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THE COZERO-DIVISOR GRAPH RELATIVE TO FINITELY GENERATED MODULES

H. ANSARI-TOROGHY, F. FARSHADIFAR, AND SH.HABIBI

Abstract. Let R be a commutative ring and let M be a finitely generated R-module. Let's denote the cozero-divisor graph of R by $\acute{\Gamma}(R)$. In this paper, we introduce a certain subgraph $\acute{\Gamma}_R(M)$ of $\acute{\Gamma}(R)$, called cozero-divisor graph relative to M, and obtain some related results.

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1. Introduction

Throughout this paper, R will denote a commutative ring with identity. We denote the set of maximal ideals of R by $Max(R)$.

A *graph* G is defined as the pair $(V(G), E(G))$, where $V(G)$ is the set of vertices of G and $E(G)$ is the set of edges of G. For two distinct vertices a and b of $V(G)$, the notation $a-b$ means that a and b are adjacent. A graph G is said to be *complete* if $a-b$ for all distinct $a,b \in V(G)$, and G is said to be *empty* if $E(G) = \varnothing$. Note that by this definition a graph may be empty even if $V(G) \neq \varnothing$. If $|V(G)| \geq 2$, a *path* from a to b is a series of adjacent vertices $a-v_1-v_2-\ldots-v_n-b$. The *length of a path* is the number of edges it contains. A *cycle* is a path that begins and ends at the same vertex in which no edge is repeated, and all vertices other than the starting and ending vertex are distinct. If a graph G has a cycle, the *girth* of G (notated $g(G)$) is defined as the length of the shortest cycle of G; otherwise, $g(G) = \infty$. A graph G is *connected* if for every pair of distinct vertices $a,b \in V(G)$, there exists a path from a to b. If there is a path from a to b with $a,b \in V(G)$, then the *distance from a to b* is the length of the shortest path from a to b and is denoted $d(a,b)$. If there is not a path between a and b, $d(a,b) = \infty$. The *diameter* of G is $diam(G) = Sup\{d(a,b) | a,b \in V(G)\}$.

The idea of a zero-divisor graph of a commutative ring was introduced by I. Beck in 1988 [8]. He assumes that all elements of the ring are vertices of the graph and was mainly interested in colorings and then this investigation of coloring of a commutative ring was continued by Anderson and Naseer in [4]. Anderson and Livingston [7], studied the zero-divisor graph whose vertices are the nonzero zero-divisors.

Let $Z(R)$ be the set of zero-divisors of R. The *zero-divisor graph* of R denoted by $\Gamma(R)$, is a graph with vertices $Z^*(R) = Z(R) \setminus \{0\}$ and for distinct $x, y \in Z^*(R)$ the vertices x and y are adjacent if and only if $xy = 0$. This graph turns out to exhibit properties of the set of the zero-divisors of a commutative ring. The zero-divisor graph helps us to study the algebraic properties of rings using graph theoretical tools. We can translate some algebraic properties of a ring to graph theory language and then the geometric properties of graphs help us explore some interesting results in algebraic structures of rings. The zero-divisor graph of a commutative ring has also been studied by several other authors (e.g., [5, 6, 10]).

In [2], Afkhami and Khashyarmanesh introduced the *cozero-divisor graph* $\acute{\Gamma}(R)$ of R, in which the vertices are precisely the nonzero, non-unit elements of R, denoted $W^*(R)$, and two vertices x and y are adjacent if and only if $x \notin yR$ and $y \notin xR$.

Now let M be a finitely generated R-module. The purpose of this paper is to introduce a certain subgraph $\acute{\Gamma}_R(M)$ of $\acute{\Gamma}(R)$, called the cozero-divisor graph relative to M and obtain some results similar to those of [2] and [3]. This graph, with a different point of view, can be regarded as a reduction of $\acute{\Gamma}(R)$, namely, we have $\acute{\Gamma}_R(R) = \acute{\Gamma}(R)$.

2. Auxiliary results

Let M be an R-module. The *support* of M is denoted by $Supp(M)$ and it is defined by

$$Supp(M) = \{P \in Spec(R) | Ann_R(N) \subseteq P \ for\ some\ cyclic\ submodule\ N\ of\ M\}.$$

In the rest of this paper $Max(Supp(M))$ (i.e., the set of all maximal elements in $Supp(M)$) is denoted by $Max(M)$.

The *Jacobson radical* of M is denoted by $J(M)$ and it is the intersection of all elements in $Max(M)$. Also, the union of all elements in $Max(M)$ is denoted by $N_R(M)$ [12].

M is said to be a *local module* if $|Max(M)| = 1$ [12].

The subset $W_R(M)$ of R is defined by $\{r \in R | rM \neq M\}$ [12] and set $W_R^*(M) = W_R(M) \setminus \{0\}$.

$Z_R(M) = \{r \in R|$ the R-module endomorphism on M defined by multiplication by r is not injective $\}$.

Remark 1 (See [12]). Let M be an R-module. Then $W_R(M) \subseteq N_R(M)$ and we have equality if M is a finitely generated R-module.

Remark 2. $Max(M) \subseteq Max(R)$.

Proof. This follows immediately from the proof of [12, 1.4]. □

3. MAIN RESULTS

In the rest of this paper M is a finitely generated R-module.

Definition 1. We define the *cozero-divisor graph relative to* M, denoted by $\acute{\Gamma}_R(M)$ as a graph with vertices $W_R^*(M) = W_R(M) \setminus \{0\}$ and two distinct vertices r and s are adjacent if and only if $r \notin (sM :_R M)$ and $s \notin (rM :_R M)$.

Definition 2. We define the *strongly cozero-divisor graph relative to* M, denoted by $\tilde{\Gamma}_R(M)$ as a graph with vertices $W_R^*(M) = W_R(M) \setminus \{0\}$ and two distinct vertices r and s are adjacent if and only if $r \notin \sqrt{(sM :_R M)}$ and $s \notin \sqrt{(rM :_R M)}$.

The following example shows that $\acute{\Gamma}(R)$, $\acute{\Gamma}_R(M)$, and $\tilde{\Gamma}_R(M)$ are different.

Example 1. Set $R = \mathbb{Z}$ (here $\mathbb{Z}$ denotes the ring of integers) and $M = \mathbb{Z}_{12}$. Then $W_R^*(R) = \mathbb{Z} \setminus \{-1, 1, 0\}$ and $W_R^*(M) = \mathbb{Z} \setminus (\{m : (m, 12) = 1\} \bigcup \{0\})$, where $(m, 12)$ denotes the greatest common divisor of m and 12. The elements 8 and 12 are adjacent in $\acute{\Gamma}(R)$ but they are not adjacent in $\acute{\Gamma}_R(M)$. Also, 6 and 8 are adjacent in $\acute{\Gamma}_R(M)$ but they are not adjacent in $\tilde{\Gamma}_R(M)$. Moreover, 6 and 10 are adjacent in $\tilde{\Gamma}_R(R)$ but they are not adjacent in $\tilde{\Gamma}_R(M)$.

An R-module L is said to be a *multiplication module* if for every submodule N of L there exists an ideal I of R such that $N = IL$

Theorem 1. (a) *$\acute{\Gamma}_R(M)$ is a subgraph of $\acute{\Gamma}(R)$.*
(b) *$\tilde{\Gamma}_R(R)$ is a subgraph of $\acute{\Gamma}(R)$.*
(c) *If M is a faithful R-module, then $W_R^*(M) = W^*(R)$.*
(d) *If M is a faithful R-module, then $\tilde{\Gamma}_R(M) = \tilde{\Gamma}_R(R)$.*
(e) *If M is a faithful multiplication R-module, then $\acute{\Gamma}_R(M) = \acute{\Gamma}(R)$.*

Proof. Parts (a) and (b) are clear.

(c) By part (a), $W_R^*(M) \subseteq W^*(R)$. Now let $r \in W^*(R)$ and $r \notin W_R^*(M)$. Then $rM = M$. Thus by Nakayama' Lemma, $1 + rt \in Ann_R(M) = 0$. Hence $Rr = R$, which is a contradiction.

(d) By part (c), $W_R^*(M) = W^*(R)$. Now let r and s be two distinct adjacent vertices of $\tilde{\Gamma}_R(R)$ and let $r \in \sqrt{(sM :_R M)}$. Then $r^n M \subseteq sM$ for some $n \in \mathbb{N}$. Thus by [11, Theorem 75], there exist $t \in R$ and $k \in \mathbb{N}$ such that $(r^{kn} + st)M = 0$. Since M is faithful, $r^{kn} + st = 0$ and so $r \in \sqrt{sR}$. This contradiction shows that $E(\tilde{\Gamma}_R(R)) \subseteq E(\tilde{\Gamma}_R M)$. The reverse inclusion is clear.

(e) By part (c), $W_R^*(M) = W^*(R)$. Now let r and s be two distinct adjacent vertices of $\acute{\Gamma}(R)$ and let $r \in (sM :_R M)$. Then $rM \subseteq sM$. Thus by [1], $Rr \subseteq sR$, which is a contradiction. Hence $E(\acute{\Gamma}(R)) \subseteq E(\acute{\Gamma}_R(M))$. The reverse inclusion is clear. $\square$

Remark 3. By using part (e) of Theorem 1, if $M = R$, then $\acute{\Gamma}_R(R) = \acute{\Gamma}(R)$.

We use the following lemma frequently.

Lemma 1. *Let M be an R-module and $P \in Max(M)$. Then $P = (PM :_R M)$.*

Proof. Assume $(PM :_R M) = R$ so that $PM = M$. Since M is finitely generated, there exists $x \in P$ such that $(1+x)M = 0$. Thus $1+x \in Ann_R(M)$ but by [12], $P \supseteq Ann_R(M)$. It follows that $1 \in P$, a contradiction. Now the results follows from $P \subseteq (PM :_R M)$ and Remark 2. □

Proposition 1.

(a) *The graph $\acute{\Gamma}_R(M)$ is not complete if and only if there exists an element $s \in W_R^*(M)$ such that $|(sM :_R M)| > 2$.*

(b) *$\acute{\Gamma}_R(M)$ is complete if and only if $(sM :_R M) = \{0, s\}$ for all elements s in $W_R^*(M)$.*

(c) *If R is an integral domain, then $\acute{\Gamma}_R(M)$ is not complete.*

Proof. Straightforward □

Theorem 2. *$\acute{\Gamma}_R(M)$ is complete if and only if $\tilde{\Gamma}_R(M)$ is complete.*

Proof. The sufficiency is clear. Conversely, we assume that $\acute{\Gamma}_R(M)$ is complete and r, s be arbitrary distinct elements in $W_R^*(M)$ and $r \in \sqrt{(sM : M)}$. Then $r^n M \subseteq sM$ for some $n \in \mathbb{N}$. Since $\acute{\Gamma}_R(M)$ is complete, r^n and s are adjacent. But this is a contradiction by the above arguments. □

We use the notation $\acute{\Gamma}_R(M) \setminus J(M)$ to denote a subgraph of $\acute{\Gamma}_R(M)$ with vertices $W_R^*(M) \setminus J(M)$.

Theorem 3. (a) *The graph $\acute{\Gamma}_R(M) \setminus J(M)$ is connected.*
(b) *If M is a non-local module, then $diam(\acute{\Gamma}_R(M) \setminus J(M)) \leq 2$.*

Proof. (a) If M is a local module, then $W_R^*(M) \setminus J(M)$ is a empty set, which is connected. So we assume that $|Max(M)| > 1$. Let r and s be arbitrary distinct elements in $W_R^*(M) \setminus J(M)$. Suppose that r is not adjacent to s. We may assume that $r \in (sM :_R M)$. Since $r \notin J(M)$, there exists $P \in Max(M)$ such that $r \notin P$. Thus $P \nsubseteq J(M) \cup (sM :_R M)$, otherwise, $P \subseteq J(M)$ or $P \subseteq (sM :_R M)$. In first case, $J(M) = P$ so that $|Max(M)| = 1$. In second case, $P = (sM :_R M)$ by Lemma 1. In either case we have a contradiction. Choose t in $P \setminus (J(M) \cup (sM :_R M))$. Now by using Lemma 1, we see that $r - t - s$ is the required path.
(b) This follows from the proof of part (a). □

Corollary 1. *Let M be a non-local R-module with $J(M) = 0$. Then $\acute{\Gamma}_R(M)$ is connected and $diam(\acute{\Gamma}_R(M)) \leq 2$.*

Theorem 4. *Let M be a non-local module such that for every element $r \in J(M)$, there exist $P \in Max(M)$ and $s \in P \setminus J(M)$ with $r \notin (sM :_R M)$. Then $\acute{\Gamma}_R(M)$ is connected and $diam(\acute{\Gamma}_R(M)) \leq 3$.*

Proof. Suppose that $r, s \in W_R^*(M)$ and r is not adjacent to s. We may assume that $r \in (sM :_R M)$. Then, we have the following cases:

Case 1. Suppose that $s \in J(M)$. We claim that $r \in J(M)$. Otherwise there exists $P \in Max(M)$ such that $r \notin P$. Then $rM \subseteq sM \subseteq PM$. Thus by Lemma 1, $r \in (PM :_R M) = P$, a contradiction. Thus by hypothesis, there exists $t \in P \setminus J(M)$ for some $P \in Max(M)$ with $r \notin (tM :_R M)$. Also $t \notin (rM :_R M)$; otherwise, we have $tM \subseteq rM \subseteq sM$. Thus $t \in (sM :_R M) \subseteq (PM :_R M) = P$ for each $P \in J(M)$ so that $t \in J(M)$, a contradiction. Thus r is adjacent to t. By similar arguments, we see that t is adjacent to s. Hence $r - t - s$ is the required path.

Case 2. Suppose that $r, s \notin J(M)$. Then $r \notin P$, for some $P \in Max(M)$. If $P = (sM :_R M)$, then since $r \in (sM :_R M)$, we have a contradiction. Choose p in $P \setminus (sM :_R M)$. By similar arguments as in part (a), we see that $r - p - s$ is the desired path.

Case 3. Assume that $s \notin J(M)$ and $r \in J(M)$. By our assumption, there exists $q \in P \setminus J(M)$, for some $P \in Max(M)$ such that $r \notin (qM :_R M)$. We claim that $q \notin (rM :_R M)$. Otherwise, $qM \subseteq rM \subseteq PM$ for every $P \in Max(M)$. Thus by Lemma 1, $q \in (PM :_R M) = P$ for every $P \in Max(M)$, a contradiction. Hence r is adjacent to q. Further, $s \notin (qM :_R M)$. If $q \notin (sM :_R M)$, then we get the the path $r - q - s$. Otherwise, we can apply case 2 for the elements q and s to get a path $q - u - s$ for some $u \in W_R^*(M)$. Hence we have $r - q - u - s$. □

Theorem 5. *Let M be a non-local module. Then $g(\acute{\Gamma}_R(M) \setminus J(M)) \leq 5$ or $g(\acute{\Gamma}_R(M) \setminus J(M)) = \infty$.*

Proof. Use the technique of [2, 2.8] and apply Theorem 3. □

Theorem 6. *Let $|Max(M)| \geq 3$. Then $g(\acute{\Gamma}_R(M)) = 3$.*

Proof. Clearly, $g(\acute{\Gamma}_R(M)) \geq 3$. Let P_1, P_2, and P_3 be distinct elements of $Max(M)$. By Remark 2, $Max(M) \subseteq Max(R)$. Choose $a_i \in P_i \setminus \cup_{j=1}^{3} P_j$, $1 \leq i \leq 3$ and $j \neq i$. Then by using 1, we see that $a_1 - a_2 - a_3 - a_1$ is a cycle. Therefore $g(\acute{\Gamma}_R(M)) = 3$. □

For a graph G, let $\chi(G)$ denote the *chromatic number of the graph* G, i.e., the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. A *clique* of a graph is its complete subgraph and the number of vertices in the largest clique of G, denoted by $clique(G)$, is called the clique number of G.

Theorem 7. (a) *Let R not be a field. Then if $Max(M)$ has an infinite number of maximal ideals, then $clique(\acute{\Gamma}_R(M))$ is also infinite; otherwise $clique(\acute{\Gamma}_R(M)) \geq |Max(M)|$.*

(b) *If $\chi(\acute{\Gamma}_R(M)) < \infty$, then $|Max(M)| < \infty$.*

Proof. Use the technique of [2, 2.14]. □

A graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends.

Theorem 8. *Assume that $|Max(M)| \geq 5$. Then $\acute{\Gamma}_R(M)$ is not planar.*

Proof. Assume that $|Max(M)| \geq 5$. Choose $a_i \in m_i \setminus \cup_{j=1}^5 m_j$, where $m_i \in Max(M)$, $1 \leq i \leq 5$, and $j \neq i$. Then $a_i \notin (a_j M :_R M)$. Otherwise, $a_i \in (a_j M :_R M) \subseteq (m_j M :_R M) = m_j$ by Lemma 1. Similarly, $a_j \notin (a_i M :_R M)$. Hence m_1, m_2, m_3, m_4, m_5 forms a complete subgraph of $\acute{\Gamma}_R(M)$ which is isomorphic to K_5. Thus by [9, p.153], $\acute{\Gamma}_R(M)$ is not planar. $\square$

For any vertex x of a connected graph G, the *eccentricity* of x, denoted by $e(x)$, is the maximum of the distances from x to the other vertices of G, and the minimum value of the eccentricity is the *radius* of G, which is denoted by $r(G)$.

Theorem 9. *Let M be a non-local module with $J(M) = 0$. Then $r(\acute{\Gamma}_R(M)) = 2$ if and only if for each $t \in W_R^*(M)$, there exists $s \in W_R^*(M)$ such that t is not adjacent to s.*

Proof. The proof is similar to that of [2, 3.14]. $\square$

Theorem 10. *Let R be a Noetherian ring. If $\acute{\Gamma}_R(M)$ is totally disconnected, then M is a local module with maximal ideal of the from $(xM :_R M)$ for some $x \in W_R^*(M)$.*

Proof. It is easy to see that M is a local module. Set $Max(M) = m$. Assume to contrary that m is not the form of $(rM :_R M)$ for every $r \in W_R^*(M)$. Set $A = \{(rM :_R M), r \in W_R^*(M)\}$. Then A has a maximal member, say $(\acute{r}M :_R M)$ for some $\acute{r} \in W_R^*(M)$. Choose $s \in m \setminus (\acute{r}M :_R M)$. We claim that $\acute{r} \notin (sM :_R M)$. Otherwise, we have $(\acute{r}M :_R M) \subseteq (sM :_R M)$, so $(\acute{r}M :_R M) = (sM :_R M)$ by maximality. Hence $s \in (\acute{r}M :_R M)$ so that $\acute{r}$ is adjacent to s, a contradiction. $\square$

Theorem 11. *Assume that M is a non-local module. Then the following conditions are equivalent.*

(a) $\acute{\Gamma}_R(M) \setminus J(M)$ *is complete bipartite.*
(b) $\acute{\Gamma}_R(M) \setminus J(M)$ *is bipartite.*
(c) $\acute{\Gamma}_R(M) \setminus J(M)$ *contains no triangles.*

Proof. Use the technique of [3, 2.13]. $\square$

Proposition 2. *If the graph $\acute{\Gamma}_R(M) \setminus J(M)$ is n-partite for some positive integer n, then $|Max(M)| \leq n$.*

Proof. Assume to the contrary that $|Max(M)| > n$. Since $\acute{\Gamma}_R(M) \setminus J(M)$ is an n-partite graph, there are maximal ideals P_1 and P_2 of $Max_R(M)$ with $(rM :_R M) \subseteq P_1 \setminus P_2$ and $(sM :_R M) \subseteq P_2 \setminus P_1$, where r, s belong to the same part. But this implies that r is adjacent to s which is a contradiction. $\square$

Theorem 12. *Let M be an R-module with $Max(M) = \{m_1, m_2\}$. Then $\acute{\Gamma}_R(M) \setminus J(M)$ is a complete bipartite graph with parts $m_i \setminus J(M)$, $i = 1,2$, if and only if every pair of ideals $(rM :_R M)$, $(sM :_R M)$ contained in $(m_1 \setminus J(M))$ or $(m_2 \setminus J(M))$, where $r,s \in R$, are totally ordered.*

Proof. Suppose that $\acute{\Gamma}_R(M) \setminus J(M)$ is a complete bipartite graph with parts $m_i \setminus J(M)$, $i = 1,2$. Further assume to the contrary that there exist ideals $(rM :_R M)$, $(sM :_R M) \subseteq m_1 \setminus J(M)$ such that $(rM :_R M) \nsubseteq (sM :_R M)$ and $(sM :_R M) \nsubseteq (rM :_R M)$. We claim that r is adjacent to s in $m_1 \setminus J(M)$. Otherwise, without loss of generality, we assume that $r \in (sM :_R M)$. Then $r,s \in m_1 \setminus J(M)$ and we have $rM \subseteq (sM :_R M)M$. Thus $(rM :_R M) \subseteq ((sM :_R M)M :_R M) = (sM :_R M)$, a contradiction. Hence r is adjacent to s in $m_1 \setminus J(M)$, which is again a contradiction by hypothesis. Conversely, assume that $i \in \{1,2\}$ and $(rM :_R M), (sM :_R M) \subseteq m_i \setminus J(M)$. We may assume that $(rM :_R M) \subseteq (sM :_R M)$. Then clearly, $r,s \in m_i \setminus J(M)$ and r is not adjacent. Now if $r \in m_1 \setminus m_2$ and $s \in m_2 \setminus m_1$, then by using 1, we see that r is adjacent to s. Therefore $\acute{\Gamma}_R(M) \setminus J(M)$ is a complete bipartite graph with parts $m_i \setminus J(M)$, $i = 1,2$. □

Theorem 13. *Let M be a faithful R-module and $Z_R(M) \neq W_R(M)$. Then $\acute{\Gamma}_R(M)$ is finite if and only if R is finite.*

Proof. Clearly if R is finite, then $\acute{\Gamma}_R(M)$ is finite. So we assume that $\acute{\Gamma}_R(M)$ is finite and show that R is finite. Suppose that R is infinite and look for a contradiction. By Remark 1, we have $Z_R(M) \subset W_R(M) = N_R(M)$. Choose $x \in W_R(M) \setminus Z_R(M)$. Since Rx is a finite R-module and $R \setminus W_R(M)$ is an infinite set, there exist distinct elements $r_1, r_2 \in R \setminus W_R(M)$ such that $r_1 x = r_2 x$. Therefore $(r_1 - r_2)x = 0$. Then we have $x((r_1 - r_2)M) = 0$. Since x is a nonzero-divisor on M, we have $(r_1 - r_2)M = 0$ so that $r_1 - r_2 \in Ann_R(M)$. Thus $r_1 = r_2$, a contradiction. □

Corollary 2. *Let R be a domain and let $Z_R(M) = \{0\}$. If $\acute{\Gamma}_R(M)$ is a finite graph, then R is a field.*

Proof. If $W_R(M) \neq \{0\}$, then by Theorem 13, R is finite so that R is a field. Otherwise, if $W_R(M) = \{0\}$, then we have $W_R(M) = \cup_{p \in Max(M)} P = \{0\}$ by Remark 1. This implies that the zero ideal of R is a maximal ideal and hence R is a field. □

Remark 4. One can see, by using the same technique, that the results about $\acute{\Gamma}_R(M)$ in this section is also true for $\tilde{\Gamma}_R(M)$.

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Authors' addresses

H. Ansari-Toroghy
Department of pure Mathematics, Faculty of mathematical Sciences, University of Guilan, P. O. Box 41335-19141, Rasht, Iran.
E-mail address: ansari@guilan.ac.ir

F. Farshadifar
University of Farhangian, Tehran, Iran.
E-mail address: f.farshadifar@gmail.com

Sh.Habibi
Department of pure Mathematics, Faculty of mathematical Sciences, University of Guilan, P. O. Box 41335-19141, Rasht, Iran.
E-mail address: sh.habibi@guilan.ac.ir