mathbb{U}$. A function $f(z)$ belonging to $\mathcal{A}$ is said

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to be strongly starlike of order β in $\mathbb{U}$, and denoted by $\mathcal{SS}^*(\mathcal{L})$ if it satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\beta \qquad (0<\beta\leq 1,\ z\in\mathbb{U}). \tag{1.2}$$

If $f(z)\in\mathcal{A}$ satisfies

$$\left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi}{2}\beta \qquad (0<\beta\leq 1,\ z\in\mathbb{U}), \tag{1.3}$$

then we say that $f(z)$ is strongly convex of order β in $\mathbb{U}$, and we denote by $\mathcal{SC}(\mathcal{L})$ the class of all such functions.
With the help of the differential operator $\Theta_{\mu}^{\lambda,s}$, we say that a function $f(z)$ belonging to $\mathcal{A}$ is said to be in the class $\mathcal{R}_{\mathcal{L}}(s,\lambda,\mu,\beta)$ if it satisfies

$$\left|\arg\left(\frac{\Theta_{\mu}^{\lambda,s+1}f(z)}{\Theta_{\mu}^{\lambda,s}f(z)}\right)\right| < \frac{\pi}{2}\beta \qquad (\mu\in\mathbb{N},\lambda,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\}), \tag{1.4}$$

for some $\beta(0<\beta\leq 1)$ and for all $z\in\mathbb{U}$. Note that $\mathcal{R}_{\mathcal{L}}(0,0,1,\beta)=\mathcal{SS}^*(\mathcal{L})$ and $\mathcal{R}_{\mathcal{L}}(1,0,1,\beta)=\mathcal{SC}(\mathcal{L})$.

For the class $\mathcal{S}$ of analytic univalent functions, [6] obtained the maximum value of $\left|a_3-\tau a_2^2\right|$ when τ is real. For various functions of $\mathcal{S}$, the upper bound for $\left|a_3-\tau a_2^2\right|$ is investigated by many different authors (see [1–5, 7, 8, 11–14, 16, 17] and [18]).

In this paper we obtain a sharp upper bounds for $\left|a_3-\tau a_2^2\right|$ when f belongs to the class of functions defined as follows:

Definition 1. Let $\beta(0<\beta\leq 1)$ and let $f\in\mathcal{A}$. Then $f\in\mathcal{T}(s,\lambda,\mu,\beta)$ if and only if there exist $g\in\mathcal{R}_{\mathcal{L}}(s,\lambda,\mu,\beta)$ such that

$$\operatorname{Re}\left(\frac{\Theta_{\mu}^{\lambda,s+1}f(z)}{\Theta_{\mu}^{\lambda,s}g(z)}\right)>0 \qquad (\mu\in\mathbb{N},\lambda,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\ z\in\mathbb{U}), \tag{1.5}$$

where $g(z)=z+b_2z^2+b_3z^3+\cdots$.

Note that $\mathcal{T}(0,0,1,\beta)=\mathcal{K}(\beta)$ the class of close-to-convex functions defined by [3], $\mathcal{T}(0,0,1,1)=\mathcal{K}(1)$ is the class of normalized close-to-convex functions defined by [10].

2. Main results

In order to derive our main results, we have to recall here the following lemma.

Lemma 1 ([17])**.** *Let $h\in\mathcal{P}$ i.e. h be analytic in $\mathbb{U}$ and be given by $h(z)=1+c_1z+c_2z^2+c_3z^3+\cdots$, and $\operatorname{Re}h(z)>0$ for $z\in\mathbb{U}$. Then*

$$\left|c_2-\frac{c_1^2}{2}\right|\leq 2-\frac{\left|c_1^2\right|}{2}. \tag{2.1}$$

Theorem 1. *Let $f(z) \in \mathcal{T}(s,\lambda,\mu,\beta)$ and be given by (1.1). Then for $\beta \geq 1$ and $\tau \geq 1$ we have the sharp inequality*

$$\left|a_3 - \tau a_2^2\right| \leq \frac{\beta^2\left[(3^s\tau\mu^2(\lambda+2) - 2^{2s}\mu(\mu+1)(\lambda+1)\right]}{2^{2s}3^s(\lambda+2)(\lambda+1)^2} + \frac{\left[(3^{s+1}\tau\mu^2(\lambda+2) - 2^{2s+1}\mu(\mu+1)(\lambda+1)\right](2\beta+1)}{2^{2s}3^{s+1}(\lambda+2)(\lambda+1)^2}. \tag{2.2}$$

Proof. Let $f(z) \in \mathcal{T}(s,\lambda,\mu,\beta)$. It follows from (1.5) that

$$\Theta_\mu^{\lambda,s+1} f(z) = \Theta_\mu^{\lambda,s} g(z) q(z), \tag{2.3}$$

for $z \in \mathbb{U}$, with $q \in \mathcal{P}$ given by $q(z) = 1 + q_1 z + q_2 z^2 + q_3 z^3 + \cdots$. Equating coefficients, we obtain

$$\frac{(\lambda+1)}{\mu} 2^{s+1} a_2 = q_1 + \frac{(\lambda+1)}{\mu} 2^s b_2, \tag{2.4}$$

and

$$\frac{(\lambda+2)(\lambda+1)}{\mu(\mu+1)} 3^{s+1} a_3 = q_2 + \frac{(\lambda+1)}{\mu} 2^s b_2 q_1 + \frac{(\lambda+2)(\lambda+1)}{\mu(\mu+1)} 3^s b_3. \tag{2.5}$$

Also, it follows from (1.4) that

$$\Theta_\mu^{\lambda,s+1} g(z) = \Theta_\mu^{\lambda,s} g(z) (p(z))^\beta.$$

where for $z \in \mathbb{U}$, $p \in \mathcal{P}$ and $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$. Thus equating coefficients, we obtain

$$\frac{(\lambda+1)}{\mu} 2^s b_2 = \beta p_1, \tag{2.6}$$

$$\frac{(\lambda+2)(\lambda+1)}{\mu(\mu+1)} (2) 3^s b_3 = \beta\left(p_2 + \frac{3\beta-1}{2} p_1^2\right). \tag{2.7}$$

From (2.4), (2.5), (2.6) and (2.7) we have

$$\begin{aligned} a_3 - \tau a_2^2 = {} & \frac{\mu(\mu+1)}{3^{s+1}(\lambda+2)(\lambda+1)}\left(q_2 - \frac{1}{2}q_1^2\right) \\ & + \frac{2^{2s+1}\mu(\mu+1)(\lambda+1) - 3^{s+1}\tau\mu^2(\lambda+2)}{3^{s+1} \times 2^{2s+2}(\lambda+2)(\lambda+1)^2} q_1^2 \\ & + \frac{\beta\mu(\mu+1)}{2 \times 3^{s+1}(\lambda+2)(\lambda+1)}\left(p_2 - \frac{1}{2}p_1^2\right) \\ & + \frac{[2^{2s+1}\mu(\mu+1)(\lambda+1) - 3^{s+1}\tau\mu^2(\lambda+2)]\beta}{3^{s+1} \times 2^{2s+1}(\lambda+2)(\lambda+1)^2} p_1 q_1 \\ & + \frac{3 \times 2^{2s}\beta^2\mu(\mu+1)(\lambda+1) - 3^{s+1}\tau\beta^2\mu^2(\lambda+2)}{3^{s+1}2^{2s+2}(\lambda+2)(\lambda+1)^2} p_1^2. \end{aligned} \tag{2.8}$$

Assume that $a_3-\tau a_2^2$ is positive. Thus we now estimate $\mathrm{Re}(a_3-\tau a_2^2)$, so from (2.8) and by using the Lemma 1 and letting $p_1=2re^{i\theta}$, $q_1=2Re^{i\phi}, 0\le r\le 1$, $0\le R\le 1, 0\le\theta<2\pi$ and $0\le\phi<2\pi$, we obtain

$$\begin{aligned}
&3^{s+1}\,\mathrm{Re}(a_3-\tau a_2^2)\\
&=\frac{\mu(\mu+1)}{(\lambda+2)(\lambda+1)}\mathrm{Re}(q_2-\frac{1}{2}q_1^2)+\frac{[2^{2s+1}\mu(\mu+1)(\lambda+1)-3^{s+1}\tau\mu^2(\lambda+2)]}{2^{2s+2}(\lambda+2)(\lambda+1)^2}\mathrm{Re}\,q_1^2\\
&\quad+\frac{\beta\mu(\mu+1)}{2(\lambda+2)(\lambda+1)}\mathrm{Re}(p_2-\frac{1}{2}p_1^2)\\
&\quad+\frac{\beta[2^{2s+1}\mu(\mu+1)(\lambda+1)-3^{s+1}\tau\mu^2(\lambda+2)]}{2^{2s+1}(\lambda+2)(\lambda+1)^2}\mathrm{Re}\,p_1q_1\\
&\quad+\frac{3\times 2^{2s}\beta^2\mu(\mu+1)(\lambda+1)-3^{s+1}\tau\beta^2\mu^2(\lambda+2)}{2^{2s+2}(\lambda+2)(\lambda+1)^2}\mathrm{Re}\,p_1^2 \qquad (2.9)\\
&\le\frac{2\mu(\mu+1)}{(\lambda+2)(\lambda+1)}(1-R^2)+\frac{[2^{2s+1}\mu(\mu+1)(\lambda+1)-3^{s+1}\tau\mu^2(\lambda+2)]}{2^{2s}(\lambda+2)(\lambda+1)^2}R^2\cos 2\phi\\
&\quad+\frac{\beta\mu(\mu+1)}{(\lambda+2)(\lambda+1)}(1-r^2)\\
&\quad+\frac{2\beta\left[2^{2s+1}\mu(\mu+1)(\lambda+1)-3^{s+1}\tau\mu^2(\lambda+2)\right]}{2^{2s}(\lambda+2)(\lambda+1)^2}rR\cos(\theta+\phi)\\
&\quad+\frac{3\beta^2[2^{2s}\mu(\mu+1)(\lambda+1)-3^{s}\tau\mu^2(\lambda+2)]}{2^{2s}(\lambda+2)(\lambda+1)^2}r^2\cos 2\theta\\
&\le\frac{3^{s+1}\tau\mu^2(\lambda+2)-2^{2s+2}\mu(\mu+1)(\lambda+1)}{2^{2s}(\lambda+2)(\lambda+1)^2}R^2\\
&\quad+\frac{2\beta\mu[3^{s+1}\tau\mu(\lambda+2)-2^{s+1}(\mu+1)(\lambda+1)]}{2^{2s}(\lambda+2)(\lambda+1)^2}rR\\
&\quad+\beta\left(\frac{3\beta\left[3^{s}\tau\mu^2(\lambda+2)-2^{2s}(\mu+1)(\lambda+1)\right]}{2^{2s}(\lambda+2)(\lambda+1)^2}-\frac{\mu(\mu+1)}{(\lambda+2)(\lambda+1)}\right)r^2\\
&\quad+\frac{\mu(\mu+1)}{(\lambda+2)(\lambda+1)}(\beta+2)\\
&=\Psi(r,R). \qquad (2.10)
\end{aligned}$$

Letting β,λ,μ and τ fixed and differentiating $\Psi(r,R)$ partially when $\mu\ge 1,\lambda\ge 0$, $\beta\ge 1$ and $\tau\ge 1$ we observe that

$$\begin{aligned}
&\Psi_{rr}\Psi_{RR}-(\Psi_{rR})^2\\
&\qquad=\frac{2^{2s+4}\beta\mu^2(\mu+1)^2}{(\lambda+2)}[2\beta+1]-\frac{4\times 3^{s+1}\tau\beta\mu^3(\mu+1)}{(\lambda+1)}[5\beta-1]<0.
\end{aligned}$$

Therefore, the maximum of $\Psi(r,R)$ occurs on the boundaries. Thus the desired inequality follows by observing that

$$\Psi(r,R) \leq \Psi(1,1) = \frac{3\beta^2\left[3^s\tau\mu^2(\lambda+2) - 2^{2s}\mu(\mu+1)(\lambda+1)\right]}{2^{2s}(\lambda+2)(\lambda+1)^2} + \frac{\left[(3^{s+1}\tau\mu^2(\lambda+2) - 2^{2s+1}\mu(\mu+1)(\lambda+1)\right](2\beta+1)}{2^{2s}(\lambda+2)(\lambda+1)^2}.$$

The equality for (2.2) is attained when $p_1 = q_1 = 2i$ and $p_2 = q_2 = -2$. □

Letting $s = \lambda = 0$ and $\mu = 1$ in the above Theorem, we have the result given by Jahangiri [9]:

Corollary 1. *Let $f(z) \in \mathcal{K}(\beta)$ and be given by (1.1). Then for $\beta \geq 1$ and $\tau \geq 1$ we have the sharp inequality*

$$\left|a_3 - \tau a_2^2\right| \leq \beta^2(\tau-1) + \frac{(2\beta+1)(3\tau-2)}{3}. \tag{2.11}$$

Letting $s = \mu = 1$ and $\lambda = 0$ in the above Theorem, we have the following result:

Corollary 2. *Let $f(z) \in \mathcal{T}(1,0,1,\beta)$ and be given by (1.1). Then for $\beta \geq 1$ and $\tau \geq 1$ we have the sharp inequality*

$$\left|a_3 - \tau a_2^2\right| \leq \frac{1}{36}\left[3\beta^2(3\tau-4) + (9\tau-8)(2\beta+1)\right].$$

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