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Some aspects of $L_v^q(\mathbb{R}^d) \cap W_k^{p,w}(\mathbb{R}^d)$

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SOME ASPECTS OF $L_v^q\left(\mathbb{R}^d\right)\cap W_k^{p,w}\left(\mathbb{R}^d\right)$

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Abstract. Let $1\leq q,p<\infty$ and v,w be Beurling's weight functions on $\mathbb{R}^d$. In this article we deal with harmonic properties of intersection space $A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)=L_v^q\left(\mathbb{R}^d\right)\cap W_k^{p,w}\left(\mathbb{R}^d\right)$ defined by aid of weighted Lebesgue space $L_v^q\left(\mathbb{R}^d\right)$ and weighted Sobolev space $W_k^{p,w}\left(\mathbb{R}^d\right)$. We research the inclusions and inequalities between the spaces $A_{k,v,w}^{q,p}(\Omega)$ where $\Omega\subset\mathbb{R}^d$ be an open set. Finally, we proved that the spaces $M\left(A_{k,w}^{1,p}\left(\mathbb{R}^d\right),L_w^1\left(\mathbb{R}^d\right)\right)$ can be identified with the weighted spaces of bounded measures $M_w\left(\mathbb{R}^d\right)$.

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1. INTRODUCTION AND PRELIMINARIES

Throughout this work, $\mathbb{R}^d$ denote $d-$dimensional real Euclidean space with Lebesgue measure dx. We use Beurling's weight function, i.e., a measurable, locally bounded function on $\mathbb{R}^d$ satisfying $\omega(x)\geq 1$ and $\omega(x+y)\leq\omega(x)\omega(y)$ for all $x,y\in\mathbb{R}^d$ [2]. We denote weighted Lebesgue space $L_\omega^p\left(\mathbb{R}^d\right)=\left\{f\mid f\omega\in L^p\left(\mathbb{R}^d\right)\right\}$ which is a Banach space under the norm

$$\|f\|_{p,\omega}=\int_{\mathbb{R}^d}|f(x)|^p\,\omega^p(x)\,dx.$$

Some well-known terms such as convolution, translation invariant, continuous embeddings, Banach algebra, Banach module, essential Banach ideal, approximate identity will be used frequently through this paper; their definitions may be found in [6], [13],[15], [22]. It is known that $L_\omega^p\left(\mathbb{R}^d\right)$ is translation invariant and

$$\|L_s f\|_{p,\omega}\leq\omega(s)\,\|f\|_{p,\omega} \tag{1.1}$$

for any $f \in L^p_\omega\left(\mathbb{R}^d\right)$. The translation operator L_s ($L_s f(x) = f(x-s)$) is continuous on $L^p_\omega\left(\mathbb{R}^d\right)$. For two weight functions ω_1 and ω_2, we write $\omega_1 \prec \omega_2$ if and only if there exists a constant $c > 0$ such that $\omega_1(x) \leq c\omega_2(x)$ for all $x \in \mathbb{R}^d$. We write $\omega_1 \approx \omega_2$ if and only if $\omega_1 \prec \omega_2$ and $\omega_2 \prec \omega_1$. Recall that one has $L^p_{\omega_1}\left(\mathbb{R}^d\right) \subset L^p_{\omega_2}\left(\mathbb{R}^d\right)$ if and only if $\omega_2 \prec \omega_1$. The space $L^p_\omega\left(\mathbb{R}^d\right)$ is a Banach module over $L^1_\omega\left(\mathbb{R}^d\right)$ under the convolution [8].

If $\alpha = (\alpha_1, \alpha_2, ..., \alpha_d) \in \mathbb{R}^d$ is an d-tuple of nonnegative integers α_i, then it is written $\alpha \in \mathbb{Z}^d_+$ and $|\alpha| = \sum_{i=1}^{d} \alpha_i$. Similarly if $D_j = \frac{\partial}{\partial x_j}$ for $1 \leq j \leq d$, then

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} ... D_d^{\alpha_d} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_d^{\alpha_d}}$$

denotes a differential operator of order α. For given two locally integrable functions u and v on $\mathbb{R}^d$, we say that v is α^{th}-weak derivative of u, written $D^\alpha u = v$, provided

$$\int_{\mathbb{R}^d} u(x) D^\alpha \varphi(x)\, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^d} v(x) \varphi(x)\, dx$$

for all $\varphi \in C_c^\infty\left(\mathbb{R}^d\right)$, where $C_c^\infty\left(\mathbb{R}^d\right)$ is the space of all infinitely differentiable functions on $\mathbb{R}^d$, each with compact support. It is known that a weak derivative, if it exists, is uniquely defined up to a set of measure zero and also it is linear [19].

Let w be Beurling's weight function. For any nonnegative integer k and $1 \leq p < \infty$, the weighted Sobolev space $W_k^{p,w}\left(\mathbb{R}^d\right)$ is defined as the space of the functions $u \in L^p_w\left(\mathbb{R}^d\right)$ such that $D^\alpha u$ exists and $D^\alpha u \in L^p_w\left(\mathbb{R}^d\right)$ for all $\alpha \in \mathbb{Z}^d_+$ with $|\alpha| \leq k$. $W_k^{p,w}\left(\mathbb{R}^d\right)$ is a Banach space with the norm

$$\|u\|_{W_k^{p,w}} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{p,w} \text{ [12],[21].}$$

Weighted Sobolev spaces are defined by aid of weighted Lebesgue space $L^p_w\left(\mathbb{R}^d\right)$ by Kufner in 1980s. Clearly, $W_k^{p,w}\left(\mathbb{R}^d\right)$ is a subspace of $L^p_w\left(\mathbb{R}^d\right)$ and also $W_0^{p,w}\left(\mathbb{R}^d\right) = L^p_w\left(\mathbb{R}^d\right)$. For any k, it is obvious the embedding $W_k^{p,w}\left(\mathbb{R}^d\right) \hookrightarrow L^p_w\left(\mathbb{R}^d\right)$. If $\omega = 1$, $W_k^{p,w}\left(\mathbb{R}^d\right) = W_k^p\left(\mathbb{R}^d\right)$. If we take norm $\|.\|_{p,w}$ instead of $\|.\|_p$, we get the following properties by using the method in [10] and [12]. If $w_2 \prec$

w_1 and $k > l$, then $W_k^{p,w_1}(\Omega) \hookrightarrow W_l^{p,w_2}(\Omega)$ for an arbitrary open set $\Omega \subset \mathbb{R}^d$. $W_k^{p,w}\left(\mathbb{R}^d\right)$ is translation invariant and

$$\|L_s f\|_{W_k^{p,w}} \leq \omega(s) \|f\|_{W_k^{p,w}} \tag{1.2}$$

for any $f \in W_k^{p,w}\left(\mathbb{R}^d\right)$. The translation operator is continuous on $W_k^{p,w}\left(\mathbb{R}^d\right)$. Also $W_k^{p,w}\left(\mathbb{R}^d\right)$ is a Banach module over $L_\omega^1\left(\mathbb{R}^d\right)$ under the convolution.

Sobolev spaces $W_k^p\left(\mathbb{R}^d\right)$ of integer order were introduced by S.L. Sobolev in [17], [18]. These spaces are defined over an arbitrary domain $\Omega \subset \mathbb{R}^d$ by using subspaces of Lebesgue spaces. Many generalizations and specializations of these spaces have been constructed and studied in years. In particular, there are extensions that allow arbitrary real values of k, weighted spaces that introduce weight functions into the L^p-norms and other generalizations involve different orders of differentitaion and different L^p-norms in different coordinate directions. Finally, there has been much work on Sobolev spaces and its related areas. To an interested reader, we can suggest our main reference book [1] and the references therein.

Let E and F be two translation invariant Banach spaces. A multiplier on E to F is a bounded linear operator commuting with all translations. We denote by $M(E,F)$ the space of all multipliers on E to F [14].

2. Some results in $L_v^q\left(\mathbb{R}^d\right) \cap W_k^{p,w}\left(\mathbb{R}^d\right)$

If one looks for Sobolev algebras in literature, one sees that there are a lot of published papers about Sobolev algebras obtained by using different function spaces that are defined over different groups or sets. These spaces have been investigated under several respects, and mostly applied to the study of strongly nonlinear variational problems and partial differential equations.

In the sense of our study, we attach importance to [3], [5], [20]. In [5], it is showed that the space $L_\alpha^p(G) \cap L^\infty(G)$ is an algebra with respect to pointwise multiplication, where G is a connected unimodular Lie group. Also, sufficient conditions for the Sobolev spaces to form an algebra under pointwise multiplication have been given in [20].

In [3], Chu defined $A_k^p\left(\mathbb{R}^d\right) = L^1\left(\mathbb{R}^d\right) \cap W^{k,p}\left(\mathbb{R}^d\right)$ spaces and showed some algebraic properties of these spaces (Segal algebras). In this section, we will generalize his results to weighted Sobolev algebras.

Let $1 \leq q, p < \infty$, k be a nonnegative integer and v, w be Beurling's weight functions on $\mathbb{R}^d$. We deal with the some harmonic properties of the intersection space $L_v^q\left(\mathbb{R}^d\right) \cap W_k^{p,w}\left(\mathbb{R}^d\right)$. This space, denoted by $A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)$, is a normed space with the norm

$$\|.\|_{k,v,w}^{q,p} = \|.\|_{q,v} + \|.\|_{W_k^{p,w}}.$$

Theorem 1. $\left(A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right), \|.\|_{k,v,w}^{q,p}\right)$ *is a Banach space.*

Proof. Assume that (f_n) be a Cauchy sequence in $A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)$. Clearly (f_n) is a Cauchy sequence in both $L_v^q\left(\mathbb{R}^d\right)$ and $W_k^{p,w}\left(\mathbb{R}^d\right)$. For this reason, (f_n) converges to $f \in L_v^q\left(\mathbb{R}^d\right)$ and $g \in W_k^{p,w}\left(\mathbb{R}^d\right)$. By using the inequalities $\|.\|_q \leq \|.\|_{q,v}$ and $\|.\|_p \leq \|.\|_{p,w} \leq \|.\|_{W_k^{p,w}}$, we can easily demonstrate that there exist a subsequence $\left(f_{n_k}\right)$ of (f_n) such that $f_{n_k} \to f$ a.e. and a subsequence $\left(f_{n_{k_l}}\right)$ of $\left(f_{n_k}\right)$ such that $f_{n_{k_l}} \to g$ a.e. Therefore, we get $f = g$ a.e. □

Theorem 2. ***(i)*** $\left(A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right), \|.\|_{k,v,w}^{q,p}\right)$ *is translation invariant and*

$$\|L_s f\|_{k,v,w}^{q,p} \leq (v+w)(s)\|f\|_{k,v,w}^{q,p}$$

for all $f \in A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)$.

(ii) *The function* $s \to L_s f$ *is continuous from* $\mathbb{R}^d$ *to* $A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)$ *for any* $f \in A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)$.

Proof. **(i)** We know that the spaces $L_v^q\left(\mathbb{R}^d\right)$ and $W_k^{p,w}\left(\mathbb{R}^d\right)$ are translation invariant. Hence $A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)$ is translation invariant. We get

$$\begin{aligned}\|L_s f\|_{k,v,w}^{q,p} &= \|L_s f\|_{q,v} + \|L_s f\|_{W_k^{p,w}} \\ &\leq v(s)\|f\|_{q,v} + w(s)\|f\|_{W_k^{p,w}} \\ &\leq (v+w)(s)\|f\|_{k,v,w}^{q,p}.\end{aligned}$$

by (1.1) and (1.2).

(ii) Since $s \to L_s f$ is continuous in $L_v^q\left(\mathbb{R}^d\right)$, for any $\varepsilon > 0$ and $s_0 \in \mathbb{R}^d$ there is a neighbourhood V_1 of s_0 such that

$$\|L_s f - L_{s_0} f\|_{q,v} < \frac{\varepsilon}{2} \tag{2.1}$$

for all $s \in V_1$. There is a neighbourhood V_2 of s_0 such that

$$\|L_s f - L_{s_0} f\|_{W_k^{p,w}} < \frac{\varepsilon}{2} \tag{2.2}$$

for all $s \in V_2$, because the function $s \to L_s f$ is continuous in $W_k^{p,w}\left(\mathbb{R}^d\right)$. Consequently $V = V_1 \cap V_2$ is a neighbourhood of s_0 and we get

$$\|L_s f - L_{s_0} f\|_{k,v,w}^{q,p} < \varepsilon$$

for $s \in V$ by (2.1) and (2.2). □

Theorem 3. *$A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)$ is a BF-space.*

Proof. Let $f \in A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)$ and any compact subset $K \subset \mathbb{R}^d$. Using Hölder inequality with $\frac{1}{p} + \frac{1}{p'} = 1$, we obtain

$$\begin{aligned}
\int_K |f(x)|\,dx &= \int_{\mathbb{R}^d} |f(x)|\,\chi_K(x)\,dx \\
&\le \left(\int_{\mathbb{R}^d} |f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^d} (\chi_K(x))^{p'}\,dx\right)^{\frac{1}{p'}} \\
&\le \|f\|_{p,w}\,\mu(K)^{\frac{1}{p'}} \le \|f\|_{W_k^{p,w}}\,\mu(K)^{\frac{1}{p'}}
\end{aligned}$$

for any $f \in W_k^{p,w}\left(\mathbb{R}^d\right)$. If we write $M_K = \mu(K)^{\frac{1}{p'}}$, there exists $M_K > 0$ such that

$$\int_K |f(x)|\,dx \le M_K\,\|f\|_{W_k^{p,w}}. \tag{2.3}$$

Also since $L_v^q\left(\mathbb{R}^d\right)$ is a BF-space, there exists $N_K > 0$ such that

$$\int_K |f(x)|\,dx \le N_K\,\|f\|_{q,v}. \tag{2.4}$$

If we write $C_K = \max\{M_K, N_K\}$, we get

$$\int_K |f(x)|\,dx \le C_K\,\|f\|_{k,v,w}^{q,p}$$

by (2.3) and (2.4). □

Theorem 4. *If $v \prec w'$ and $w \prec w'$, then $A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)$ is Banach module over $L_{w'}^1\left(\mathbb{R}^d\right)$ under the convolution.*

Proof. Assume that $v \prec w'$ and $w \prec w'$. Then we know that $L^1_{w'}\left(\mathbb{R}^d\right) \subset L^1_v\left(\mathbb{R}^d\right)$ and $L^1_{w'}\left(\mathbb{R}^d\right) \subset L^1_w\left(\mathbb{R}^d\right)$. Consequently there exist $c_1, c_2 > 0$ such that $\|g\|_{1,v} \leq c_1 \|g\|_{1,w'}$ and $\|g\|_{1,w} \leq c_2 \|g\|_{1,w'}$ for any $g \in L^1_{w'}\left(\mathbb{R}^d\right)$. Since $L^q_v\left(\mathbb{R}^d\right)$ is a Banach module over $L^1_v\left(\mathbb{R}^d\right)$ and $W^{p,w}_k\left(\mathbb{R}^d\right)$ is a Banach module over $L^1_w\left(\mathbb{R}^d\right)$ under the convolution, we get

$$\begin{aligned}
\|f*g\|^{q,p}_{k,v,w} &= \|f*g\|_{q,v} + \|f*g\|_{W^{p,w}_k} \\
&\leq \|f\|_{q,v}\|g\|_{1,v} + \|f\|_{W^{p,w}_k}\|g\|_{1,w} \\
&\leq \|f\|_{q,v}c_1\|g\|_{1,w'} + \|f\|_{W^{p,w}_k}c_2\|g\|_{1,w'} \\
&\leq \max\{c_1,c_2\}\|f\|^{q,p}_{k,v,w}\|g\|_{1,w'}
\end{aligned}$$

for any $f \in A^{q,p}_{k,v,w}\left(\mathbb{R}^d\right)$ and $g \in L^1_{w'}\left(\mathbb{R}^d\right)$. □

Theorem 5. *If $1 \leq p < \infty$ and $w \prec v$, then $A^{1,p}_{k,v,w}\left(\mathbb{R}^d\right)$ is Banach algebra under the convolution.*

Proof. Suppose that $w \prec v$. So, there is a constant $c > 0$ such that $\|f\|_{1,w} \leq c\|f\|_{1,v}$ for any $f \in L^1_v\left(\mathbb{R}^d\right)$. Now we take any $f, g \in A^{1,p}_{k,v,w}\left(\mathbb{R}^d\right)$. Since $L^1_v\left(\mathbb{R}^d\right)$ is a Beurling algebra and $W^{p,w}_k\left(\mathbb{R}^d\right)$ is $L^1_w\left(\mathbb{R}^d\right)$−module, we find

$$\begin{aligned}
\|f*g\|^{1,p}_{k,v,w} &= \|f*g\|_{1,v} + \|f*g\|_{W^{p,w}_k} \leq \|f\|_{1,v}\|g\|_{1,v} + \|f\|_{W^{p,w}_k}\|g\|_{1,w} \\
&\leq \|f\|_{1,v}\|g\|_{1,v} + \|f\|_{W^{p,w}_k}c\|g\|_{1,v} \leq \max\{1,c\}\|f\|^{1,p}_{k,v,w}\|g\|_{1,v} \\
&\leq \max\{1,c\}\|f\|^{1,p}_{k,v,w}\|g\|^{1,p}_{k,v,w}.
\end{aligned}$$

If we define a new function on $A^{1,p}_{k,v,w}\left(\mathbb{R}^d\right)$ such that $|||.||| = \max\{1,c\}\|.\|^{1,p}_{k,v,w}$, then we can see easily that it is a norm. Moreover, the norms $|||.|||$ and $\|.\|^{1,p}_{k,v,w}$ on $A^{1,p}_{k,v,w}\left(\mathbb{R}^d\right)$ are equivalent. Hence we obtain

$$\begin{aligned}
|||f*g||| &= \max\{1,c\}\|f*g\|^{1,p}_{k,v,w} \\
&\leq \max\{1,c\}\max\{1,c\}\|f\|^{1,p}_{k,v,w}\|g\|^{1,p}_{k,v,w} \\
&\leq |||f|||\,|||g|||.
\end{aligned}$$

□

Definition 1. A sequence of functions φ_n in $C_c^\infty\left(\mathbb{R}^d\right)$ satisfies the following conditions:

(i) $\varphi_n(x) \geq 0$ for all $x \in \mathbb{R}^d$
(ii) $\int\limits_{\mathbb{R}^d} \varphi_n(x)\,dx = 1$
(iii) The support of φ_n is in $[-\varepsilon_n, \varepsilon_n]^d$, $\varepsilon_n > 0$ and $\lim\limits_{n\to\infty} \varepsilon_n = 0$ [9].

Theorem 6. *The sequence of functions φ_n is an approximate identity for $A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)$.*

Proof. Since φ_n is an approximate identity, for any $f \in L_v^q\left(\mathbb{R}^d\right)$ and $\varepsilon > 0$ there exists $n_1 \in \mathbb{N}$ such that

$$\|f * \varphi_n - f\|_{q,v} < \frac{\varepsilon}{2} \tag{2.5}$$

for all $n \geq n_1$. Also we can see that there exists a $n_2 \in \mathbb{N}$ such that

$$\|f * \varphi_n - f\|_{W_{k,w}^p} < \frac{\varepsilon}{2} \tag{2.6}$$

for all $n \geq n_2$ by using the method in [4],[22]. If we set $n_0 = \max\{n_1, n_2\}$, then by (2.5) and (2.6) we obtain

$$\|f * \varphi_n - f\|_{k,v,w}^{q,p} < \varepsilon$$

for all $n \geq n_0$. □

Theorem 7. *For each $f \neq 0$, $f \in A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)$ there exists $c(f) > 0$ such that*

$$c(f)(v+w)(s) \leq \|L_s f\|_{k,v,w}^{q,p} \leq (v+w)(s)\,\|f\|_{k,v,w}^{q,p}.$$

Proof. For given $f \in A_{k,v,w}^{q,p}\left(\mathbb{R}^d\right)$, we write $f \in L_v^q\left(\mathbb{R}^d\right)$ and $f \in W_k^{p,w}\left(\mathbb{R}^d\right)$. Let K be any compact subset of $\mathbb{R}^d$. Since $\|L_s f\|_{W_k^{p,w}} \geq \|D^\alpha L_s f\|_{p,w}$ for all $f \in W_k^{p,w}\left(\mathbb{R}^d\right)$, we find

$$\|L_s f\|_{W_k^{p,w}} \geq \|D^\alpha L_s f\|_{p,w} = \left\{\int\limits_{\mathbb{R}^d} |D^\alpha f(x-s)|^p\, w^p(x)\,dx\right\}^{\frac{1}{p}}$$

$$= \left\{\int\limits_{\mathbb{R}^d} |D^\alpha f(u)|^p\, w^p(u+s)\,du\right\}^{\frac{1}{p}} \geq \left\{\int\limits_K |D^\alpha f(u)|^p \frac{w^p(s)}{w^p(-u)}du\right\}^{\frac{1}{p}}$$

$$\geq \left\{\int\limits_K |D^\alpha f(u)|^p \frac{w^p(s)}{\sup\limits_{u\in K} w^p(-u)}du\right\}^{\frac{1}{p}} \geq \frac{w(s)}{\sup\limits_{u\in K} w(-u)}\|D^\alpha f\chi_K\|_p.$$

If we set $c_1(f) = \frac{\|D^\alpha f \chi_K\|_p}{\sup_{u\in K} w(-u)}$, then there exists a constant $c_1(f) > 0$ such that

$$\|L_s f\|_{W_k^{p,w}} \geq c_1(f)\, w(s). \tag{2.7}$$

Also we know that there exists a constant $c_2(f) > 0$ such that

$$\|L_s f\|_{q,v} \geq c_2(f)\, v(s) \tag{2.8}$$

for all $f \in L_v^q\left(\mathbb{R}^d\right)$. If we set $c(f) = \min\{c_1(f), c_2(f)\}$, then we get

$$\|L_s f\|_{k,v,w}^{q,p} \geq c(f)(v+w)(s) \tag{2.9}$$

by inequalities (2.7) and (2.8). Also we know that

$$\|L_s f\|_{k,v,w}^{q,p} \leq (v+w)(s)\, \|f\|_{k,v,w}^{q,p} \tag{2.10}$$

by Theorem 2. Hence the proof is completed from (2.9) and (2.10). □

Proposition 1. *Let* $1 \leq q_1, q_2, p_1, p_2 < \infty$ *and* v_1, v_2, w_1, w_2 *be weight functions on* $\mathbb{R}^d$*. Then*

$$A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right) \subset A_{k,v_2,w_2}^{q_2,p_2}\left(\mathbb{R}^d\right)$$

if and only if there is a constant $M > 0$ *such that*

$$\|f\|_{k,v_2,w_2}^{q_2,p_2} \leq M\, \|f\|_{k,v_1,w_1}^{q_1,p_1}$$

for every $f \in A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right)$.

Proof. Assume that $A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right) \subset A_{k,v_2,w_2}^{q_2,p_2}\left(\mathbb{R}^d\right)$. We define the norm

$$\|f\| = \|f\|_{k,v_1,w_1}^{q_1,p_1} + \|f\|_{k,v_2,w_2}^{q_2,p_2}$$

for all $f \in A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right)$. Let (f_n) be a Cauchy sequence in $\left(A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right), \|.\|\right)$. Hence (f_n) is a Cauchy sequence in

$$\left(A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right), \|.\|_{k,v_1,w_1}^{q_1,p_1}\right) \text{ and } \left(A_{k,v_2,w_2}^{q_2,p_2}\left(\mathbb{R}^d\right), \|.\|_{k,v_2,w_2}^{q_2,p_2}\right).$$

Since $\left(A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right), \|.\|_{k,v_1,w_1}^{q_1,p_1}\right)$ and $\left(A_{k,v_2,w_2}^{q_2,p_2}\left(\mathbb{R}^d\right), \|.\|_{k,v_2,w_2}^{q_2,p_2}\right)$ are Banach spaces, there exist $f \in A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right)$ and $g \in A_{k,v_2,w_2}^{q_2,p_2}\left(\mathbb{R}^d\right)$ such that

$$\|f_n - f\|_{k,v_1,w_1}^{q_1,p_1} \to 0 \text{ and } \|f_n - g\|_{k,v_2,w_2}^{q_2,p_2} \to 0.$$

If we use the inequalities $\|.\|_{p_1} \leq \|.\|_{k,v_1,w_1}^{q_1,p_1}$ and $\|.\|_{p_2} \leq \|.\|_{k,v_2,w_2}^{q_2,p_2}$, then we find $\|f_n - f\|_{p_1} \to 0$ and $\|f_n - g\|_{p_2} \to 0$. Thus there is a subsequence $\left(f_{n_k}\right)$ of (f_n) such that $f_{n_k} \to f$ a.e. and also there is a subsequence $\left(f_{n_{k_l}}\right)$ of $\left(f_{n_k}\right)$ such that $f_{n_{k_l}} \to g$ a.e. Therefore we find $f = g$ a.e., consequently we get $\|f_n - f\| \to 0$.

Hence $\left(A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right),\|.\|\right)$ is a Banach space. We consider the unit function I from $\left(A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right),\|.\|\right)$ onto $\left(A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right),\|.\|_{k,v_1,w_1}^{q_1,p_1}\right)$. Since $\|I(f)\|_{k,v_1,w_1}^{q_1,p_1}=\|f\|_{k,v_1,w_1}^{q_1,p_1}\leq\|f\|$, the unit function is continuous. Then it is homeomorphism by Banach Theorem. This means that $\|.\|$ and $\|.\|_{k,v_1,w_1}^{q_1,p_1}$ are equivalent, so there is a constant $M>0$ such that

$$\|f\|\leq M\|f\|_{k,v_1,w_1}^{q_1,p_1} \tag{2.11}$$

for all $f\in A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right)$. If we use the definiton of $\|.\|$ and the inequality (2.11), then we obtain $\|f\|_{k,v_2,w_2}^{q_2,p_2}\leq M\|f\|_{k,v_1,w_1}^{q_1,p_1}$.

Conversely, if $\|f\|_{k,v_2,w_2}^{q_2,p_2}\leq M\|f\|_{k,v_1,w_1}^{q_1,p_1}$ for all $f\in A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right)$, we can easily that the inclusion $A_{k,v_1,w_1}^{q_1,p_1}\left(\mathbb{R}^d\right)\subset A_{k,v_2,w_2}^{q_2,p_2}\left(\mathbb{R}^d\right)$ holds. □

It is easy to obtain the following proposition by aid of Proposition 1.

Proposition 2. *Let v_1,v_2,w_1,w_2 be weight functions on $\mathbb{R}^d$ and $1\leq q,p<\infty$. If $v_2\prec v_1$ and $w_2\prec w_1$, then $A_{k,v_1,w_1}^{q,p}\left(\mathbb{R}^d\right)\subset A_{k,v_2,w_2}^{q,p}\left(\mathbb{R}^d\right)$.*

Theorem 8. *Let $\Omega\subset\mathbb{R}^d$ be an open set and v_1,v_2,w_1,w_2 be weight functions on $\mathbb{R}^d$ satisfying $v_2\prec v_1$ and $w_2\prec w_1$. Then*

$$A_{k,v_1,w_1}^{q,p}(\Omega)\hookrightarrow A_{l,v_2,w_2}^{q,p}(\Omega)$$

for all $k,l\in\mathbb{Z}^+$ where $k>l$.

Proof. Let $f\in A_{k,v_1,w_1}^{q,p}(\Omega)$ be given, so we write
$f\in L_{v_1}^q(\Omega)$ and $f\in W_k^{p,w_1}(\Omega)$. It is known that $L_{v_1}^q(\Omega)\subset L_{v_2}^q(\Omega)$ where $v_2\prec v_1$. Also we know that $W_k^{p,w_1}(\Omega)\subset W_l^{p,w_2}(\Omega)$ where $w_2\prec w_1$ and $k>l$. Therefore we obtain
$f\in L_{v_2}^q(\Omega)\cap W_l^{p,w_2}(\Omega)=A_{l,v_2,w_2}^{q,p}(\Omega)$. So we find $A_{k,v_1,w_1}^{q,p}(\Omega)\subset A_{l,v_2,w_2}^{q,p}(\Omega)$.
There exists a constant $c_1>0$ such that

$$\|f\|_{q,v_2}\leq c_1\|f\|_{q,v_1} \tag{2.12}$$

for all $f\in L_{v_1}^q(\Omega)$, because $v_2\prec v_1$. Moreover, since $W_k^{p,w_1}(\Omega)\hookrightarrow W_l^{p,w_2}(\Omega)$ where $k>l$ and $w_2\prec w_1$, there exists a constant $c_2>0$ such that

$$\|f\|_{W_l^{p,w_2}}\leq c_2\|f\|_{W_k^{p,w_1}} \tag{2.13}$$

for all $f\in W_k^{p,w_1}(\Omega)$. If we set $c=\max\{c_1,c_2\}$, we get

$$\begin{aligned}\|f\|_{l,v_2,w_2}^{q,p}&\leq c\left(\|f\|_{q,v_1}+\|f\|_{W_k^{p,w_1}}\right)\\&\leq c\|f\|_{k,v_1,w_1}^{q,p}\end{aligned}$$

from the inequalities (2.12) and (2.13) This completes the proof. □

We prove the following theorem with using method in [23].

Theorem 9. *Let v_1, v_2, w_1, w_2 be weight functions on $\mathbb{R}^d$ satisfying $v_2 \prec v_1$, $w_2 \prec w_1$ and $k, l \in \mathbb{Z}^+$ with $k > l$. If $\Omega \subset \mathbb{R}^d$ be an open set such that $\mu(\Omega) < \infty$, then*

$$A^{s,r}_{k,v_1,w_1}(\Omega) \subset A^{q,p}_{l,v_2,w_2}(\Omega)$$

where $1 \leq q < s < \infty$ and $1 \leq p < r < \infty$.

Proof. Assume that $f \in A^{s,r}_{k,v_1,w_1}(\Omega)$, so we write $f \in L^s_{v_1}(\Omega)$ and $f \in W^{r,w_1}_k(\Omega)$. If we set $\alpha = \frac{s}{q}$ where $1 \leq q < s < \infty$ and let β be conjugate exponent of α, then we find

$$\begin{aligned}\|f\|^q_{q,v_1} &= \int_\Omega |f(x)|^q v_1^q(x)\,dx \leq \left\{\int_\Omega \left[|f(x)|^q v_1^q(x)\right]^{\frac{s}{q}} dx\right\}^{\frac{q}{s}} \left\{\int_\Omega (\chi_\Omega)^\beta\,dx\right\}^{\frac{1}{\beta}} \\ &\leq \left(\int_\Omega |f(x)|^s v_1^s(x)\,dx\right)^{\frac{q}{s}} [\mu(\Omega)]^{\frac{1}{\beta}} = \|f\|^q_{s,v_1} [\mu(\Omega)]^{\frac{1}{\beta}} \qquad (2.14)\end{aligned}$$

by Hölder inequality. Since $\mu(\Omega) < \infty$ and $f \in L^s_{v_1}(\Omega)$, we obtain $f \in L^q_{v_1}(\Omega)$ from (2.14). Hence we have $f \in L^q_{v_2}(\Omega)$, because $v_2 \prec v_1$. Also we can see that $W^{r,w_1}_k(\Omega) \subset W^{p,w_1}_k(\Omega)$ where $1 \leq p < r < \infty$ and $\mu(\Omega) < \infty$ by similar method. Since $w_2 \prec w_1$ and $k > l$, we find $W^{p,w_1}_k(\Omega) \subset W^{p,w_2}_l(\Omega)$. So we get $W^{r,w_1}_k(\Omega) \subset W^{p,w_2}_l(\Omega)$, therefore we write $f \in W^{p,w_2}_l(\Omega)$. Thus we obtain $f \in L^q_{v_2}(\Omega) \cap W^{p,w_2}_l(\Omega) = A^{q,p}_{l,v_2,w_2}(\Omega)$. This completes the proof. □

Theorem 10. *Let v_1, v_2, w_1, w_2 be weight functions on $\mathbb{R}^d$ satisfying $v_2 \prec v_1$, $w_2 \prec w_1$ and $k, l \in \mathbb{Z}^+$ with $k > l$. If $\Omega \subset \mathbb{R}^d$ be an open set such that $\mu(\Omega) < \infty$, then there exist $c(f) > 0$ and $c > 0$ such that*

$$c(f)(v_2 + w_2)(s) \leq \|L_s f\|^{q,p}_{l,v_2,w_2} \leq c(v_2 + w_2)(s)\ \ \|f\|^{s,r}_{k,v_1,w_1}$$

for all $f \in A^{s,r}_{k,v_1,w_1}(\Omega)$, $f \neq 0$ where $1 \leq q < s < \infty$ and $1 \leq p < r < \infty$.

Proof. For given $f \in A^{s,r}_{k,v_1,w_1}(\Omega)$, there exists a constant $c > 0$ such that

$$c(f)(v_2 + w_2)(s) \leq \|L_s f\|^{q,p}_{l,v_2,w_2} \qquad (2.15)$$

by Theorem 7 and Theorem 9. Since $v_2 \prec v_1$, there is a constant $c_1 > 0$ such that

$$\|f\|_{q,v_2} \leq c_1 \|f\|_{q,v_1}. \qquad (2.16)$$

Also since $W^{p,w_1}_k(\Omega) \hookrightarrow W^{p,w_2}_l(\Omega)$ where $w_2 \prec w_1$ and $k > l$, there is a constant $c_2 > 0$ such that

$$\|f\|_{W^{p,w_2}_l} \leq c_2 \|f\|_{W^{p,w_1}_k}. \qquad (2.17)$$

If we set $m_1 = \max\{c_1, c_2\}$, we obtain

$$\begin{aligned}
\|L_s f\|_{l,v_2,w_2}^{q,p} &= \|L_s f\|_{q,v_2} + \|L_s f\|_{W_l^{p,w_2}} \\
&\le v_2(s)\|f\|_{q,v_2} + w_2(s)\|f\|_{W_l^{p,w_2}} \\
&\le v_2(s)c_1\|f\|_{q,v_1} + w_2(s)c_2\|f\|_{W_k^{p,w_1}} \\
&\le m_1(v_2+w_2)(s)\|f\|_{k,v_1,w_1}^{q,p}
\end{aligned}$$

by using (2.16) and (2.17). Also we can see that $A_{k,v_1,w_1}^{s,r}(\Omega) \subset A_{k,v_1,w_1}^{q,p}(\Omega)$ by Theorem 9 and so there exists a constant $m_2 > 0$ such that

$$\|f\|_{k,v_1,w_1}^{q,p} \le m_2\|f\|_{k,v_1,w_1}^{s,r}$$

by Proposition 1. Thus there exists a constant $c > 0$ such that

$$\|L_s f\|_{l,v_2,w_2}^{q,p} \le c(v_2+w_2)(s)\|f\|_{k,v_1,w_1}^{s,r} \tag{2.18}$$

for all $f \in A_{k,v_1,w_1}^{s,r}(\Omega)$. If we combine (2.15) with (2.18), the proof is completed. □

We prove the following theorem with using method in [1].

Theorem 11. *Let $\Omega \subset \mathbb{R}^d$ be open set, v_1, v_2, w_1, w_2 be weight functions on $\mathbb{R}^d$ satisfying $v_2 \prec v_1$, $w_2 \prec w_1$ and $k, l \in \mathbb{Z}^+$ with $k > l$. If $\frac{1}{s} = \frac{\lambda}{q_1} + \frac{1-\lambda}{q_2}$, $\frac{1}{r} = \frac{\lambda}{p_1} + \frac{1-\lambda}{p_2}$ for some λ with $0 < \lambda < 1$, then*

$$A_{k,v_1,w_1}^{q_1,p_1}(\Omega) \cap A_{k,v_1,w_1}^{q_2,p_2}(\Omega) \subset A_{l,v_2,w_2}^{s,r}(\Omega)$$

where $1 \le q_1 < s < q_2 < \infty$ and $1 \le p_1 < r < p_2 < \infty$.

Proof. Suppose that $f \in A_{k,v_1,w_1}^{q_1,p_1}(\Omega) \cap A_{k,v_1,w_1}^{q_2,p_2}(\Omega)$, so we write $f \in L_{v_1}^{q_1}(\Omega) \cap L_{v_1}^{q_2}(\Omega)$ and $f \in W_k^{p_1,w_1}(\Omega) \cap W_k^{p_2,w_1}(\Omega)$. If we set $t = \frac{q_1}{s\lambda}$, then we see $t' = \frac{q_2}{s(1-\lambda)}$ is conjugate exponent of t. Thus we obtain

$$\begin{aligned}
\|f\|_{s,v_1}^{s} &= \int_\Omega |f(x)|^s v_1^s(x)\,dx \\
&= \int_\Omega |f(x)|^{s\lambda} v_1^{s\lambda}(x)\,|f(x)|^{s(1-\lambda)} v_1^{s(1-\lambda)}(x)\,dx \\
&\le \left\{\int_\Omega \left[|f(x)|^{s\lambda} v_1^{s\lambda}(x)\right]^{\frac{q_1}{s\lambda}} dx\right\}^{\frac{s\lambda}{q_1}}
\end{aligned}$$

$$\left\{ \int_{\Omega} \left[|f(x)|^{s(1-\lambda)} v_1^{s(1-\lambda)}(x) \right]^{\frac{q_2}{s(1-\lambda)}} dx \right\}^{\frac{s(1-\lambda)}{q_2}}$$
$$= \|f\|_{q_1,v_1}^{s\lambda} \|f\|_{q_2,v_1}^{s(1-\lambda)}$$

by Hölder inequality. Since $f \in L_{v_1}^{q_1}(\Omega) \cap L_{v_1}^{q_2}(\Omega)$, we get $f \in L_{v_1}^{s}(\Omega)$. Also we can show that $f \in W_k^{r,w_1}(\Omega)$ by similar method under the hypothesis. Hence we find $f \in L_{v_1}^{s}(\Omega) \cap W_k^{r,w_1}(\Omega) = A_{k,v_1,w_1}^{s,r}(\Omega)$. We know that $A_{k,v_1,w_1}^{s,r}(\Omega) \subset A_{l,v_2,w_2}^{s,r}(\Omega)$ where $v_2 \prec v_1$, $w_2 \prec w_1$ and $k > l$ by Theorem 8, therefore we get $f \in A_{l,v_2,w_2}^{s,r}(\Omega)$. □

3. Multiplier spaces of $\left(A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right), L_w^1\left(\mathbb{R}^d\right)\right)$

In this section we call the intersection space $L_w^1\left(\mathbb{R}^d\right) \cap W_k^{p,w}\left(\mathbb{R}^d\right)$ as $A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ and equipped with the sum norm $\|f\|_{k,w}^{1,p} = \|f\|_{1,w} + \|f\|_{W_k^{p,w}}$. We denote the space of multipliers from $A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ to $L_w^1\left(\mathbb{R}^d\right)$ by $M\left(A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right), L_w^1\left(\mathbb{R}^d\right)\right)$. It is known that $L_w^1\left(\mathbb{R}^d\right)$ is a closed ideal in the space $M_w\left(\mathbb{R}^d\right)$ which is defined by

$$M_w\left(\mathbb{R}^d\right) = \left\{ \mu : \mu \text{ is a bounded measure and } \|\mu\|_{\omega} = \int_{\mathbb{R}^d} w \, d|\mu| < \infty \right\}.$$

We will show that $M\left(A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right), L_w^1\left(\mathbb{R}^d\right)\right) \cong M_w\left(\mathbb{R}^d\right)$ by using results in the second section.

Proposition 3. *If $\mu \in M_w\left(\mathbb{R}^d\right)$ and $f \in A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$, then $\mu * f \in A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ and $\|\mu * f\|_{k,w}^{1,p} \le \|\mu\|_{\omega} \|f\|_{k,w}^{1,p}$.*

Proof. Since $s \to L_s f$ is a continuous function from $\mathbb{R}^d$ to $A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ for $f \in A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ and μ is a bounded Borel measure, then $\int_{\mathbb{R}^d} \|L_s f\|_{k,w}^{1,p} \, d|\mu|(s) < \infty$. So, the integral $\int_{\mathbb{R}^d} L_s f d\mu(s)$ belong to $A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ by [16, Proposition 3.2.62]. Therefore we get

$$\|\mu * f\|_{k,w}^{1,p} = \left\| \int_{\mathbb{R}^d} L_s f d\mu(s) \right\|_{k,w}^{1,p} \leq \int_{\mathbb{R}^d} \|L_s f\|_{k,w}^{1,p} d|\mu|(s)$$
$$\leq \int_{\mathbb{R}^d} \|f\|_{k,w}^{1,p} w(s) d|\mu|(s) = \|f\|_{k,w}^{1,p} \ \|\mu\|_{\omega}.$$

□

Proposition 4. *$A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ is an essential Banach ideal in $L_w^1\left(\mathbb{R}^d\right)$.*

Proof. Let $f \in A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ and $g \in L_w^1\left(\mathbb{R}^d\right)$. By Theorem 4, we can easily see that $f * g \in A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ and we find

$$\begin{aligned} \|f * g\|_{k,w}^{1,p} &= \|f * g\|_{1,w} + \|f * g\|_{W_k^{p,w}} \\ &\leq \|f\|_{1,w} \|g\|_{1,w} + \|f\|_{W_k^{p,w}} \|g\|_{1,w} \\ &\leq \|f\|_{k,w}^{1,p} \|g\|_{1,w}. \end{aligned}$$

We known that $C_c^\infty\left(\mathbb{R}^d\right)$ is a dense subset of $L_w^1\left(\mathbb{R}^d\right)$ [10] and we can easily see that $C_c^\infty\left(\mathbb{R}^d\right) \subset A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$. Hence we find that $A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ is a dense subset of $L_w^1\left(\mathbb{R}^d\right)$. So we get that $A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ is a dense Banach ideal in $L_w^1\left(\mathbb{R}^d\right)$. Now let $f \in A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ and $\varepsilon > 0$. By Theorem 2, there is a neighbourhood U of the unit element e of $\mathbb{R}^d$ such that

$$\|L_s f - f\|_{k,w}^{1,p} < \varepsilon$$

for all $s \in \mathbb{R}^d$. Let $(\varphi_n)_{n\in\mathbb{N}}$ be as in Definition 1, so there exists $n_0 \in \mathbb{N}$ such that $\operatorname{supp}\varphi_{n_0} \subset U$. Thus

$$\begin{aligned} \|\varphi_n * f - f\|_{k,w}^{1,p} &= \left\| \int_{\mathbb{R}^d} \varphi_n(s)(L_s f - f) ds \right\|_{k,w}^{1,p} \\ &\leq \|L_s f - f\|_{k,w}^{1,p} \int_{\mathbb{R}^d} \varphi_n(s) ds \\ &= \|L_s f - f\|_{k,w}^{1,p} < \varepsilon \end{aligned}$$

for all $n \geq n_0$. Therefore $A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ is an essential Banach ideal in $L_w^1\left(\mathbb{R}^d\right)$. □

Theorem 12. *Let $T : A^{1,p}_{k,w,w}\left(\mathbb{R}^d\right) \to L^1_w\left(\mathbb{R}^d\right)$ be a linear transformation, then the following are equivalent.*

i) *$T \in M\left(A^{1,p}_{k,w,w}\left(\mathbb{R}^d\right), L^1_w\left(\mathbb{R}^d\right)\right)$.*

ii) *There exists a unique measure $\mu \in M_w\left(\mathbb{R}^d\right)$ such that $Tf = \mu * f$ for each $f \in A^{1,p}_{k,w,w}\left(\mathbb{R}^d\right)$.*

Moreover the correspondence between T and μ defines an isometric algebra isomorphism of $M\left(A^{1,p}_{k,w,w}\left(\mathbb{R}^d\right), L^1_w\left(\mathbb{R}^d\right)\right)$ onto $M_w\left(\mathbb{R}^d\right)$.

Proof. Let $\mu \in M_w\left(\mathbb{R}^d\right)$ and $Tf = \mu * f$ for each $f \in A^{1,p}_{k,w,w}\left(\mathbb{R}^d\right)$. Then,

$$\begin{aligned}\|Tf\|_{1,w} = \|\mu * f\|_{1,w} &= \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d} f(x-s)\,\mu(s)\,ds\right| w(x)\,dx \\ &\le \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |f(x-s)|\,|\mu(s)|\,ds\right) w(x)\,dx \\ &\le \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |f(x)|\,|\mu(s)|\,ds\right) w(x+s)\,dx \\ &\le \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |f(x)|\,w(x)\,dx\right) w(s)\,|\mu(s)|\,ds \\ &\le \|f\|_{1,w}\,\|\mu\|_w \le \|f\|^{1,p}_{k,w}\,\|\mu\|_w\,.\end{aligned}$$

Hence we get $T \in M\left(A^{1,p}_{k,w,w}\left(\mathbb{R}^d\right), L^1_w\left(\mathbb{R}^d\right)\right)$ and $\|T\| \le \|\mu\|_\omega$.

Conversely, suppose that $T \in M\left(A^{1,p}_{k,w,w}\left(\mathbb{R}^d\right), L^1_w\left(\mathbb{R}^d\right)\right)$. Therefore we have

$$\|Tf\|_{1,w} \le \|T\|\,\|f\|^{1,p}_{k,w} = \|T\|\left(\|f\|_{1,w} + \|f\|_{W^{p,w}_k}\right)$$

for each $f \in A^{1,p}_{k,w,w}\left(\mathbb{R}^d\right)$. In [7, Lemma 2.1], it is obtained $\lim_{s\to\infty}\|f + L_s f\|_{p,w} = 2^{\frac{1}{p}}\|f\|_{p,w}$ for all $f \in L^p_w\left(\mathbb{R}^d\right)$ using the method in [11]. Since the norm $\|.\|_{W^{p,w}_k}$ is a finite sum of L^p_w norms, we find $\lim_{s\to\infty}\|f + L_s f\|_{W^{p,w}_k} = 2^{\frac{1}{p}}\|f\|_{W^{p,w}_k}$. So we get

$$2\|Tf\|_{1,w} = \lim_{s\to\infty}\|Tf + TL_s f\|_{1,w} = \lim_{s\to\infty}\|T(f + L_s f)\|_{1,w}$$

$$\leq \lim_{s\to\infty} \|T\| \left(\|f + L_s f\|_{1,w} + \|f + L_s f\|_{W_k^{p,w}} \right)$$
$$\leq \|T\| \left(2\|f\|_{1,w} + 2^{\frac{1}{p}} \|f\|_{W_k^{p,w}} \right).$$

Therefore we have

$$\|Tf\|_{1,w} \leq \|T\| \left(\|f\|_{1,w} + 2^{\frac{1}{p}-1} \|f\|_{W_k^{p,w}} \right) .$$

Repeating this process n times, we see that

$$\|Tf\|_{1,w} \leq \|T\| \left(\|f\|_{1,w} + 2^{n\left(\frac{1}{p}-1\right)} \|f\|_{W_k^{p,w}} \right) .$$

Since $p > 1$ we obtain $\lim_{n\to\infty} 2^{n\left(\frac{1}{p}-1\right)} = 0$ and so we conclude that

$$\|Tf\|_{1,w} \leq \|T\| \|f\|_{1,w} .$$

Hence T is continuous on $A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$, considered as a subspace of $L_w^1\left(\mathbb{R}^d\right)$. Thus T defines a continuous linear transformation from $A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ as a subspace of $L_w^1\left(\mathbb{R}^d\right)$ to $L_w^1\left(\mathbb{R}^d\right)$ which commutes with translation. Since $A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ is dense in $L_w^1\left(\mathbb{R}^d\right)$, T determines a unique element T' of $M\left(L_w^1\left(\mathbb{R}^d\right)\right)$ and $\|T'\| \leq \|T\|$. There exists a unique element $\mu \in M_w\left(\mathbb{R}^d\right)$ such that $T'f = \mu * f$ for each $f \in L_w^1\left(\mathbb{R}^d\right)$ and $\|\mu\|_w = \|T'\|$. Consequently $Tf = \mu * f$ for each $f \in A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right)$ and $\|\mu\|_w \leq \|T\|$. Hence (i) and (ii) are equivalent. It is evident that the correspondence between T and μ defines isometric algebra isomorhism from $M\left(A_{k,w,w}^{1,p}\left(\mathbb{R}^d\right), L_w^1\left(\mathbb{R}^d\right)\right)$ onto $M_w\left(\mathbb{R}^d\right)$. □

REFERENCES

[1] R. Adams and J. Fournier, *Sobolev spaces*, 2nd ed., ser. Pure and Applied Mathematics. London: Academic Press, 2003.

[2] A. Beurling, "Sur les intégrales de Fourier absolument convergentes et leur application à une transformation fonctionelle," in *Proceedings of the Ninth Scandinavian Mathematical Congress*, Helsinki, 1938, pp. 345–366.

[3] C. Chu, "Some properties of sobolev algebras," *Soochow J. Math.*, vol. 9, pp. 47–52, 1983.

[4] J. Conway, *A course in functional analysis*, 1st ed., ser. Graduate Texts in Mathematics. New York: Springer-Verlag Berlin Heidelberg, 1985.

[5] T. Coulhon, E. Russ, and V. Tardivel-Nachef, "Sobolev algebras on Lie groups and Riemannian manifolds," *Amer.J. Math.*, vol. 123, no. 2, pp. 283–342, 2001.

[6] R. Doran and J. Wichmann, *Approximate Identities and Factorization in Banach Modules*, 1st ed., ser. Lecture Notes in Mathematics. New York: Springer-Verlag Berlin Heidelberg, 1979.

[7] C. Duyar and B. Sağır, "A note on multipliers of weighted Lebesgue spaces," *to submitted Thai Journal Of Mathematics*, 2011.

[8] R. Fischer, A. T. Gürkanlı, and T. Liu, "On a family of weighted spaces," *Math. Slovaca*, vol. 46, pp. 71–82, 1996.
[9] C. Gasquet and P. Witomski, *Fourier analysis and applications: filtering, numerical computation, wavelets*, ser. Texts in Applied Mathematics. New York: Springer-Verlag, 1999.
[10] N. Güngör, "Ağırlıklı sobolev cebirleri ve bazı özellikleri," Master's thesis, Ondokuz Mayıs University, Samsun, 2012.
[11] L. Hörmander, "Estimates for translation invariant operators in L^p-spaces," *Acta Math*, vol. 104, pp. 93–140, 1960.
[12] A. Kufner, *Weighted Sobolev spaces*, 1st ed., ser. Teubner-Texte zur Mathematik. B. G. Teubner Verlagsgesellschaft Leipzig, 1980.
[13] A. Kufner, O. John, and S. Fucik, *Function spaces*, 1st ed., ser. Czechoslovak Academy of Sciences. New York: Noordhoff İnternational Publishing, 1977.
[14] R. Larsen, *An introduction to the theory of multipliers*, 2nd ed., ser. Grundlehren der mathematischen Wissenschaften. Die Grundlehren der mathematischen wissenschaften, 1971.
[15] R. Larsen, *Banach algebras. An introduction*, ser. Pure and Applied Mathematics. New York: Marcel Dekker Inc., 1977.
[16] H. Reiter and J. Stegeman, *Classical harmonic analysis and locally compact groups*, 1st ed., ser. London Mathematical Society monographs. Oxford University Press, 2000.
[17] S. Sobolev, "On a theorem of functional analysis," *Mat. Sb.*, vol. 46, pp. 471–496, 1938.
[18] S. Sobolev, *Some applications of functional analysis in mathematical physics*, 3rd ed., ser. Translation of Mathematical Monographs. Moscow: English transl.:Amer. Mat. Soc., 1988.
[19] E. Stein, *Singular integrals and differentiability properties of functions*, 2nd ed. New Jersey: Princeton University Press, 1970.
[20] R. Strichartz, "A note on sobolev algebras," *Proc. of the Amer. Math. Soc.*, vol. 29, no. 1, pp. 205–207, 1971.
[21] B. Turesson, *Nonlinear potential theory and weighted Sobolev spaces*, 1st ed., ser. Lecture Notes in Mathematics. New York: Springer - Verlag, 2000.
[22] H. Wang, *Homogeneous Banach algebras*, ser. Lecture Notes Pure and Applied Mathematics. New York: Marcel Dekker Inc., 1977.
[23] J. Yeh, *Real analysis theory of measure and integration*, 2nd ed. World Scientific Publishing, 2006.

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