Some aspects of $L^q_v(\mathbb{R}^d) \cap W^{p,w}_k(\mathbb{R}^d)$

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SOME ASPECTS OF $L^q_v(\mathbb{R}^d) \cap W^{p,w}_k(\mathbb{R}^d)$

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Abstract. Let $1 < q, p < \infty$ and $v, w$ be Beurling’s weight functions on $\mathbb{R}^d$. In this article we deal with harmonic properties of intersection space $A^{q,p}_{k,v,w}(\mathbb{R}^d) = L^q_v(\mathbb{R}^d) \cap W^{p,w}_k(\mathbb{R}^d)$ defined by aid of weighted Lebesgue space $L^q_v(\mathbb{R}^d)$ and weighted Sobolev space $W^{p,w}_k(\mathbb{R}^d)$. We research the inclusions and inequalities between the spaces $A^{q,p}_{k,v,w}(\Omega)$ where $\Omega \subset \mathbb{R}^d$ be an open set. Finally, we proved that the spaces $M(A^{1,p}_{k,v,w}(\mathbb{R}^d), L^1_w(\mathbb{R}^d))$ can be identified with the weighted spaces of bounded measures $M_w(\mathbb{R}^d)$.

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1. INTRODUCTION AND PRELIMINARIES

Throughout this work, $\mathbb{R}^d$ denote $d$–dimensional real Euclidean space with Lebesgue measure $dx$. We use Beurling’s weight function, i.e., a measurable, locally bounded function on $\mathbb{R}^d$ satisfying $\omega(x) \geq 1$ and $\omega(x+y) \leq \omega(x)\omega(y)$ for all $x, y \in \mathbb{R}^d$ [2]. We denote weighted Lebesgue space $L^p_\omega(\mathbb{R}^d) = \{ f \mid f \in L^p(\mathbb{R}^d) \}$ which is a Banach space under the norm

$$\| f \|_{p,\omega} = \int_{\mathbb{R}^d} |f(x)|^p \omega^p(x) \, dx.$$ 

Some well-known terms such as convolution, translation invariant, continuous embeddings, Banach algebra, Banach module, essential Banach ideal, approximate identity will be used frequently throughout this paper; their definitions may be found in [6], [13],[15], [22]. It is known that $L^p_\omega(\mathbb{R}^d)$ is translation invariant and

$$\| L_s f \|_{p,\omega} \leq \omega(s) \| f \|_{p,\omega} \quad (1.1)$$

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for any \( f \in L^p_\omega (\mathbb{R}^d) \). The translation operator \( L_s \) \( (L_s f(x) = f(x-s)) \) is continuous on \( L^p_\omega (\mathbb{R}^d) \). For two weight functions \( \omega_1 \) and \( \omega_2 \), we write \( \omega_1 \prec \omega_2 \) if and only if there exists a constant \( c > 0 \) such that \( \omega_1 (x) \leq c \omega_2 (x) \) for all \( x \in \mathbb{R}^d \). We write \( \omega_1 \asymp \omega_2 \) if and only if \( \omega_1 \prec \omega_2 \) and \( \omega_2 \prec \omega_1 \). Recall that one has \( L^p_{\omega_1} (\mathbb{R}^d) \subset L^p_{\omega_2} (\mathbb{R}^d) \) if and only if \( \omega_2 \prec \omega_1 \). The space \( L^p_\omega (\mathbb{R}^d) \) is a Banach module over \( L^1_\omega (\mathbb{R}^d) \) under the convolution \([8]\).

If \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{R}^d \) is an \( d \)-tuple of nonnegative integers \( \alpha_i \), then it is written \( \alpha \in \mathbb{Z}_+^d \) and \( |\alpha| = \sum_{i=1}^d \alpha_i \). Similarly if \( D_j = \frac{\partial}{\partial x_j} \) for \( 1 \leq j \leq d \), then \( D^\alpha \) \( (D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_d^{\alpha_d}) \) denotes a differential operator of order \( \alpha \). For given two locally integrable functions \( u \) and \( v \) on \( \mathbb{R}^d \), we say that \( v \) is \( \alpha \)-th weak derivative of \( u \), written \( D^\alpha u = v \), provided

\[
\int_{\mathbb{R}^d} u(x) D^\alpha \varphi(x) \, dx = (-1)^{|\alpha|} \int_{\mathbb{R}^d} v(x) \varphi(x) \, dx
\]

for all \( \varphi \in C^\infty_c (\mathbb{R}^d) \), where \( C^\infty_c (\mathbb{R}^d) \) is the space of all infinitely differentiable functions on \( \mathbb{R}^d \), each with compact support. It is known that a weak derivative, if it exists, is uniquely defined up to a set of measure zero and also it is linear \([19]\).

Let \( w \) be Beurling’s weight function. For any nonnegative integer \( k \) and \( 1 \leq p < \infty \), the weighted Sobolev space \( W^p_{k,w} (\mathbb{R}^d) \) is defined as the space of the functions \( u \in L^p_w (\mathbb{R}^d) \) such that \( D^\alpha u \) exists and \( D^\alpha u \in L^p_w (\mathbb{R}^d) \) for all \( \alpha \in \mathbb{Z}_+^d \) with \( |\alpha| \leq k \). \( W^p_{k,w} (\mathbb{R}^d) \) is a Banach space with the norm

\[
\|u\|_{W^p_{k,w}} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p_w} \quad [12],[21].
\]

Weighted Sobolev spaces are defined by aid of weighted Lebesgue space \( L^p_w (\mathbb{R}^d) \) by Kufner in 1980s. Clearly, \( W^p_{k,w} (\mathbb{R}^d) \) is a subspace of \( L^p_w (\mathbb{R}^d) \) and also \( W^p_{0,w} (\mathbb{R}^d) = L^p_w (\mathbb{R}^d) \). For any \( k \), it is obvious the embedding \( W^p_{k,w} (\mathbb{R}^d) \hookrightarrow L^p_w (\mathbb{R}^d) \). If \( \omega = 1 \), \( W^p_{k,w} (\mathbb{R}^d) = W^p_k (\mathbb{R}^d) \). If we take norm \( \|\cdot\|_{p,w} \) instead of \( \|\cdot\|_p \), we get the following properties by using the method in \([10]\) and \([12]\). If \( w_2 < \
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If one looks for Sobolev algebras in literature, one sees that there are a lot of published papers about Sobolev algebras obtained by using different function spaces that are defined over different groups or sets. These spaces have been investigated under several respects, and mostly applied to the study of strongly nonlinear variational problems and partial differential equations.

In the sense of our study, we attach importance to \cite{3}, \cite{5}, \cite{20}. In \cite{5}, it is showed that the space $L^p_\alpha(G) \cap L^\infty(G)$ is an algebra with respect to pointwise multiplication, where $G$ is a connected unimodular Lie group. Also, sufficient conditions for the Sobolev spaces to form an algebra under pointwise multiplication have been given in \cite{20}.

In \cite{3}, Chu defined $A^p_k(\mathbb{R}^d) = L^1(\mathbb{R}^d) \cap W^{k,p}(\mathbb{R}^d)$ spaces and showed some algebraic properties of these spaces (Segal algebras). In this section, we will generalize his results to weighted Sobolev algebras.

Let $1 \leq q, p < \infty$, $k$ be a nonnegative integer and $v, w$ be Beurling’s weight functions on $\mathbb{R}^d$. We deal with the some harmonic properties of the intersection space $L^q_v(\mathbb{R}^d) \cap W^{p,w}_k(\mathbb{R}^d)$. This space, denoted by $A^{q,p}_{k,v,w}(\mathbb{R}^d)$, is a normed space
with the norm
\[ \|f\|_{q,p,k,v,w} = \|f\|_{q,v} + \|f\|_{W^p,w}. \]

**Theorem 1.** \( A_{q,p}^{k,v,w} \) is a Banach space.

**Proof.** Assume that \((f_n)\) be a Cauchy sequence in \( A_{q,p}^{k,v,w} \). Clearly \((f_n)\) is a Cauchy sequence in both \( L_v^q(\mathbb{R}^d) \) and \( W_k^{p,w}(\mathbb{R}^d) \). For this reason, \((f_n)\) converges to \( f \in L_v^q(\mathbb{R}^d) \) and \( g \in W_k^{p,w}(\mathbb{R}^d) \). By using the inequalities \( \|\cdot\|_q \leq \|\cdot\|_{q,v} \) and \( \|\cdot\|_p \leq \|\cdot\|_{p,w} \leq \|\cdot\|_{W_k^{p,w}} \), we can easily demonstrate that there exist a subsequence \( (f_{n_k}) \) of \((f_n)\) such that \( f_{n_k} \to f \) a.e. and a subsequence \( (f_{n_{k_l}}) \) of \((f_{n_k})\) such that \( f_{n_{k_l}} \to g \) a.e. Therefore, we get \( f \equiv g \) a.e. \( \square \)

**Theorem 2.**

(i) \( A_{q,p}^{k,v,w} \) translation invariant and
\[ \|L_sf\|_{q,p,k,v,w} \leq (v+w)(s)\|f\|_{q,p,k,v,w} \]
for all \( f \in A_{q,p}^{k,v,w} \).

(ii) The function \( s \to L_sf \) is continuous from \( \mathbb{R}^d \) to \( A_{q,p}^{k,v,w} \) for any \( f \in A_{q,p}^{k,v,w} \).

**Proof.** (i) We know that the spaces \( L_v^q(\mathbb{R}^d) \) and \( W_k^{p,w}(\mathbb{R}^d) \) are translation invariant. Hence \( A_{q,p}^{k,v,w} \) is translation invariant. We get
\[ \|L_sf\|_{q,p,k,v,w} = \|L_sf\|_{q,v} + \|L_sf\|_{W_k^{p,w}} \]
\[ \leq v(s)\|f\|_{q,v} + w(s)\|f\|_{W_k^{p,w}} \]
\[ \leq (v+w)(s)\|f\|_{q,p,k,v,w}. \]
by (1.1) and (1.2).

(ii) Since \( s \to L_sf \) is continuous in \( L_v^q(\mathbb{R}^d) \), for any \( \varepsilon > 0 \) and \( s_0 \in \mathbb{R}^d \) there is a neighbourhood \( V_1 \) of \( s_0 \) such that
\[ \|L_sf - L_{s_0}f\|_{q,v} < \frac{\varepsilon}{2} \]
for all \( s \in V_1 \). There is a neighbourhood \( V_2 \) of \( s_0 \) such that
\[ \|L_sf - L_{s_0}f\|_{W_k^{p,w}} < \frac{\varepsilon}{2} \]
for all $s \in V_2$, because the function $s \to L_s f$ is continuous in $W_{k}^{p,w}(\mathbb{R}^d)$. Consequently $V = V_1 \cap V_2$ is a neighbourhood of $s_0$ and we get
\[ \|L_s f - L_{s_0} f\|_{k,v,w}^{q,p} < \varepsilon \]
for $s \in V$ by (2.1) and (2.2).

\[ \square \]

**Theorem 3.** $A_{k,v,w}^{q,p}(\mathbb{R}^d)$ is a BF-space.

**Proof.** Let $f \in A_{k,v,w}^{q,p}(\mathbb{R}^d)$ and any compact subset $K \subset \mathbb{R}^d$. Using Hölder inequality with $\frac{1}{p} + \frac{1}{p'} = 1$, we obtain
\[ \int_K |f(x)| dx = \int_{\mathbb{R}^d} |f(x)| \chi_K(x) dx \]
\[ \leq \left( \int_{\mathbb{R}^d} |f(x)|^p dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^d} (\chi_K(x))^{p'} dx \right)^{\frac{1}{p'}} \]
\[ \leq \|f\|_{p,w} \mu(K)^{\frac{1}{p'}} \leq \|f\|_{W_k^{p,w}} \mu(K)^{\frac{1}{p'}} \]
for any $f \in W_{k}^{p,w}(\mathbb{R}^d)$. If we write $M_K = \mu(K)^{\frac{1}{p'}}$, there exists $M_K > 0$ such that
\[ \int_K |f(x)| dx \leq M_K \|f\|_{W_k^{p,w}}. \quad (2.3) \]
Also since $L_v^q(\mathbb{R}^d)$ is a BF-space, there exists $N_K > 0$ such that
\[ \int_K |f(x)| dx \leq N_K \|f\|_{q,v}. \quad (2.4) \]
If we write $C_K = \max\{M_K, N_K\}$, we get
\[ \int_K |f(x)| dx \leq C_K \|f\|_{k,v,w}^{q,p} \]
by (2.3) and (2.4).

\[ \square \]

**Theorem 4.** If $v \prec w'$ and $w \prec w'$, then $A_{k,v,w}^{q,p}(\mathbb{R}^d)$ is Banach module over $L_{w'}^1(\mathbb{R}^d)$ under the convolution.
Proof. Assume that \( v < w' \) and \( w < w' \). Then we know that \( L^1_{w'}(\mathbb{R}^d) \subset L^1_v(\mathbb{R}^d) \) and \( L^1_{w'}(\mathbb{R}^d) \subset L^1_w(\mathbb{R}^d) \). Consequently there exist \( c_1, c_2 > 0 \) such that \( \| g \|_{1,v} \leq c_1 \| g \|_{1,w'} \) and \( \| g \|_{1,w} \leq c_2 \| g \|_{1,w'} \) for any \( g \in L^1_{w'}(\mathbb{R}^d) \). Since \( L^q_v(\mathbb{R}^d) \) is a Banach module over \( L^1_v(\mathbb{R}^d) \) and \( W^p_w(\mathbb{R}^d) \) is a Banach module over \( L^1_w(\mathbb{R}^d) \) under the convolution, we get

\[
\| f * g \|_{q,p}^{k,v,w} = \| f * g \|_{q,v} + \| f * g \|_{W^p_w} \\
\leq \| f \|_{q,v} \| g \|_{1,v} + \| f \|_{W^p_w} \| g \|_{1,w} \\
\leq \| f \|_{q,v} c_1 \| g \|_{1,w'} + \| f \|_{W^p_w} c_2 \| g \|_{1,w'} \\
\leq \max \{ c_1, c_2 \} \| f \|_{q,u} \| g \|_{1,w'}
\]

for any \( f \in A_{k,v,w}^p(\mathbb{R}^d) \) and \( g \in L^1_w(\mathbb{R}^d) \).

**Theorem 5.** If \( 1 \leq p < \infty \) and \( w < v \), then \( A_{k,v,w}^{1,p}(\mathbb{R}^d) \) is Banach algebra under the convolution.

Proof. Suppose that \( w < v \). So, there is a constant \( c > 0 \) such that \( \| f \|_{1,w} \leq c \| f \|_{1,v} \) for any \( f \in L^1_v(\mathbb{R}^d) \). Now we take any \( f, g \in A_{k,v,w}^{1,p}(\mathbb{R}^d) \). Since \( L^1_v(\mathbb{R}^d) \) is a Beurling algebra and \( W^p_w(\mathbb{R}^d) \) is a Banach module over \( L^1_w(\mathbb{R}^d) \) under the convolution, we find

\[
\| f * g \|_{1,p}^{k,v,w} = \| f * g \|_{1,v} + \| f * g \|_{W^p_w} \leq \| f \|_{1,v} \| g \|_{1,v} + \| f \|_{W^p_w} \| g \|_{1,w} \\
\leq \| f \|_{1,v} \| g \|_{1,v} + \| f \|_{W^p_w} c \| g \|_{1,v} \leq \max \{ 1, c \} \| f \|_{k,v,w} \| g \|_{1,v} \\
\leq \max \{ 1, c \} \| f \|_{k,v,w} \| g \|_{k,v,w}^{1,p}.
\]

If we define a new function on \( A_{k,v,w}^{1,p}(\mathbb{R}^d) \) such that \( \| \cdot \| = \max \{ 1, c \} \| \cdot \|_{k,v,w}^{1,p} \), then we can see easily that it is a norm. Moreover, the norms \( \| \cdot \| \) and \( \| \cdot \|_{k,v,w}^{1,p} \) on \( A_{k,v,w}^{1,p}(\mathbb{R}^d) \) are equivalent. Hence we obtain

\[
\| f * g \| = \max \{ 1, c \} \| f * g \|_{k,v,w}^{1,p} \\
\leq \max \{ 1, c \} \max \{ 1, c \} \| f \|_{k,v,w}^{1,p} \| g \|_{k,v,w}^{1,p} \\
\leq \| f \| \| g \|.
\]
Definition 1. A sequence of functions $\varphi_n$ in $C_c^\infty(\mathbb{R}^d)$ satisfies the following conditions:

(i) $\varphi_n(x) \geq 0$ for all $x \in \mathbb{R}^d$
(ii) $\int_{\mathbb{R}^d} \varphi_n(x) \, dx = 1$
(iii) The support of $\varphi_n$ is in $[-\varepsilon_n, \varepsilon_n]^d$, $\varepsilon_n > 0$ and $\lim_{n \to \infty} \varepsilon_n = 0$ [9].

Theorem 6. The sequence of functions $\varphi_n$ is an approximate identity for $A_{k,v,w}^{q,p} (\mathbb{R}^d)$.

Proof. Since $\varphi_n$ is an approximate identity, for any $f \in L^q_0 (\mathbb{R}^d)$ and $\varepsilon > 0$ there exists $n_1 \in \mathbb{N}$ such that

$$\| f * \varphi_n - f \|_{q,v} < \frac{\varepsilon}{2}$$

for all $n \geq n_1$. Also we can see that there exists a $n_2 \in \mathbb{N}$ such that

$$\| f * \varphi_n - f \|_{W^{p,w}_k} < \frac{\varepsilon}{2}$$

for all $n \geq n_2$ by using the method in [4],[22]. If we set $n_0 = \max \{ n_1, n_2 \}$, then by (2.5) and (2.6) we obtain

$$\| f * \varphi_n - f \|_{q,p}^{q,p} < \varepsilon$$

for all $n \geq n_0$.

Theorem 7. For each $f \neq 0$, $f \in A_{k,v,w}^{q,p} (\mathbb{R}^d)$ there exists $c(f) > 0$ such that

$$c(f)(v+w)(s) \leq \| L_s f \|_{q,p}^{q,p} \leq (v+w)(s) \| f \|_{q,p}^{q,p}.$$

Proof. For given $f \in A_{k,v,w}^{q,p} (\mathbb{R}^d)$, we write $f \in L^q_0 (\mathbb{R}^d)$ and $f \in W^{p,w}_k (\mathbb{R}^d)$. Let $K$ be any compact subset of $\mathbb{R}^d$. Since $\| L_s f \|_{W^{p,w}_k} \geq \| D^\alpha L_s f \|_{p,w}$ for all $f \in W^{p,w}_k (\mathbb{R}^d)$, we find

$$\| L_s f \|_{W^{p,w}_k} \geq \| D^\alpha L_s f \|_{p,w} = \left\{ \int_{\mathbb{R}^d} |D^\alpha f (x-s)|^p w(s) \, dx \right\}^{\frac{1}{p}}$$

$$= \left\{ \int_{\mathbb{R}^d} |D^\alpha f (u)|^p w(u) \, du \right\}^{\frac{1}{p}} \geq \left\{ \int_{K} |D^\alpha f (u)|^p \sup_{w \in K} (u) \, du \right\}^{\frac{1}{p}}$$

$$\geq \left\{ \int_{K} |D^\alpha f (u)|^p \frac{w(u)}{w(-u)} \, du \right\}^{\frac{1}{p}} \geq \frac{w(s)}{w(-s)} \| D^\alpha f \|_{p,K}.$$
If we set \( c_1(f) = \|D^\alpha f\|_{K}^p \), then there exists a constant \( c_1(f) > 0 \) such that
\[
\|L_s f\|_{W_k^{p, w}} \geq c_1(f) w(s).
\] (2.7)
Also we know that there exists a constant \( c_2(f) > 0 \) such that
\[
\|L_s f\|_{q, v} \geq c_2(f) v(s)
\] (2.8)
for all \( f \in L^q_k(\mathbb{R}^d) \). If we set \( c(f) = \min\{c_1(f), c_2(f)\} \), then we get
\[
\|L_s f\|_{k, v, w} \geq c(f)(v + w)(s)
\] (2.9)
by inequalities (2.7) and (2.8). Also we know that
\[
\|L_s f\|_{k, v, w}^q \leq (v + w)(s) \|f\|_{k, v, w}^q
\] (2.10)
by Theorem 2. Hence the proof is completed from (2.9) and (2.10).

**Proposition 1.** Let \( 1 \leq q_1, q_2, p_1, p_2 < \infty \) and \( v_1, v_2, w_1, w_2 \) be weight functions on \( \mathbb{R}^d \). Then
\[
A^{q_1, p_1}_{k, v_1, w_1}(\mathbb{R}^d) \subset A^{q_2, p_2}_{k, v_2, w_2}(\mathbb{R}^d)
\]
if and only if there is a constant \( M > 0 \) such that
\[
\|f\|_{k, v_2, w_2} \leq M \|f\|_{k, v_1, w_1}^{q_1, p_1}
\]
for every \( f \in A^{q_1, p_1}_{k, v_1, w_1}(\mathbb{R}^d) \).

**Proof.** Assume that \( A^{q_1, p_1}_{k, v_1, w_1}(\mathbb{R}^d) \subset A^{q_2, p_2}_{k, v_2, w_2}(\mathbb{R}^d) \). We define the norm
\[
\|f\| = \|f\|_{k, v_1, w_1}^{q_1, p_1} + \|f\|_{k, v_2, w_2}^{q_2, p_2}
\]
for all \( f \in A^{q_1, p_1}_{k, v_1, w_1}(\mathbb{R}^d) \). Let \((f_n)\) be a Cauchy sequence in \( A^{q_1, p_1}_{k, v_1, w_1}(\mathbb{R}^d) \). Hence \((f_n)\) is a Cauchy sequence in
\[
\left(A^{q_1, p_1}_{k, v_1, w_1}(\mathbb{R}^d), \|\|_{k, v_1, w_1}^{q_1, p_1}\right)\text{ and }\left(A^{q_2, p_2}_{k, v_2, w_2}(\mathbb{R}^d), \|\|_{k, v_2, w_2}^{q_2, p_2}\right).
\]
Since \( A^{q_1, p_1}_{k, v_1, w_1}(\mathbb{R}^d), \|\|_{k, v_1, w_1}^{q_1, p_1}\) and \( A^{q_2, p_2}_{k, v_2, w_2}(\mathbb{R}^d), \|\|_{k, v_2, w_2}^{q_2, p_2}\) are Banach spaces, there exist \( f \in A^{q_1, p_1}_{k, v_1, w_1}(\mathbb{R}^d) \) and \( g \in A^{q_2, p_2}_{k, v_2, w_2}(\mathbb{R}^d) \) such that
\[
\|f_n - f\|_{k, v_1, w_1}^{q_1, p_1} \to 0 \text{ and } \|f_n - g\|_{k, v_2, w_2}^{q_2, p_2} \to 0.
\]
If we use the inequalities \( \|\|_{p_1} \leq \|\|_{k, v_1, w_1}^{q_1, p_1} \) and \( \|\|_{p_2} \leq \|\|_{k, v_2, w_2}^{q_2, p_2} \), then we find
\[
\|f_n - f\|_{p_1} \to 0 \text{ and } \|f_n - g\|_{p_2} \to 0.
\]
Thus there is a subsequence \((f_{n_k})\) of \((f_n)\) such that \(f_{n_k} \to f\) a.e. and also there is a subsequence \((f_{n_k})\) of \((f_n)\) such that \(f_{n_k} \to g\) a.e. Therefore we find \(f = g\) a.e., consequently we get \(\|f_n - f\| \to 0\).
Hence \( \left( A_{k,v_1,w_1}^{q_1,p_1} \left( \mathbb{R}^d \right), \| \cdot \| \right) \) is a Banach space. We consider the unit function \( I \) from 
\( \left( A_{k,v_1,w_1}^{q_1,p_1} \left( \mathbb{R}^d \right), \| \cdot \| \right) \) onto \( \left( A_{k,v_1,w_1}^{q_1,p_1} \left( \mathbb{R}^d \right), \| \cdot \|_{k,v_1,w_1}^{q_1,p_1} \right) \). Since \( \| I(f) \|_{k,v_1,w_1}^{q_1,p_1} = \| f \|_{k,v_1,w_1}^{q_1,p_1} \), the unit function is continuous. Then it is homeomorphism by Banach Theorem. This means that \( \| \cdot \| \) and \( \| \cdot \|_{k,v_1,w_1}^{q_1,p_1} \) are equivalent, so there is a constant \( M > 0 \) such that
\[
\| f \| \leq M \| f \|_{k,v_1,w_1}^{q_1,p_1} \tag{2.11}
\]
for all \( f \in A_{k,v_1,w_1}^{q_1,p_1} \left( \mathbb{R}^d \right) \). If we use the definition of \( \| \cdot \| \) and the inequality (2.11), then we obtain
\[
\| f \|_{k,v_2,w_2}^{q_2,p_2} \leq M \| f \|_{k,v_1,w_1}^{q_1,p_1}.
\]
Conversely, if \( \| f \|_{k,v_2,w_2}^{q_2,p_2} \leq M \| f \|_{k,v_1,w_1}^{q_1,p_1} \) for all \( f \in A_{k,v_1,w_1}^{q_1,p_1} \left( \mathbb{R}^d \right) \), we can easily that the inclusion \( A_{k,v_1,w_1}^{q_1,p_1} \left( \mathbb{R}^d \right) \subset A_{k,v_2,w_2}^{q_2,p_2} \left( \mathbb{R}^d \right) \) holds.

It is easy to obtain the following proposition by aid of Proposition 1.

**Proposition 2.** Let \( v_1, v_2, w_1, w_2 \) be weight functions on \( \mathbb{R}^d \) and \( 1 \leq q, p < \infty \). If \( v_2 < v_1 \) and \( w_2 < w_1 \), then \( A_{k,v_1,w_1}^{q,p}(\mathbb{R}^d) \subset A_{k,v_2,w_2}^{q,p}(\mathbb{R}^d) \).

**Theorem 8.** Let \( \Omega \subset \mathbb{R}^d \) be an open set and \( v_1, v_2, w_1, w_2 \) be weight functions on \( \mathbb{R}^d \) satisfying \( v_2 < v_1 \) and \( w_2 < w_1 \). Then
\[
A_{k,v_1,w_1}^{q,p}(\Omega) \prec A_{l,v_2,w_2}^{q,p}(\Omega)
\]
for all \( k, l \in \mathbb{Z}^+ \) where \( k > l \).

**Proof.** Let \( f \in A_{k,v_1,w_1}^{q,p}(\Omega) \) be given, so we write \( f \in L_v^q(\Omega) \) and \( f \in W_k^p,w_1(\Omega) \). It is known that \( L_v^q(\Omega) \subset L_v^q(\Omega) \) where \( v_2 < v_1 \). Also we know \( W_k^p,w_1(\Omega) \subset W_k^p,w_2(\Omega) \) where \( w_2 < w_1 \) and \( k > l \). Therefore we obtain
\[
f \in L_v^q(\Omega) \cap W_k^p,w_2(\Omega) = A_{k,v_2,w_2}^{p,q}(\Omega).
\]
So we find \( A_{k,v_1,w_1}^{q,p}(\Omega) \subset A_{l,v_2,w_2}^{q,p}(\Omega) \).

There exists a constant \( c_1 > 0 \) such that
\[
\| f \|_{q,v_2} \leq c_1 \| f \|_{q,v_1} \tag{2.12}
\]
for all \( f \in L_v^q(\Omega) \), because \( v_2 < v_1 \). Moreover, since \( W_k^p,w_1(\Omega) \subset W_k^p,w_2(\Omega) \) where \( k > l \) and \( w_2 < w_1 \), there exists a constant \( c_2 > 0 \) such that
\[
\| f \|_{W_k^p,w_2} \leq c_2 \| f \|_{W_k^p,w_1} \tag{2.13}
\]
for all \( f \in W_k^p,w_1(\Omega) \). If we set \( c = \max \{ c_1, c_2 \} \), we get
\[
\| f \|_{l,v_2,w_2} \leq c \left( \| f \|_{q,v_1} + \| f \|_{W_k^p,w_1} \right)
\leq c \| f \|_{k,v_1,w_1}
\]
from the inequalities (2.12) and (2.13). This completes the proof.
\( \square \)
We prove the following theorem with using method in [23].

**Theorem 9.** Let $v_1, v_2, w_1, w_2$ be weight functions on $\mathbb{R}^d$ satisfying $v_2 < v_1$, $w_2 < w_1$ and $k, l \in \mathbb{Z}^+$ with $k > l$. If $\Omega \subset \mathbb{R}^d$ be an open set such that $\mu(\Omega) < \infty$, then

$$A_{k,v_1,w_1}^q(\Omega) \subset A_{l,v_2,w_2}^p(\Omega)$$

where $1 \leq q < s < \infty$ and $1 \leq p < r < \infty$.

**Proof.** Assume that $f \in A_{k,v_1,w_1}^q(\Omega)$, so we write $f \in L_{v_1}^s(\Omega)$ and $f \in W_{p,w_1}^r(\Omega)$. If we set $\alpha = \frac{q}{p}$ where $1 \leq q < s < \infty$ and let $\beta$ be conjugate exponent of $\alpha$, then we find

$$
\left\| f \right\|_{q,v_1,1}^q = \int_{\Omega} |f(x)|^q v_1^q(x) \, dx \leq \left\{ \int_{\Omega} \left( \left| \int_{\Omega} |f(x)|^q v_1^q(x) \right|^\frac{q}{s} \, dx \right)^\frac{s}{q} \right\}^{\frac{1}{\beta}} \left( \int_{\Omega} (|f|)^\beta \, dx \right)^{\frac{1}{\beta}}
$$

by Hölder inequality. Since $\mu(\Omega) < \infty$ and $f \in L_{v_1}^s(\Omega)$, we obtain $f \in L_{v_2}^s(\Omega)$ from (2.14). Hence we have $f \in L_{v_2}^s(\Omega)$, because $v_2 < v_1$. Also we can see that $W_{p,w_1}^r(\Omega) \subset W_{p,w_2}^r(\Omega)$ where $1 \leq p < r < \infty$ and $\mu(\Omega) < \infty$ by similar method. Since $w_2 < w_1$ and $k > l$, we find $W_{k,w_1}^r(\Omega) \subset W_{l,w_2}^r(\Omega)$. So we get $W_{k,w_1}^r(\Omega) \subset W_{l,w_2}^r(\Omega)$, therefore we write $f \in W_{l,w_2}^r(\Omega)$. Thus we obtain $f \in L_{v_2}^s(\Omega) \cap W_{l,w_2}^r(\Omega) = A_{l,v_2,w_2}^{q,p}(\Omega)$. This completes the proof. 

**Theorem 10.** Let $v_1, v_2, w_1, w_2$ be weight functions on $\mathbb{R}^d$ satisfying $v_2 < v_1$, $w_2 < w_1$ and $k, l \in \mathbb{Z}^+$ with $k > l$. If $\Omega \subset \mathbb{R}^d$ be an open set such that $\mu(\Omega) < \infty$, then there exist $c(f) > 0$ and $c > 0$ such that

$$c(f)(v_2 + w_2)(s) \leq \left\| L_s f \right\|_{l,v_2,w_2}^q \leq c(v_2 + w_2)(s) \left\| f \right\|_{k,v_1,w_1}^q$$

for all $f \in A_{k,v_1,w_1}^q(\Omega)$, $f \neq 0$ where $1 \leq q < s < \infty$ and $1 \leq p < r < \infty$.

**Proof.** For given $f \in A_{k,v_1,w_1}^q(\Omega)$, there exists a constant $c > 0$ such that

$$c(f)(v_2 + w_2)(s) \leq \left\| L_s f \right\|_{l,v_2,w_2}^q$$

by Theorem 7 and Theorem 9. Since $v_2 < v_1$, there is a constant $c_1 > 0$ such that

$$\left\| f \right\|_{q,v_1} \leq c_1 \left\| f \right\|_{q,v_1}.$$

Also since $W_{k,w_1}^p(\Omega) \ni \rightarrow W_{l,w_2}^p(\Omega)$ where $w_2 < w_1$ and $k > l$, there is a constant $c_2 > 0$ such that

$$\left\| f \right\|_{W_{l,w_2}^p} \leq c_2 \left\| f \right\|_{W_{k,w_1}^p}.$$
If we set $m_1 = \max\{c_1, c_2\}$, we obtain
\[
\|L_s f\|_{l^p, u^2, w^2} = \|L_s f\|_{q, u^2} + \|L_s f\|_{w^p, u^2} \\ \leq \nu_2(s) \|f\|_{q, u^2} + \nu_2(s) \|f\|_{w^p, u^2} \\ \leq \nu_2(s) c_1 \|f\|_{q, v^1} + \nu_2(s) c_2 \|f\|_{w^p, v^1} \\ \leq m_1 (\nu_2 + \nu_2) (s) \|f\|_{q, v^1, v^2} \\
\]
by using (2.16) and (2.17). Also we can see that $A^{s,r}_{k,v^1, w^1}(\Omega) \subset A^{q,p}_{k,v^1, w^1}(\Omega)$ by Theorem 9 and so there exists a constant $m_2 > 0$ such that
\[
\|f\|_{q,p} \leq m_2 \|f\|_{s,r}^{v^1, v^1} \\
\]
by Proposition 1. Thus there exists a constant $c > 0$ such that
\[
\|L_s f\|_{l^p, u^2, w^2} \leq c (\nu_2 + \nu_2) (s) \|f\|_{s,r}^{v^1, v^1} \tag{2.18}
\]
for all $f \in A^{s,r}_{k,v^1, w^1}(\Omega)$. If we combine (2.15) with (2.18), the proof is completed.

We prove the following theorem with using method in [1].

**Theorem 11.** Let $\Omega \subset \mathbb{R}^d$ be open set, $v_1, v_2, w_1, w_2$ be weight functions on $\mathbb{R}^d$ satisfying $v_2 < v_1$, $w_2 < w_1$ and $k, l \in \mathbb{Z}^+$ with $k > l$. If $\frac{1}{p} = \frac{\lambda}{q_1} + \frac{1-\lambda}{q_2}$, then
\[
A^{q_1,p_1}_{k,v^1, w^1} (\Omega) \cap A^{q_2,p_2}_{k,v^1, w^1} (\Omega) \subset A^{s,r}_{l,v^2, w^2} (\Omega)
\]
where $1 \leq q_1 < q_2 < \infty$ and $1 \leq p_1 < r < p_2 < \infty$.

**Proof.** Suppose that $f \in A^{q_1,p_1}_{k,v^1, w^1} (\Omega) \cap A^{q_2,p_2}_{k,v^1, w^1} (\Omega)$, so we write $f \in L_{q_1}^{q_1}(\Omega) \cap L_{q_2}^{q_2}(\Omega)$ and $f \in W_{k}^{p_1, w_1}(\Omega) \cap W_{k}^{p_2, w_1}(\Omega)$. If we set $t = \frac{q_1}{q_2}$, then we see $t' = \frac{q_2}{q_2 - (1-\lambda)}$ is conjugate exponent of $t$. Thus we obtain
\[
\|f\|_{s,v^1} = \int_{\Omega} |f(x)|^{\lambda} v_1^s(x) dx \\
\leq \left\{ \int_{\Omega} \left[ |f(x)|^{\lambda} v_1^s(x) \right]^{\frac{q_1}{\lambda - 1}} \right\}^{\frac{\lambda - 1}{q_1}}
\]
\[
\left\{ \int_\Omega \left[ \left| f(x) \right|^p (1-\lambda) v_1^{s(1-\lambda)}(x) \right]^{q_2/(1-\lambda)} \, dx \right\}^{1/(1-\lambda)} = \| f \|_{q_1, v_1}^{s \lambda} \| f \|_{q_2, v_1}^{s(1-\lambda)}
\]

by Hölder inequality. Since \( f \in L_{q_1}^1(\Omega) \cap L_{q_2}^1(\Omega) \), we get \( f \in L_{q_1}^2(\Omega) \). Also we can show that \( f \in W_k^{r, w}(\Omega) \) by similar method under the hypothesis. Hence we find \( f \in L_{q_1}^2(\Omega) \cap W_k^{r, w}(\Omega) = A_{k,v_1,w_1}^{s,r}(\Omega) \). We know that \( A_{k,v_1,w_1}^{s,r}(\Omega) \subset A_{k,v_2,w_2}^{s,r}(\Omega) \) where \( v_2 \leq v_1 \), \( w_2 \leq w_1 \) and \( k > l \) by Theorem 8, therefore we get \( f \in A_{k,v_2,w_2}^{s,r}(\Omega) \).

\[\square\]

3. Multiplier Spaces of \( A_{k,w_1}^{1,p}(\mathbb{R}^d), L_1^w(\mathbb{R}^d) \)

In this section we call the intersection space \( L_{q_1}^1(\mathbb{R}^d) \cap W_k^{r, w}(\mathbb{R}^d) \) as \( A_{k,w_1}^{1,p}(\mathbb{R}^d) \) and equipped with the sum norm \( \| f \|_{k,w_1}^{1,p} = \| f \|_{1,w} + \| f \|_{W_k^{r, w}} \). We denote the space of multipliers from \( A_{k,w_1}^{1,p}(\mathbb{R}^d) \) to \( L_1^w(\mathbb{R}^d) \) by \( M \left( A_{k,w_1}^{1,p}(\mathbb{R}^d), L_1^w(\mathbb{R}^d) \right) \). It is known that \( L_1^w(\mathbb{R}^d) \) is a closed ideal in the space \( M_w(\mathbb{R}^d) \) which is defined by

\[ M_w(\mathbb{R}^d) = \left\{ \mu : \mu \text{ is a bounded measure and } \| \mu \|_w = \int_{\mathbb{R}^d} w \, d|\mu| < \infty \right\} \]

We will show that \( M \left( A_{k,w_1}^{1,p}(\mathbb{R}^d), L_1^w(\mathbb{R}^d) \right) \cong M_w(\mathbb{R}^d) \) by using results in the second section.

**Proposition 3.** If \( \mu \in M_w(\mathbb{R}^d) \) and \( f \in A_{k,w_1}^{1,p}(\mathbb{R}^d) \), then \( \mu * f \in A_{k,w_1}^{1,p}(\mathbb{R}^d) \) and \( \| \mu * f \|_{k,w}^{1,p} \leq \| \mu \|_w \| f \|_{k,w}^{1,p} \).

**Proof.** Since \( s \rightarrow L_s f \) is a continuous function from \( \mathbb{R}^d \) to \( A_{k,w_1}^{1,p}(\mathbb{R}^d) \) for \( f \in A_{k,w_1}^{1,p}(\mathbb{R}^d) \) and \( \mu \) is a bounded Borel measure, then \( \int_{\mathbb{R}^d} \| L_s f \|_{k,w}^{1,p} \, d|\mu| (s) < \infty \). So, the integral \( \int_{\mathbb{R}^d} L_s f d\mu(s) \) belong to \( A_{k,w_1}^{1,p}(\mathbb{R}^d) \) by \[16, Proposition 3.2.62\]. Therefore we get
Proposition 4. \( A_{k,w}^{1,p}(\mathbb{R}^d) \) is an essential Banach ideal in \( L_{w}^{1} \left( \mathbb{R}^d \right) \).

Proof. Let \( f \in A_{k,w}^{1,p}(\mathbb{R}^d) \) and \( g \in L_{w}^{1} \left( \mathbb{R}^d \right) \). By Theorem 4, we can easily see that \( f * g \in A_{k,w}^{1,p}(\mathbb{R}^d) \) and we find

\[
\|f * g\|_{k,w}^{1,p} = \|f\|_{1,w} \cdot \|g\|_{1,w} + \|f * g\|_{W^{1,p},w}
\leq \|f\|_{1,w} \cdot \|g\|_{1,w} + \|f\|_{W^{1,p},w} \cdot \|g\|_{1,w}
\leq \|f\|_{k,w}^{1,p} \cdot \|g\|_{1,w}.
\]

We known that \( C_{c}^{\infty}(\mathbb{R}^d) \) is a dense subset of \( L_{w}^{1} \left( \mathbb{R}^d \right) \) \cite{10} and we can easily see that \( C_{c}^{\infty}(\mathbb{R}^d) \subset A_{k,w}^{1,p}(\mathbb{R}^d) \). Hence we find that \( A_{k,w}^{1,p}(\mathbb{R}^d) \) is a dense subset of \( L_{w}^{1} \left( \mathbb{R}^d \right) \). So we get that \( A_{k,w}^{1,p}(\mathbb{R}^d) \) is a dense Banach ideal in \( L_{w}^{1} \left( \mathbb{R}^d \right) \). Now let \( f \in A_{k,w}^{1,p}(\mathbb{R}^d) \) and \( \varepsilon > 0 \). By Theorem 2, there is a neighbourhood \( U \) of the unit element \( e \) of \( \mathbb{R}^d \) such that

\[
\|L_{s} f - f\|_{k,w}^{1,p} < \varepsilon
\]
for all \( s \in \mathbb{R}^d \). Let \( (\varphi_n)_{n \in \mathbb{N}} \) be as in Definition 1, so there exists \( n_0 \in \mathbb{N} \) such that \( \text{supp} \varphi_{n_0} \subset U \). Thus

\[
\|\varphi_n * f - f\|_{k,w}^{1,p} = \left\| \int_{\mathbb{R}^d} \varphi_n (s) (L_{s} f - f) d s \right\|_{k,w}^{1,p}
\leq \|L_{s} f - f\|_{k,w}^{1,p} \int_{\mathbb{R}^d} \varphi_n (s) d s
\leq \|L_{s} f - f\|_{k,w}^{1,p} < \varepsilon
\]
for all \( n \geq n_0 \). Therefore \( A_{k,w}^{1,p}(\mathbb{R}^d) \) is an essential Banach ideal in \( L_{w}^{1} \left( \mathbb{R}^d \right) \). \( \square \)
Theorem 12. Let $T : A^{1,p}_{k,w,w} \left( \mathbb{R}^d \right) \to L^1_{w} \left( \mathbb{R}^d \right)$ be a linear transformation, then the following are equivalent.

i) $T \in M \left( A^{1,p}_{k,w,w} \left( \mathbb{R}^d \right), L^1_{w} \left( \mathbb{R}^d \right) \right)$.

ii) There exists a unique measure $\mu \in M_{w} \left( \mathbb{R}^d \right)$ such that $Tf = \mu * f$ for each $f \in A^{1,p}_{k,w,w} \left( \mathbb{R}^d \right)$.

Moreover the correspondence between $T$ and $\mu$ defines an isometric algebra isomorphism of $M \left( A^{1,p}_{k,w,w} \left( \mathbb{R}^d \right), L^1_{w} \left( \mathbb{R}^d \right) \right)$ onto $M_{w} \left( \mathbb{R}^d \right)$.

Proof. Let $\mu \in M_{w} \left( \mathbb{R}^d \right)$ and $Tf = \mu * f$ for each $f \in A^{1,p}_{k,w,w} \left( \mathbb{R}^d \right)$. Then,

$$\|Tf\|_{1,w} = \|\mu * f\|_{1,w} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-s) \mu(s) ds \ w(x) dx$$

$$\leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |f(x-s)||\mu(s)| ds \right) w(x) dx$$

$$\leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |f(x)||\mu(s)| ds \right) w(x+s) dx$$

$$\leq \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |f(x)|w(x) dx \right) w(s)|\mu(s)| ds$$

$$\leq \|f\|_{1,w} \|\mu\|_{w} \leq \|f\|^{1,p}_{k,w} \|\mu\|_{w}.$$ 

Hence we get $T \in M \left( A^{1,p}_{k,w,w} \left( \mathbb{R}^d \right), L^1_{w} \left( \mathbb{R}^d \right) \right)$ and $\|T\| \leq \|\mu\|_{w}$.

Conversely, suppose that $T \in M \left( A^{1,p}_{k,w,w} \left( \mathbb{R}^d \right), L^1_{w} \left( \mathbb{R}^d \right) \right)$. Therefore we have

$$\|Tf\|_{1,w} \leq \|T\| \|f\|^{1,p}_{k,w} = \|T\| \left( \|f\|_{1,w} + \|f\|_{W_p^{w,w}} \right)$$

for each $f \in A^{1,p}_{k,w,w} \left( \mathbb{R}^d \right)$. In [7, Lemma 2.1], it is obtained $\lim_{s \to \infty} \|f + L_s f\|_{p,w} = 2 \frac{1}{p} \|f\|_{p,w}$ for all $f \in L^p_{w} \left( \mathbb{R}^d \right)$ using the method in [11]. Since the norm $\|\cdot\|_{W_p^{w,w}}$ is a finite sum of $L^p_{w}$ norms, we find $\lim_{s \to \infty} \|f + L_s f\|_{W_p^{w,w}} = 2 \frac{1}{p} \|f\|_{W_p^{w,w}}$. So we get

$$2\|Tf\|_{1,w} = \lim_{s \to \infty} \|Tf + TL_s f\|_{1,w} = \lim_{s \to \infty} \|T \left( f + L_s f \right)\|_{1,w}$$
SOME ASPECTS OF $L^q_v(\mathbb{R}^d) \cap W^{p,w}_k(\mathbb{R}^d)$

\[
\begin{align*}
&\leq \lim_{s \to \infty} \|T\| (\|f + L_s f\|_{1,w} + \|f + L_s f\|_{W^{p,w}_k}) \\
&\leq \|T\| (2\|f\|_{1,w} + 2^{\frac{1}{p}} \|f\|_{W^{p,w}_k}).
\end{align*}
\]

Therefore we have

\[
\|Tf\|_{1,w} \leq \|T\| (\|f\|_{1,w} + 2^{\frac{1}{p}-1} \|f\|_{W^{p,w}_k}).
\]

Repeating this process $n$ times, we see that

\[
\|Tf\|_{1,w} \leq \|T\| (\|f\|_{1,w} + 2^{n(\frac{1}{p}-1)} \|f\|_{W^{p,w}_k}).
\]

Since $p > 1$ we obtain $\lim_{n \to \infty} 2^n(\frac{1}{p}-1) = 0$ and so we conclude that

\[
\|Tf\|_{1,w} \leq \|T\| \|f\|_{1,w}.
\]

Hence $T$ is continuous on $A^{1,p}_{k,w,w} (\mathbb{R}^d)$, considered as a subspace of $L^1_w (\mathbb{R}^d)$.

Thus $T$ defines a continuous linear transformation from $A^{1,p}_{k,w,w} (\mathbb{R}^d)$ as a subspace of $L^1_w (\mathbb{R}^d)$ to $L^1_w (\mathbb{R}^d)$ which commutes with translation. Since $A^{1,p}_{k,w,w} (\mathbb{R}^d)$ is dense in $L^1_w (\mathbb{R}^d)$, $T$ determines a unique element $T'$ of $M\left(L^1_w (\mathbb{R}^d)\right)$ and $\|T'\| \leq \|T\|$. There exists a unique element $\mu \in M_w (\mathbb{R}^d)$ such that $T'f = \mu * f$ for each $f \in L^1_w (\mathbb{R}^d)$ and $\|\mu\|_w = \|T'\|$. Consequently $Tf = \mu * f$ for each $f \in A^{1,p}_{k,w,w} (\mathbb{R}^d)$ and $\|\mu\|_w \leq \|T\|$. Hence (i) and (ii) are equivalent. It is evident that the correspondence between $T$ and $\mu$ defines isometric algebra isomorphism from $M\left(A^{1,p}_{k,w,w} (\mathbb{R}^d), L^1_w (\mathbb{R}^d)\right)$ onto $M_w (\mathbb{R}^d)$.

\[\square\]

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