Gradient estimates in Morrey spaces of weak solutions to quasilinear parabolic systems of Hörmander's vector fields

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GRADIENT ESTIMATES IN MORREY SPACES OF WEAK SOLUTIONS TO QUASILINEAR PARABOLIC SYSTEMS OF HÖRMANDER'S VECTOR FIELDS

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Abstract. This paper is concerned with higher integrability for gradients of weak solutions to quasilinear parabolic systems of Hörmander's vector fields. We establish L^p estimates for gradients of weak solutions by deriving a parabolic Caccioppli inequality and using the reverse Hölder inequality in parabolic cylinders, and then obtain L^p estimates for gradients of weak solutions to homogeneous parabolic system. At last higher integrability of gradients in Morrey space with $p \geq 2$ is proved.

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1. INTRODUCTION

Let X $= (X_1, \cdots, X_q)$ be a family of real smooth vector fields in a neighborhood $\tilde{\Omega}$ of some bounded domain $\Omega \subset \mathrm{R}^n (q \leq n)$ with the form

$$X_j = \sum_{k=1}^{n} b_{jk}(x) \frac{\partial}{\partial x_k}, \quad (b_{jk}(x) \in C^\infty(\Omega), j = 1, 2, \ldots, q)$$

and satisfying Hörmander's condition, namely, the Lie algebra generated by the family $X_1, \ldots, X_q$ at any point of $\tilde{\Omega}$ spans $\mathbb{R}^n$.

Regularity for weak solutions to parabolic equations and systems in Euclidean spaces (i.e., $q = n, X_i = \frac{\partial}{\partial x_i}, i = 1, 2, \cdots, n$) had fairly complete results. When coefficients are discontinuous and belong to some VMO spaces, Mcbride in [15] derived the generalized Morrey estimates for gradients of weak solutions to linear parabolic systems. Giaquinta and Struwe in [12] treated partial regularity for weak solutions to quasilinear parabolic systems under the natural condition and got a sharp Hölder

exponent by establishing a Caccioppoli inequality and the reverse Hölder inequality. For some other studies, we quote [2, 3, 11, 17] and references therein.

Divergence equations or systems constituted by Hörmander's vector fields have received extensive attention. Di Fazio and Fanciullo in [4] studied gradient estimates in Morrey spaces when $p = 2$ for weak solutions to linear degenerate elliptic system with coefficients in some VMO spaces. Dong and Niu in [6] generalized the result and derived gradient estimates in Morrey spaces when $p > 2$ for weak solutions to linear elliptic systems by establishing energy estimate and using the reverse Hölder inequality on the homogeneous spaces, and got Hölder regularity. Xu and Zuily in [20] attained interior regularity for weak solutions to quasilinear elliptic systems with natural condition. Partial Hölder regularity for weak solutions to quasilinear elliptic systems under quadratic natural growth condition and controllable growth condition was deirved by Gao and Niu in [8]. Related results can also be found in [18, 19].

The purpose of this paper is to investigate quasilinear parabolic systems of Hörmander's vector fields

$$u_t^i + X_\alpha^* \left(a_i^{\alpha\beta}(z,u) X_\beta u^i \right) = g_i(z,u,Xu) + X_\alpha^* f_i^\alpha(z), \qquad (1.1)$$

where $\alpha, \beta = 1, 2, \ldots, q, i = 1, 2, \ldots, N, X_j^* = -X_j + c_j (c_j = -\sum_{k=1}^{n} \frac{\partial b_{jk}}{\partial x_k} \in C^\infty(\Omega))$ is the transposed vector field of X_j, $z = (x,t) \in Q_T \subset \mathbb{R}^{n+1}$, $Q_T = \Omega \times (0,T)$, $g_i(z,u,Xu)$ satisfies natural growth condition. We will establish higher integrability of gradients in the Morrey spaces $L_X^{p,\lambda}(p \geqslant 2)$ for weak solutions to (1.1).

Definition 1 (BMO and VMO spaces)**.** For any $f \in L^1(Q_T)$, we set

$$\eta_R(f) = \sup_{z_0 \in Q_T, 0 \leqslant \rho \leqslant R} \left(\frac{1}{|Q_T \cap Q_\rho(z_0)|} \iint_{Q_T \cap Q_\rho(z_0)} |f(z) - f_{Q_T \cap Q_\rho(z_0)}(z)| dz \right),$$

where $f_{Q_T \cap Q_\rho(z_0)} = \frac{1}{|Q_T \cap Q_\rho(z_0)|} \iint_{Q_T \cap Q_\rho(z_0)} f(z) dz$, and the meaning of $Q_\rho(z_0)$ sees the next section. If $\sup_{R>0} \eta_R(f) < +\infty$, we say $f \in \mathrm{BMO}(Q_T)$ (Bounded Mean Oscillation). Moreover, if $\eta_R(f) \to 0$ as $R \to 0$, then we say $f \in \mathrm{VMO}(Q_T)$ (Vanishing Mean Oscillation).

Now we introduce the relevant assumptions.

(H1) (ellipticity condition) Let coefficients $a_i^{\alpha\beta}(z,u)$ satisfy the ellipticity condition, namely, there exists a constant $\delta > 0$ such that for any $\xi \in \mathbb{R}^{(q+1)N}$,

$$a_i^{\alpha\beta}(z,u)\xi_\alpha^i \xi_\beta^i \geqslant \delta |\xi|^2, \qquad (1.2)$$

where $z = (x,t) \in Q_T$, $Q_T = \Omega \times (0,T)$, $\Omega \subset \mathbb{R}^n$.

(H2) ($VMO \cap L^\infty$) Assume $a_i^{\alpha\beta}(z, u_0) \in VMO$ uniformly on $z \in Q_T$ for some fixed $u_0 \in \mathbb{R}^N$, that is,

$$\lim_{R \to 0} \eta_R(a_i^{\alpha\beta}(\cdot, u_0)) = 0,$$

and there exists a constant $L > 0$, such that for any $z \in Q_T$ and $u \in \mathbb{R}^N$, one has $\left| a_i^{\alpha\beta}(z, u) \right| \leqslant L$.

(H3) (Continuity) There exist a positive constant c and a continuous concave function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$, $\omega(0) = 0$, $0 \leqslant \omega \leqslant 1$, such that for any $z \in Q_T$, $u, v \in \mathbb{R}^N$,

$$\left| a_i^{\alpha\beta}(z, u) - a_i^{\alpha\beta}(z, v) \right| \leqslant c\omega\left(|u - v|^2 \right).$$

(H4) (Natural growth condition) Let $u \in W_2^{1,1}(Q_T, \mathbb{R}^N) \cap L^\infty(Q_T, \mathbb{R}^N)$ (the definition of $W_2^{1,1}(Q_T, \mathbb{R}^N)$ sees Section 2), $M = \sup\limits_{z \in Q_T} |u(z)|$ and $g_i(z, u, Xu)$ satisfy

$$|g_i(z, u, Xu)| \leqslant \mu(M)\left(|Xu|^2 + g^i(z) \right),$$

where $\mu(M) > 0$, $g^i(z) \in L_X^{m,\lambda}(Q_T)$, $f_i^\alpha(z) \in L_X^{m,\lambda}(Q_T)$, $i = 1, 2, \cdots, N$, $m > p$ (for definition of p see Theorem 3.2), $0 < \lambda < Q + 2$ (Q is the homogeneous dimension relative to Ω), and $2\mu(M)M < \delta$.

Definition 2. If $u \in W_2^{1,1}(Q_T, \mathbb{R}^N) \cap L^\infty(Q_T, \mathbb{R}^N)$ satisfies

$$\iint_{Q_T} \left[u_t^i \varphi^i + a_i^{\alpha\beta}(z, u) X_\alpha \varphi^i X_\beta u^i \right] dz = \iint_{Q_T} \left[g_i(z, u, Xu)\varphi^i + f_i^\alpha(z) X_\alpha \varphi^i \right] dz,$$

for any $\varphi \in C_0^\infty(Q_T, \mathbb{R}^N)$, then u is said a weak solution to (1.1).

Now the main result of the paper is stated as follows.

Theorem 1. *Suppose that assumptions (H1)-(H4) hold. If $u \in W_2^{1,1}(Q_T, \mathbb{R}^N) \cap L^\infty(Q_T, \mathbb{R}^N)$ is a weak solution of (1.1) in Q_T, then there exists a constant $\varepsilon_0 > 0$ such that for any $p \in \left[2, 2 + \frac{2Q}{Q+2}\varepsilon_0 \right)$, $Q' \subset\subset Q'' \subset\subset Q_T$, we have $Xu \in L_{X,loc}^{p,\lambda}(Q_T)$ and*

$$\|Xu\|_{L_X^{p,\lambda}(Q')} \leqslant c\left(\|Xu\|_{L^p(Q'')} + \sum_i \left\| g^i \right\|_{L_X^{p,\lambda}(Q_T)} + \|f\|_{L_X^{p,\lambda}(Q_T)} \right). \quad (1.3)$$

Remark 1. These conclusions are new even in quasilinear parabolic systems under Euclidean spaces. The proofs depend on some a priori estimates, e.g., parabolic Caccioppoli inequalities. We note that in the study of regularity to parabolic equations in Euclidean spaces, a normal tool is a parabolic Poincaré inequality. But for (1.1), a corresponding parabolic Poincaré inequality is still not proved, we introduce the average $\bar{u}(t)$ of $u(x, t)$ as done in [5] and [12], and apply (2.2) below to deduce a

parabolic Caccioppoli inequality. In order to get higher integrability, we employ a reverse Hölder inequality in parabolic cylinders.

The plan of the paper is organized as follows. In Section 2, we describe the notions of Carnot-Carathéodory distance, ball, doubling property, parabolic cylinder, and some function spaces related to vector fields, such as Morrey space, Sobolev space, and then recall the known iterative lemma, Sobolev-Poincaré inequality in Ω, reverse Hölder inequality relate to parabolic cylinder. In Section 3, we establish a parabolic Caccioppoli inequality with the average $\bar{u}(t)$ of $u(x, t)$ and get the L^p estimates of gradients for weak solutions (Theorem 3) by using the reverse Hölder inequality. In Section 4, higher integrability of gradients for weak solutions to homogeneous parabolic system are derived. In Section 5, the proof of Theorem 1 is given with the help of previous results.

2. PRELIMINARIES

Let us denote the commutator of vector fields $X_1, \cdots, X_q$ by

$$X_\beta = \left[X_{\beta_d}, \left[X_{\beta_{d-1}}, \cdots \left[X_{\beta_2}, X_{\beta_1} \right] \cdots \right] \right],$$

where $\beta = (\beta_1, \cdots, \beta_d)$, $1 \leqslant \beta_i \leqslant q$, $i = 1, \cdots, d$, $|\beta| = d$, and recall that d is the length of X_β.

Definition 3. If $\left\{ X_\beta(x_0) \right\}_{|\beta| \leqslant s}$ spans $\mathbb{R}^n$ at every $x_0 \in \Omega \subset \mathbb{R}^n$, then we say that the system $X = (X_1, \cdots, X_q)$ satisfies Hörmander's condition of step s.

Following [20], we can assume that Hörmander's vector fields $X_1, \cdots, X_q$ are free up to the order s.

Definition 4 (Carnot-Carathéodory distance). Let Ω be a bounded set in $\mathbb{R}^n$. An absolutely continuous curve $\gamma : [0, T] \to \Omega$ is called a sub-unit curve with respect to the system $X = (X_1, \cdots, X_q)$, if $\gamma'(t)$ exists for a.e. $t \in [0, T]$ and satisfies

$$< \gamma'(t), \xi >^2 \leqslant \sum_{j=1}^{q} < X_j(\gamma(t)), \xi >^2, \quad \text{for any } \xi \in \mathbb{R}^n.$$

We denote the length of this curve by $l_S(\gamma) = T$. Given any $x, y \in \Omega$, let $\Phi(x, y)$ be the collection of all sub-unit curves connecting x and y, and define the Carnot-Carathéodory distance induced by X as follows

$$d_X(x, y) = \inf\{l_S(\gamma) : \gamma \in \Phi(x, y)\}.$$

With this distance, we denote a metric ball of radius R centered at x by

$$B_R(x) = B(x, R) = \{y \in \Omega : d(x, y) < R\}.$$

If one does not need to consider the center of the ball, then we also write B_R instead of $B(x, R)$.

It is well known that the doubling property (see [16]) for metric balls holds true, i.e., there exist positive constants c_D and R_D, such that for any $x_0 \in \Omega$, $0 < 2R < R_D$, $B(x_0, 2R) \subset \Omega$,

$$|B(x_0, 2R)| \leqslant c_D |B(x_0, R)|.$$

Furthermore, it follows that for any $R \leqslant R_D$ and $\tau \in (0, 1)$,

$$|B_{\tau R}| \geqslant c_D^{-1} \tau^Q |B_R|.$$

The number $Q = \log_2 c_D$ is called a locally homogeneous dimension relative to Ω. Clearly, $Q \geqslant n$.

As in [20] we assume that for any small positive R there exist two positive constants c_1 and c_2, such that

$$c_1 R^Q \leqslant |B_R| \leqslant c_2 R^Q. \tag{2.1}$$

Throughout this paper, we denote $z_0 = (x_0, t_0) \in Q_T \subset \mathbb{R}^{n+1}$. A parabolic cylinder with vertex at z_0 is defined by

$$Q_R(z_0) = B_R(x_0) \times \left(t_0 - \frac{R^2}{2}, t_0 + \frac{R^2}{2} \right]$$

In what follows, let us denote $I_R(t_0) = (t_0 - \frac{R^2}{2}, t_0 + \frac{R^2}{2}]$, and the parabolic boundary of Q_R by

$$\partial_p Q_R(z_0) = \left(\partial B_R(x_0) \times \left(t_0 - \frac{R^2}{2}, t_0 + \frac{R^2}{2} \right] \right) \cup B_R(x_0) \times \left\{ t_0 - \frac{R^2}{2} \right\}.$$

We denote by $|B(x, R)|$ the Lebesgue measure of $B(x, R)$ in the n-dimensional space, and the Lebesgue measure of $Q_R(z_0)$ in the $(n+1)$-dimensional space by $|Q_R(z_0)|$. To simplify the notations, in the sequel, $Q_R(z_0)$, $B_R(x_0)$, $I_R(t_0)$, $\sqrt{\sum_{i=1}^{q} |X_i u|^2}$ and (x, t) are written as Q_R, B_R, I_R, $|Xu|$ and z, respectively.

Definition 5 (Morrey space $L_X^{p,\lambda}(Q_T)$). For $1 \leqslant p < +\infty$, $\lambda > 0$, we say that $f \in L^p(Q_T)$ belongs to the Morrey space $L_X^{p,\lambda}(Q_T)$ if

$$\|f\|_{L_X^{p,\lambda}} = \sup_{z_0 \in Q_T, 0 \leqslant \rho \leqslant d_0} \left(\frac{\rho^\lambda}{|Q_T \cap Q_\rho(z_0)|} \iint_{Q_T \cap Q_\rho(z_0)} |f|^p dz \right)^{\frac{1}{p}} < \infty,$$

where d_0 is the diameter of Q_T.

It is easy to prove as in [15] that the spaces $L_X^{p,\lambda}(Q_T)$ are Banach spaces.

Definition 6 (parabolic Sobolev space $W_p^{m,k}(Q_T)$). Let m and k be non-negative integers, $1 \leqslant p < +\infty$. The set

$$W_p^{m,k}(Q_T) = \{u; X_\alpha u, \partial_t^r u \in L^p(Q_T), \ 0 \leqslant |\alpha| \leqslant m, \ 0 \leqslant r \leqslant k\}$$

is called a parabolic Sobolev space related to Hörmander's vector fields, with the norm

$$\|u\|_{W_p^{m,k}(Q_T)} = \sum_{|\alpha| \leqslant m} \|X_\alpha u\|_{L^p(Q_T)} + \sum_{r \leqslant k} \|X_t^r u\|_{L^p(Q_T)}.$$

The space $W_{p,0}^{m,k}(Q_T)$ is the closure of $C_0^\infty(\bar{Q}_T)$ in $W_p^{m,k}(Q_T)$, where $C_0^\infty(\bar{Q}_T)$ is the set of all infinitely differentiable functions in $\bar{Q}_T$ vanishing on the parabolic boundary of Q_T.

Lemma 1 (see [13]). *Let $H(\rho)$ be a non-negative increasing function. Suppose that for any $0 < \rho < R \leqslant R_0 = dist(x_0, \partial\Omega)$,*

$$H(\rho) \leqslant A\left[\left(\frac{\rho}{R}\right)^a + \varepsilon\right]H(R) + BR^b,$$

where A, a and b are positive constants with $a > b$. Then there exist $\varepsilon_1 = \varepsilon_1(A, a, b) > 0$ and $c = c(A, a, b) > 0$, such that if $\varepsilon < \varepsilon_1$, then one has

$$H(\rho) \leqslant c\left[\left(\frac{\rho}{R}\right)^b H(R) + B\rho^b\right].$$

Lemma 2 (Sobolev-Poincaré inequality, see [9,14]). *For any open set Ω', $\bar{\Omega}' \subset\subset \Omega$, there exist positive constants R_0 and c, such that for any $0 < R \leqslant R_0$, $B_R \subset \Omega$, and $u \in C^\infty(\overline{B_R})$, it holds*

$$\left(\frac{1}{|B_R|}\int_{B_R}|u - u_R|^{p'}dx\right)^{\frac{1}{p'}} \leqslant cR\left(\frac{1}{|B_R|}\int_{B_R}|Xu|^p dx\right)^{\frac{1}{p}},$$

where $1 < p < Q$, $1 \leqslant p' \leqslant \frac{pQ}{Q-p}$, $u_R = \frac{1}{|B_R|}\int_{B_R}u(x)dx$, R_0 and c depend on Ω and Ω'.

In particular, if $p = p'$, then

$$\int_{B_R}|u - u_R|^p dx \leqslant cR^p\int_{B_R}|Xu|^p dx. \tag{2.2}$$

If $p = 2$, $p' = \frac{2Q}{Q-2}$, then

$$\left(\frac{1}{|B_R|}\int_{B_R}|u - u_R|^{\frac{2Q}{Q-2}}dx\right)^{\frac{Q-2}{2Q}} \leqslant cR\left(\frac{1}{|B_R|}\int_{B_R}|Xu|^2 dx\right)^{\frac{1}{2}}. \tag{2.3}$$

For $u \in C_0^\infty(\overline{B_R})$, it follows

$$\left(\frac{1}{|B_R|}\int_{B_R}|u|^{p'}dx\right)^{\frac{1}{p'}} \leqslant cR\left(\frac{1}{|B_R|}\int_{B_R}|Xu|^p dx\right)^{\frac{1}{p}}. \tag{2.4}$$

We need to define a parabolic distance d_p corresponding to d_X. For $(x,t),(y,s) \in Q_T$, set

$$d_p((x,t),(y,s)) = \sqrt{d_X(x,y)^2 + |t-s|}.$$

Denote a ball under the distance d_p:

$$B_p((x_0,t_0),R) = \{(x,t) \in Q_T : d_p((x_0,t_0),(x,t)) < R\}.$$

Since $B_p((x_0,t_0),R)$ is a homogeneous space ([1]) and it holds

$$Q_R(z) \subset B_p(z,2R) \subset Q_{2R}(z),$$

we immediately follow the following result, which is the generalization on Q_T of results in [10].

Lemma 3 (reverse Hölder inequality). *Let $\hat{g}$, $\hat{f}$ be nonnegative on Q_T and satisfy*

$$\hat{g} \in L^{\hat{q}}(Q_T) \text{ and } \hat{f} \in L^r(Q_T), \quad 1 < \hat{q} < r.$$

Assume that there exist constants $b > 0$ and $\theta \in [0,1)$ such that for any $Q_{2R} \subset Q_T$ the following inequality holds

$$\frac{1}{|Q_R|} \iint_{Q_R} \hat{g}^{\hat{q}} dz \leq b \left[\left(\frac{1}{|Q_{4R/3}|} \iint_{Q_{4R/3}} \hat{g} dz \right)^{\hat{q}} + \frac{1}{|Q_{4R/3}|} \iint_{Q_{4R/3}} \hat{f}^{\hat{q}} dz \right]$$
$$+ \theta \frac{1}{|Q_{4R/3}|} \iint_{Q_{4R/3}} \hat{g}^{\hat{q}} dz,$$

then there exist positive constants ε_0 and $\theta_0 = \theta_0(\hat{q}, Q_T)$, such that if $\theta < \theta_0$, then $\hat{g} \in L^{\hat{p}}_{loc}(Q_T)$ for any $\hat{p} \in [\hat{q}, \hat{q} + \varepsilon_0)$, and

$$\left(\frac{1}{|Q_R|} \iint_{Q_R} \hat{g}^{\hat{p}} dz \right)^{\frac{1}{\hat{p}}} \leq c \left[\left(\frac{1}{|Q_{2R}|} \iint_{Q_{2R}} \hat{g}^{\hat{q}} dz \right)^{\frac{1}{\hat{q}}} + \left(\frac{1}{|Q_{2R}|} \iint_{Q_{2R}} \hat{f}^{\hat{p}} dz \right)^{\frac{1}{\hat{p}}} \right],$$

where c and ε_0 depend on $b, \hat{q}, \theta$ and Q.

Lemma 4 (Jensen inequality). *For any bounded open subset $U \subset \mathbb{R}^n$, suppose that $f : U \to \mathbb{R}$ is an integrable function and $\omega : \mathbb{R} \to \mathbb{R}$ is a concave function. Then*

$$\frac{1}{|U|} \iint_U \omega(f) dz \leq \omega \left(\frac{1}{|U|} \iint_U f dz \right).$$

3. PARABOLIC CACCIOPPOLI INEQUALITY AND L^p ESTIMATES

In this section, we first establish a parabolic Caccioppoli inequality (Theorem 2) for the weak solution u to (1.1) with the help of the average $\bar{u}(t)$ of u and then get a L^p estimates for the gradient of the weak solution (Theorem 3) by using Lemma 3.

Let two cut-off functions $\xi(x)$ and $\eta(t)$ satisfy that for any $0 < \rho < R$, $B_\rho \subset B_R \subset \Omega$,

$$\xi(x) \in C_0^\infty(B_R),\ 0 \leqslant \xi \leqslant 1,\ |X\xi| \leqslant \frac{c}{R-\rho},\ \text{and } \xi = 1 \text{ in } B_\rho,$$

$$\eta(t) = \begin{cases} \frac{2t-2\left(t_0-\frac{R^2}{2}\right)}{R^2-\rho^2}, t \in \left(t_0-\frac{R^2}{2}, t_0-\frac{\rho^2}{2}\right), \\ \qquad 1, \quad t \in \left[t_0-\frac{\rho^2}{2}, t_0+\frac{R^2}{2}\right]. \end{cases}$$

and set $\frac{1}{|B_R|}\int_{B_R} \xi^2 dx = N_1$. Similarly to [5, 12], we denote the average on B_R of $u(x,t)$ by

$$\bar{u}(t) = \left(\int_{B_R} \xi^2 dx\right)^{-1} \int_{B_R} u\xi^2 dx = \frac{1}{N_1 |B_R|} \int_{B_R} u\xi^2 dx.$$

Theorem 2 (parabolic Caccioppoli inequality). *Let $u \in W_2^{1,1}(Q_T, \mathbb{R}^N) \cap L^\infty(Q_T, \mathbb{R}^N)$ be a weak solution of (1.1). Then for any $0 < \rho < R$, $Q_R \subset Q_T$, one has*

$$\sup_{I_\rho} \int_{B_\rho} |u-\bar{u}(t)|^2\, dx + \iint_{Q_\rho} |Xu|^2 dz$$

$$\leqslant \frac{c}{(R-\rho)^2} \iint_{Q_R} |u-\bar{u}(t)|^2 dz + c \iint_{Q_R} \left(\sum_i \left|g^i\right|^2 + |f|^2\right) dz, \qquad (3.1)$$

where the positive constant c depends on Q, δ, L, M and $\mu(M)$, $I_\rho = \left(t_0 - \frac{\rho^2}{2}, t_0 + \frac{\rho^2}{2}\right]$.

Proof. Noting $B_\rho \subset B_R \subset \Omega$, multiplying both sides of (1.1) by $(u-\bar{u}(t))\xi^2$ $(x)\eta(t)$, and integrating on $Q'_R = B_R(x_0) \times \left(t_0 - \frac{R^2}{2}, s\right] (s \leqslant t_0 + \frac{R^2}{2})$, we get

$$\iint_{Q'_R} \left[u_t^i + X_\alpha^*\left(a_i^{\alpha\beta} X_\beta u^i\right)\right]\left(u^i - \bar{u}^i(t)\right)\xi^2\eta dz$$

$$= \iint_{Q'_R} \left[g_i + X_\alpha^* f_i^\alpha\right]\left(u^i - \bar{u}^i(t)\right)\xi^2\eta dz. \qquad (3.2)$$

By properties of $\xi(x)$ and $\eta(t)$, and the definition of $\bar{u}(t)$, we have

$$\int_{B_R} (u-\bar{u}(t))\xi^2\eta dx = \int_{B_R} u\xi^2\eta dx - \bar{u}(t)\int_{B_R} \xi^2\eta dx = 0,$$

and

$$\iint_{Q_R} \bar{u}(t)_t \left(u - \bar{u}(t)\right)\xi^2\eta dz = \int_{I_R} \bar{u}(t)_t dt \int_{B_R} \left(u - \bar{u}(t)\right)\xi^2\eta = 0.$$

It yields

$$\iint_{Q_R} u_t \left(u - \bar{u}(t)\right)\xi^2\eta dz = \iint_{Q_R} \left(u_t - \bar{u}(t)_t\right)\left(u - \bar{u}(t)\right)\xi^2\eta dz$$

$$= \iint_{Q_R} \left[\left(\frac{1}{2}|u - \bar{u}(t)|^2\eta\right)_t \xi^2 - \frac{1}{2}|u - \bar{u}(t)|^2\xi^2\eta_t\right]dz.$$

Since the left and right hand sides of (3.2) have the form

$$\iint_{Q'_R} \left[u^i_t + X^*_\alpha\left(a^{\alpha\beta}_i X_\beta u^i\right)\right]\left(u^i - \bar{u}^i(t)\right)\xi^2\eta dz$$

$$= \iint_{Q'_R} \left[\left(\frac{1}{2}\left|u^i - \bar{u}^i(t)\right|^2\eta\right)_t \xi^2 - \frac{1}{2}\left|u^i - \bar{u}^i(t)\right|^2\xi^2\eta_t + a^{\alpha\beta}_i\xi^2\eta X_\alpha u^i X_\beta u^i\right.$$

$$\left. + 2a^{\alpha\beta}_i\left(u^i - \bar{u}^i(t)\right)\xi\eta X_\alpha\xi X_\beta u^i\right]dz.$$

and

$$\iint_{Q'_R} \left[g_i + X^*_\alpha f^\alpha_i\right]\left(u^i - \bar{u}^i(t)\right)\xi^2\eta dz.$$

$$= \iint_{Q'_R} \left[g_i\left(u^i - \bar{u}^i(t)\right)\xi^2\eta + f^\alpha_i\xi^2\eta X_\alpha u^i + 2\xi\eta\left(u^i - \bar{u}^i(t)\right)f^\alpha_i X_\alpha\xi\right]dz,$$

respectively, it follows that (3.2) becomes

$$\iint_{Q'_R} \left[\left(\frac{1}{2}\left|u^i - \bar{u}^i(t)\right|^2\eta\right)_t \xi^2 + a^{\alpha\beta}_i\xi^2\eta X_\alpha u^i X_\beta u^i\right]dz$$

$$= \iint_{Q'_R} \left[\frac{1}{2}\left|u^i - \bar{u}^i(t)\right|^2\xi^2\eta_t - 2a^{\alpha\beta}_i\left(u^i - \bar{u}^i(t)\right)\xi\eta X_\alpha\xi X_\beta u^i\right]dz$$

$$+ \iint_{Q'_R} \left[g_i\left(u^i - \bar{u}^i(t)\right)\xi^2\eta + f^\alpha_i\xi^2\eta X_\alpha u^i\right]dz$$

$$+ \iint_{Q'_R} \left[2\xi\eta\left(u^i - \bar{u}^i(t)\right)f^\alpha_i X_\alpha\xi\right]dz. \tag{3.3}$$

By (H1), (H4), $|u| \leqslant M$ and the Young inequality, it implies from (3.3) that

$$
\iint_{Q'_R} \left(\frac{1}{2}|u-\bar{u}(t)|^2 \eta \right)_t \xi^2 dz + \delta \iint_{Q'_R} |Xu|^2 \xi^2 \eta dz
$$

$$
\leqslant \iint_{Q'_R} \frac{1}{2}|u-\bar{u}(t)|^2 \xi^2 \eta_t dz + \varepsilon \iint_{Q'_R} |Xu|^2 \xi^2 \eta dz
$$

$$
+ c_\varepsilon \iint_{Q'_R} |u-\bar{u}(t)|^2 |X\xi|^2 \eta dz + 2\mu(M)M \iint_{Q'_R} |Xu|^2 \xi^2 \eta dz
$$

$$
+ \mu(M)c_\varepsilon \iint_{Q'_R} \sum_i \left| g^i \right|^2 \xi^2 \eta dz + \mu(M)\varepsilon \iint_{Q'_R} |u-\bar{u}(t)|^2 \xi^2 \eta dz
$$

$$
+ 2c_\varepsilon \iint_{Q'_R} |f|^2 \xi^2 \eta dz + \varepsilon \iint_{Q'_R} |Xu|^2 \xi^2 \eta dz. \tag{3.4}
$$

Using

$$
\iint_{Q'_R} \left(\frac{1}{2}|u-\bar{u}(t)|^2 \eta \right)_t \xi^2 dz = \int_{B_R} \frac{1}{2}|u-\bar{u}(s)|^2 \xi^2 \eta dx,
$$

it follows by (3.4) that

$$
\int_{B_R} \frac{1}{2}|u-\bar{u}(s)|^2 \xi^2 \eta dx + (\delta - 2(\varepsilon + \mu(M)M)) \iint_{Q'_R} |Xu|^2 \xi^2 \eta dz
$$

$$
\leqslant \iint_{Q'_R} \frac{1}{2}|u-\bar{u}(t)|^2 \xi^2 \eta_t dz + c_\varepsilon \iint_{Q'_R} |u-\bar{u}(t)|^2 |X\xi|^2 \eta dz
$$

$$
+ 2c_\varepsilon \iint_{Q'_R} |f|^2 \xi^2 \eta dz + \mu(M)\varepsilon \iint_{Q'_R} |u-\bar{u}(t)|^2 \xi^2 \eta dz
$$

$$
+ \mu(M)c_\varepsilon \iint_{Q'_R} \sum_i \left| g^i \right|^2 \xi^2 \eta dz. \tag{3.5}
$$

Because $2\mu(M)M < \delta$, we can choose a small positive ε such that $2(\varepsilon + \mu(M)M) < \delta$, and by (3.5) we obtain (3.1). The proof is completed. $\square$

Theorem 3. *Let $u \in W_2^{1,1}(Q_T, \mathbb{R}^N) \cap L^\infty(Q_T, \mathbb{R}^N)$ be a weak solution of (1.1) in Q_T. Then there exists a positive constants ε_0 such that $Xu \in L_{loc}^p(Q_T)$, $p \in \left[2, 2 + \frac{2Q}{Q+2}\varepsilon_0\right)$, and for any $Q_R \subset Q_{2R} \subset Q_T$, it holds*

$$
\left(\frac{1}{|Q_R|} \iint_{Q_R} |Xu|^p dz \right)^{\frac{1}{p}} \leqslant
$$

$$\leq c\left[\left(\frac{1}{|Q_{2R}|}\iint_{Q_{2R}}|Xu|^2 dz\right)^{\frac{1}{2}}+\left(\frac{1}{|Q_{2R}|}\iint_{Q_{2R}}\left(\sum_i\left|g^i\right|^2+|f|^2\right)^{\frac{p}{2}}dz\right)^{\frac{1}{p}}\right]$$

(3.6)

where the positive constant c depends on Q, δ, L, M *and* $\mu(M)$.

Proof. Denote $2^* = \frac{2Q}{Q-2}$, $\tilde{q} = \frac{2Q}{Q+2}$. Clearly,

$$\iint_{Q_{4R/5}}|u-\bar{u}(t)|^2 dz$$

$$\leq \sup_{I_{4R/5}}\left(\int_{B_{4R/5}}|u-\bar{u}(t)|^2 dx\right)^{\frac{1}{2}}\cdot\left(\int_{I_{4R/5}}\left(\int_{B_{4R/5}}|u-\bar{u}(t)|^2 dx\right)^{\frac{1}{2}}dt\right)$$

$$\equiv A\cdot B.$$

(3.7)

In virtue of (2.2) and (3.1),

$$A\leq\frac{c}{R}\left(\iint_{Q_R}|u-\bar{u}(t)|^2 dz\right)^{\frac{1}{2}}+c\left(\iint_{Q_R}\left(\sum_i\left|g^i\right|^2+|f|^2\right)dz\right)^{\frac{1}{2}}$$

$$\leq c\left(\iint_{Q_R}|Xu|^2 dz\right)^{\frac{1}{2}}+c\left(\iint_{Q_R}\left(\sum_i\left|g^i\right|^2+|f|^2\right)dz\right)^{\frac{1}{2}}.$$

(3.8)

By (2.1), (2.2), (2.3) and the Hölder inequality, we have

$$B\leq\int_{I_R}\left(\int_{B_R}|u-\bar{u}(t)|^{\tilde{q}}dx\right)^{\frac{1}{2\tilde{q}}}\left(\int_{B_R}|u-\bar{u}(t)|^{2^*}dx\right)^{\frac{1}{2\cdot2^*}}dt$$

$$\leq cR^{\frac{1}{2}}\int_{I_R}\left(\int_{B_R}|Xu|^{\tilde{q}}dx\right)^{\frac{1}{2\tilde{q}}}\left(\int_{B_R}|Xu|^2 dx\right)^{\frac{1}{4}}dt$$

$$\leq cR^{\frac{3}{2}-\frac{1}{Q}}\left(\iint_{Q_R}|Xu|^{\tilde{q}}dz\right)^{\frac{1}{2\tilde{q}}}\left(\iint_{Q_R}|Xu|^2 dz\right)^{\frac{1}{4}}.$$

(3.9)

Inserting (3.8) and (3.9) into (3.7) follows

$$\iint_{Q_{4R/5}}|u-\bar{u}(t)|^2 dz\leq cR^{\frac{3}{2}-\frac{1}{Q}}\left(\iint_{Q_R}|Xu|^2 dz\right)^{\frac{3}{4}}\cdot\left(\iint_{Q_R}|Xu|^{\tilde{q}}dz\right)^{\frac{1}{2\tilde{q}}}$$

$$+cR^{\frac{3}{2}-\frac{1}{Q}}\left(\iint_{Q_R}|Xu|^{\tilde{q}}dz\right)^{\frac{1}{2\tilde{q}}}\left(\iint_{Q_R}|Xu|^2 dz\right)^{\frac{1}{4}}\cdot\left(\iint_{Q_R}\left(\sum_i\left|g^i\right|^2+|f|^2\right)dz\right)^{\frac{1}{2}}$$

$$\equiv I+II.$$

(3.10)

To estimate I and II, it uses the Young inequality to get

$$I \leqslant \varepsilon R^2 \iint_{Q_R} |Xu|^2 dz + c_\varepsilon R^{-\frac{4}{Q}} \left(\iint_{Q_R} |Xu|^{\tilde{q}} dz \right)^{\frac{2}{\tilde{q}}}, \qquad (3.11)$$

$$II \leqslant \varepsilon R^2 \iint_{Q_R} |Xu|^2 dz + \varepsilon c_\varepsilon R^{-\frac{4}{Q}} \left(\iint_{Q_R} |Xu|^{\tilde{q}} dz \right)^{\frac{2}{\tilde{q}}}$$
$$+ c_\varepsilon R^2 \iint_{Q_R} \left(\sum_i \left| g^i \right|^2 + |f|^2 \right) dz. \qquad (3.12)$$

Putting (3.11) and (3.12) into (3.10),

$$\iint_{Q_{4R/5}} |u - \bar{u}(t)|^2 dz \leqslant \varepsilon R^2 \iint_{Q_R} |Xu|^2 dz + c_\varepsilon R^{-\frac{4}{Q}} \left(\iint_{Q_R} |Xu|^{\tilde{q}} dz \right)^{\frac{2}{\tilde{q}}}$$
$$+ c_\varepsilon R^2 \iint_{Q_R} \left(\sum_i \left| g^i \right|^2 + |f|^2 \right) dz. \qquad (3.13)$$

Let us return (3.1) with (3.13), then

$$\frac{1}{|Q_{3R/4}|} \iint_{Q_{3R/4}} |Xu|^2 dz \leqslant \frac{\varepsilon}{|Q_R|} \iint_{Q_R} |Xu|^2 dz + c_\varepsilon \left(\frac{1}{|Q_R|} \iint_{Q_R} |Xu|^{\tilde{q}} dz \right)^{\frac{2}{\tilde{q}}}$$
$$+ \frac{c_\varepsilon}{|Q_R|} \iint_{Q_R} \left(\sum_i \left| g^i \right|^2 + |f|^2 \right) dz.$$

Denoting $\hat{g} = |Xu|^{\tilde{q}}$ $(\hat{q} = \frac{2}{\tilde{q}} = \frac{Q+2}{Q} > 1)$, $\hat{f} = \left(\sum_i |g^i|^2 + |f|^2 \right)^{\frac{Q}{Q+2}}$, $\theta = \varepsilon$, the above inequality is

$$\frac{1}{|Q_{3R/4}|} \iint_{Q_{3R/4}} \hat{g}^{\hat{q}} dz \leqslant c \left[\left(\frac{1}{|Q_R|} \iint_{Q_R} \hat{g} dz \right)^{\hat{q}} + \frac{1}{|Q_R|} \iint_{Q_R} \hat{f}^{\hat{q}} dz \right]$$
$$+ \frac{\theta}{|Q_R|} \iint_{Q_R} \hat{g}^{\hat{q}} dz. \qquad (3.14)$$

By Lemma 3, we know that there exist positive constant c and ε_0, such that for $\hat{p} \in [\hat{q}, \hat{q} + \varepsilon_0)$,

$$\left(\frac{1}{|Q_R|} \iint_{Q_R} |Xu|^{\hat{p}\tilde{q}} dz \right)^{\frac{1}{\hat{p}}} \leqslant$$

$$\leqslant c \left[\left(\frac{1}{|Q_{2R}|} \iint_{Q_{2R}} |Xu|^2 dz \right)^{\frac{Q}{Q+2}} + \left(\frac{1}{|Q_{2R}|} \iint_{Q_{2R}} \left(\sum_i |g^i|^2 + |f|^2 \right)^{\frac{\hat{p}\tilde{q}}{2}} dz \right)^{\frac{1}{\hat{p}}} \right].$$

Setting $p = \hat{p}\tilde{q} \in \left[2, 2 + \frac{2Q}{Q+2} \varepsilon_0 \right)$, the proof is finished. $\qquad\square$

4. Homogeneous parabolic system

In this section, we study the following homogeneous parabolic system

$$u_t^i + X_\alpha^* (a_i^{\alpha\beta}(z, u) X_\beta u^i) = 0. \tag{4.1}$$

For convenience, we divide (4.1) into two parts. Concretely, let $v \in W_2^{1,1} \cap L^\infty$ be a weak solution of the following constant coefficients system:

$$\begin{cases} v_t^i + X_\alpha^* \left(\left(a_i^{\alpha\beta}(\cdot, \bar{u}(t)) \right)_R X_\beta v^i \right) = 0, & \text{in } Q_R, \\ v - u \in W_{2,0}^{1,1} \cap L^\infty, & \text{on } \partial_p Q_R, \end{cases} \tag{4.2}$$

then $w = u - v$ satisfies:

$$\begin{cases} w_t^i + X_\alpha^* \left(\left(a_i^{\alpha\beta}(\cdot, \bar{u}(t)) \right)_R X_\beta w^i \right) \\ = X_\alpha^* \left(\left(\left(a_i^{\alpha\beta}(\cdot, \bar{u}(t)) \right)_R - a_i^{\alpha\beta}(z, u) \right) X_\beta u^i \right), & \text{in } Q_R, \\ w \in W_{2,0}^{1,1} \cap L^\infty, & \text{on } \partial_p Q_R, \end{cases} \tag{4.3}$$

where $\left(a_i^{\alpha\beta}(\cdot, \bar{u}(t)) \right)_R = \frac{1}{|Q_R|} \iint_{Q_R} a_i^{\alpha\beta}(z, \bar{u}(t)) dz$.

Lemma 5 ([7]). *Let $v \in W_2^{1,1}(Q_T, \mathbb{R}^N) \cap L^\infty(Q_T, \mathbb{R}^N)$ be a weak solution of (4.2). Then for any $0 < \rho < R$, $Q_R \subset\subset Q_T$, one has*

$$\iint_{Q_\rho} |Xv|^2 dz \leqslant c \left(\frac{\rho}{R} \right)^{Q+2} \iint_{Q_R} |Xv|^2 dz.$$

Theorem 4. *Let $u \in W_2^{1,1}(Q_T, \mathbb{R}^N) \cap L^\infty(Q_T, \mathbb{R}^N)$ be a weak solution of (4.1). Then for any $0 < \rho < R$, $Q_R \subset\subset Q_T$, there exists $\varepsilon_0 > 0$, such that for any $p \in \left[2, 2 + \frac{2Q}{Q+2} \varepsilon_0 \right)$, $\frac{p-2}{p}(Q + 2) < \mu < Q + 2$, we have*

$$\iint_{Q_\rho} |Xu|^p dz \leqslant c \left(\frac{\rho}{R} \right)^{\frac{2(Q+2) - p(Q+2-\mu)}{2}} \iint_{Q_R} |Xu|^p dz.$$

Proof. When $\frac{1}{2} R \leqslant \rho < R$, the result is clearly true. Now it is enough to treat the case $\rho < \frac{1}{2} R$.

Multiplying both sides of (4.3) by w^i and integrating on Q_R, we obtain

$$\iint_{Q_R} \left(w_t^i w^i + \left(a_i^{\alpha\beta}(\cdot, \bar{u}(t)) \right)_R X_\beta w^i X_\alpha w^i \right) dz$$

$$= \iint_{Q_R} \left(\left(a_i^{\alpha\beta} \left(\cdot, \bar{u}(t) \right) \right)_R - a_i^{\alpha\beta} (z, u) \right) X_\beta u^i \, X_\alpha w^i \, dz. \tag{4.4}$$

Noting $\iint_{Q_R} w_t^i w^i \, dz = \int_{B_R} dx \int_{t_0 - R^2/2}^{t_0 + R^2/2} w^i \, d\, w^i \geqslant 0$ and (H1), we get

$$\delta \iint_{Q_R} |Xw|^2 dz \leqslant \iint_{Q_R} \left| \left(a_i^{\alpha\beta} \left(\cdot, \bar{u}(t) \right) \right)_R - a_i^{\alpha\beta} (z, \bar{u}(t)) \right| |Xu| |Xw| dz$$

$$+ \iint_{Q_R} \left| a_i^{\alpha\beta} (z, \bar{u}(t)) - a_i^{\alpha\beta} (z, u) \right| |Xu| |Xw| dz$$

$$= I + II. \tag{4.5}$$

Because $a_i^{\alpha\beta} (z, u)$ satisfies (H2), it follows

$$\iint_{Q_R} \left| a_i^{\alpha\beta} (z, \bar{u}(t)) - \left(a_i^{\alpha\beta} (\cdot, \bar{u}(t)) \right)_R \right|^{\frac{2p}{p-2}} dz$$

$$\leqslant c \iint_{Q_R} \left| a_i^{\alpha\beta} (z, \bar{u}(t)) - \left(a_i^{\alpha\beta} (\cdot, \bar{u}(t)) \right)_R \right| dz \leqslant c |Q_R| \eta_R \left(a_i^{\alpha\beta} \right).$$

By the Young inequality and the Hölder inequality, we have

$$I \leqslant c_\varepsilon \left(\iint_{Q_R} \left| a_i^{\alpha\beta} (z, \bar{u}(t)) - \left(a_i^{\alpha\beta} (\cdot, \bar{u}(t)) \right)_R \right|^{\frac{2p}{p-2}} dz \right)^{\frac{p-2}{p}} \left(\iint_{Q_R} |Xu|^p dz \right)^{\frac{2}{p}}$$

$$+ \varepsilon \iint_{Q_R} |Xw|^2 dz$$

$$\leqslant c_\varepsilon \left(|Q_R| \eta_R \left(a_i^{\alpha\beta} \right) \right)^{\frac{p-2}{p}} \left(\iint_{Q_R} |Xu|^p dz \right)^{\frac{2}{p}} + \varepsilon \iint_{Q_R} |Xw|^2 dz. \tag{4.6}$$

By the Young inequality, (H3), Lemma 4 and the Hölder inequality, it shows

$$II \leqslant c_\varepsilon \iint_{Q_R} \left| a_i^{\alpha\beta} (z, \bar{u}(t)) - a_i^{\alpha\beta} (z, u) \right|^2 |Xu|^2 dz + \varepsilon \iint_{Q_R} |Xw|^2 dz$$

$$\leqslant c_\varepsilon \iint_{Q_R} \omega^2 \left(|u - \bar{u}(t)|^2 \right) |Xu|^2 dz + \varepsilon \iint_{Q_R} |Xw|^2 dz$$

$$\leqslant c_\varepsilon \left(\iint_{Q_R} \left(\omega \left(|u - \bar{u}(t)|^2 \right) \right)^{\frac{2p}{p-2}} dz \right)^{\frac{p-2}{p}} \left(\iint_{Q_R} |Xu|^p dz \right)^{\frac{2}{p}}$$

$$+ \varepsilon \iint_{Q_R} |Xw|^2 dz$$

$$\leq c_\varepsilon \left(|Q_R| \omega \left(\frac{1}{|Q_R|} \iint_{Q_R} |u - \bar{u}(t)|^2 dz \right) \right)^{\frac{p-2}{p}} \left(\iint_{Q_R} |Xu|^p dz \right)^{\frac{2}{p}}$$

$$+ \varepsilon \iint_{Q_R} |Xw|^2 dz. \tag{4.7}$$

Inserting (4.6) and (4.7) into (4.5), we have

$$\delta \iint_{Q_R} |Xw|^2 dz$$

$$\leq c_\varepsilon |Q_R| \left[\left(\eta_R \left(a_i^{\alpha\beta} \right) \right)^{\frac{p-2}{p}} + \left(\omega \left(\frac{1}{|Q_R|} \iint_{Q_R} |u - \bar{u}(t)|^2 dz \right) \right)^{\frac{p-2}{p}} \right]$$

$$\cdot \left(\frac{1}{|Q_R|} \iint_{Q_R} |Xu|^p dz \right)^{\frac{2}{p}} + 2\varepsilon \iint_{Q_R} |Xw|^2 dz.$$

Let $G(z_0, R) = \left(\eta_R \left(a_i^{\alpha\beta} \right) \right)^{\frac{p-2}{p}} + \left(\omega \left(\frac{1}{|Q_R|} \iint_{Q_R} |u - \bar{u}(t)|^2 dz \right) \right)^{\frac{p-2}{p}}$ and choose ε small enough such that $\delta - 2\varepsilon > 0$, then the above can be rewritten as

$$\iint_{Q_R} |Xw|^2 dz \leq c\, G(z_0, R) |Q_R| \left(\frac{1}{|Q_R|} \iint_{Q_R} |Xu|^p dz \right)^{\frac{2}{p}}.$$

By the above, Lemma 5 and the Hölder inequality, we know that for any $\rho < \frac{1}{2} R$,

$$\iint_{Q_{2\rho}} |Xu|^2 dz \leq c \left(\frac{\rho}{R} \right)^{Q+2} \iint_{Q_R} |Xv|^2 dz + c \iint_{Q_R} |Xw|^2 dz$$

$$\leq c \left(\frac{\rho}{R} \right)^{Q+2} \iint_{Q_R} |Xu|^2 dz + c \iint_{Q_R} |Xw|^2 dz$$

$$\leq c |Q_R| \left[\left(\frac{\rho}{R} \right)^{Q+2} + G(z_0, R) \right] \left(\frac{1}{|Q_R|} \iint_{Q_R} |Xu|^p dz \right)^{\frac{2}{p}}. \tag{4.8}$$

Due to Theorem 3 with $g_i = f = 0$ and (4.8), it holds

$$\iint_{Q_\rho} |Xu|^p dz \leq c |Q_\rho| \left(\frac{1}{|Q_{2\rho}|} \iint_{Q_{2\rho}} |Xu|^2 dz \right)^{\frac{p}{2}}$$

$$\leq c \left(\left(\frac{\rho}{R} \right)^{Q+2} + G(z_0, R) \right)^{\frac{p}{2}} \left(\frac{|Q_R|}{|Q_\rho|} \right)^{\frac{p-2}{2}} \iint_{Q_R} |Xu|^p dz.$$

Taking $H(\rho) = \left(|Q_\rho|^{\frac{p-2}{2}} \iint_{Q_\rho} |Xu|^p dz \right)^{\frac{2}{p}}$, $H(R) = \left(|Q_R|^{\frac{p-2}{2}} \iint_{Q_R} |Xu|^p dz \right)^{\frac{2}{p}}$, $a = Q + 2$, $B = 0$. We have by Lemma 1 with $b = \mu$ $\left(\frac{p-2}{p} (Q + 2) < \mu < Q + 2 \right)$ that

$$\left(|Q_\rho|^{\frac{p-2}{2}} \iint_{Q_\rho} |Xu|^p dz \right)^{\frac{2}{p}} \leq c \left(\frac{\rho}{R} \right)^\mu \left(|Q_R|^{\frac{p-2}{2}} \iint_{Q_R} |Xu|^p dz \right)^{\frac{2}{p}}.$$

Inserting $\frac{|Q_R|}{|Q_\rho|} \leq c \left(\frac{\rho}{R} \right)^{-Q-2}$ into the above, we get the result. $\qquad\square$

5. PROOFS OF THE MAIN RESULTS

In this section, we will prove Theorem 1. Let $v \in W_2^{1,1} \cap L^\infty$ be a weak solution of the following system:

$$\begin{cases} v_t^i + X_\alpha^* \left(a_i^{\alpha\beta} (\cdot, \bar{u}(t)) X_\beta v^i \right) = 0, & \text{in } Q_R, \\ \quad v - u \in W_{2,0}^{1,1} \cap L^\infty, & \text{on } \partial_p Q_R, \end{cases} \tag{5.1}$$

then $w = u - v$ satisfies:

$$\begin{cases} w_t^i + X_\alpha^* \left(a_i^{\alpha\beta} (\cdot, \bar{u}(t)) X_\beta w^i \right) = g_i + X_\alpha^* f_i^\alpha, & \text{in } Q_R, \\ \quad w \in W_{2,0}^{1,1} \cap L^\infty, & \text{on } \partial_p Q_R. \end{cases} \tag{5.2}$$

Lemma 6. *Let $w \in W_{2,0}^{1,1}(Q_T, \mathbb{R}^N) \cap L^\infty(Q_T, \mathbb{R}^N)$ be a weak solution of (5.2) in Q_T. Then there exists a positive constant ε_0, such that $Xw \in L_{loc}^p(Q_T)$, $p \in \left[2, 2 + \frac{2Q}{Q+2}\varepsilon_0 \right)$, and it holds*

$$\iint_{Q_R} |Xw|^p dz \leq c \iint_{Q_{2R}} \left(\sum_i \left| g^i \right|^p + |f|^p \right) dz + c \iint_{Q_{2R}} |Xu|^p dz,$$

for any $Q_{2R} \subset\subset Q_T$.

Proof. Since $w \in W_{2,0}^{1,1}(Q_T, \mathbb{R}^N) \cap L^\infty(Q_T, \mathbb{R}^N)$, there exists a constant $M_1 > 0$, such that $|w| \leq M_1$.

Multiplying both sides of (1.1) by w^i and integrating on Q_{2R} leads to

$$\iint_{Q_{2R}} \left[w_t^i w^i + a_i^{\alpha\beta} X_\beta w^i X_\alpha w^i \right] dz = \iint_{Q_{2R}} \left[g_i w^i + f_i^\alpha X_\alpha w^i \right] dz.$$

Since

$$\iint_{Q_{2R}} w_t w \, dz = \int_{B_{2R}} dx \int_{I_{2R}} w \frac{\partial w}{\partial t} dt = \int_{B_{2R}} \frac{1}{2} w^2 dx \geq 0,$$

we have by (H1), (H4), the Young inequality and (2.4) that

$$\delta \iint_{Q_{2R}} |Xw|^2 dz \leq \iint_{Q_{2R}} w_t w \, dz + \delta \iint_{Q_{2R}} |Xw|^2 dz$$

$$\leq \iint_{Q_{2R}} \mu(M) \left(|Xu|^2 + \sum_i |g^i| \right) |w| \, dz + c_\varepsilon \iint_{Q_{2R}} |f|^2 dz + \varepsilon \iint_{Q_{2R}} |Xw|^2 dz$$

$$\leq \mu(M) M_1 \iint_{Q_{2R}} |Xu|^2 dz + c \iint_{Q_{2R}} \left(\sum_i |g^i|^2 + |f|^2 \right) dz + 2\varepsilon \iint_{Q_{2R}} |Xw|^2 dz,$$

where ε is small enough such that $\delta - 2\varepsilon > 0$. It gets

$$\iint_{Q_{2R}} |Xw|^2 dz \leq c \iint_{Q_{2R}} |Xu|^2 dz + c \iint_{Q_{2R}} \left(\sum_i |g^i|^2 + |f|^2 \right) dz.$$

Using the above estimate into (3.6) and the Hölder inequality, it follows

$$\frac{1}{|Q_R|} \iint_{Q_R} |Xw|^p dz$$

$$\leq c \left[\frac{1}{|Q_{2R}|} \iint_{Q_{2R}} |Xu|^2 dz + \frac{1}{|Q_{2R}|} \iint_{Q_{2R}} \left(\sum_i |g^i|^2 + |f|^2 \right) dz \right]^{\frac{p}{2}}$$

$$+ \frac{c}{|Q_{2R}|} \iint_{Q_{2R}} \left(\sum_i |g^i|^p + |f|^p \right) dz$$

$$\leq \frac{c}{|Q_{2R}|} \iint_{Q_{2R}} |Xu|^p dz + \frac{c}{|Q_{2R}|} \iint_{Q_{2R}} \left(\sum_i |g^i|^p + |f|^p \right) dz.$$

Hence the conclusion is proved. $\qquad\square$

Theorem 5. *Suppose that assumptions (H1)-(H4) hold. Let $u \in W_2^{1,1}(Q_T, \mathbb{R}^N) \cap L^\infty(Q_T, \mathbb{R}^N)$ be a weak solution of (1.1). Then there exists a positive constant ε_0, such that for any $p \in \left[2, 2 + \frac{2Q}{Q+2}\varepsilon_0\right)$, we have*

$$Xu \in L_{X,loc}^{p,\lambda}(Q_T, \mathbb{R}^{qn}).$$

Moreover, there exists a constant $R_0 \leq d_0$ such that for any ρ and R, $0 < \rho < R \leq R_0$, $Q_{2R} \subset\subset Q_T$, one has

$$\|Xu\|_{L_X^{p,\lambda}(Q_\rho)}^p \leq c \left[\|Xu\|_{L^p(Q_R)}^p + \sum_i \left\| g^i \right\|_{L_X^{p,\lambda}}^p + \|f\|_{L_X^{p,\lambda}}^p \right].$$

Proof. By Lemma 6 and Theorem 4, it shows

$$\iint_{Q_\rho} |Xu|^p dz \leqslant c \iint_{Q_\rho} |Xv|^p dz + c \iint_{Q_\rho} |Xw|^p dz$$

$$\leqslant c \left(\frac{\rho}{R}\right)^{\frac{2(Q+2)-p(Q+2-\mu)}{2}} \iint_{Q_R} |Xu|^p dz + c \iint_{Q_{2R}} \left(\sum_i \left|g^i\right|^p + |f|^p\right) dz$$

$$+ c \iint_{Q_{2R}} |Xu|^p dz$$

$$\leqslant \left[c \left(\frac{\rho}{R}\right)^{\frac{2(Q+2)-p(Q+2-\mu)}{2}} + c \right] \iint_{Q_{2R}} |Xu|^p dz$$

$$+ c R^{Q+2-\lambda} \left(\sum_i \left\|g^i\right\|_{L_X^{p,\lambda}}^p + \|f\|_{L_X^{p,\lambda}}^p\right).$$

Denote $H(\rho) = \iint_{Q_\rho} |Xu|^p dz$, $H(R) = \iint_{Q_R} |Xu|^p dz$, $a = \frac{2(Q+2)-p(Q+2-\mu)}{2}$, $b = Q + 2 - \lambda$, $B = c \left(\sum_i \left\|g^i\right\|_{L_X^{p,\lambda}}^p + \|f\|_{L_X^{p,\lambda}}^p\right)$, $0 < \lambda < Q + 2$. Then there exists μ, $Q + 2 - \frac{2\lambda}{p} < \mu < Q + 2$, such that $a > b$. By Lemma 1, we have

$$\iint_{Q_\rho} |Xu|^p dz$$

$$\leqslant c \left[\left(\frac{\rho}{R}\right)^{Q+2-\lambda} \iint_{Q_{2R}} |Xu|^p dz + \rho^{Q+2-\lambda} \left(\sum_i \left\|g^i\right\|_{L_X^{p,\lambda}}^p + \|f\|_{L_X^{p,\lambda}}^p\right) \right].$$

This proof is completed. □

Proof. (of Theorem 1) By Theorem 5 and the cutoff function technique, it is easy to see that Theorem 1 is true. □

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