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ON GREEN'S EQUIVALENCES IN Γ-GROUPOIDS

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Abstract. This paper deals with Γ-groupoids which are generalizations of groupoids and Γ-semigroups. The main purpose of this paper is to extend Green's equivalences and Green's Lemma to suitably restricted Γ-groupoids. We study only Γ-groupoids satisfying some additional conditions and we show that these are sufficient for the statement of Green's equivalences in case of Γ-groupoids. Additional condition sufficient to prove Green's Lemma for Γ-groupoids is provided and some illustrative examples are presented.

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1. Introduction and preliminaries

In 1964, N. Nobusawa [16] introduced the notion of a Γ-ring, more general than a ring. In 1966, W. E. Barnes [1] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa. Many fundamental results in ring theory have been extended to Γ-rings by different authors obtaining various generalization analogous to corresponding parts in ring theory. In 1981, M. K. Sen [19] and later in 1986, Sen and Saha [20] introduced the concept of the Γ-semigroup as a generalization of semigroup and ternary semigroup. Many classical notions and results of the theory of semigroups have been extended and generalized to Γ-semigroups by a lot of mathematicians. Green's relations for semigroups were introduced by J. A. Green in a paper of 1951 [7]. Green's relations for Γ-semigroups defined in [6, 18], play an important role in studying of the structure of Γ-semigroups as well as in case of the plain semigroups and become a familiar tool among Γ-semigroups. Several treatments and contributions concerning Green's relations for Γ-semigroups have been made by a lot of mathematicians, for instant [2–4, 6, 8–10, 12, 13, 15, 17, 18, 20]. Recently, in [11] we have introduced and studied the hyperversion of Green's relations in Γ-semihypergroups. The Green's equivalence relations, Green's Lemma and its corollaries are important tools in the theory of Γ-semigroups as well as in the case of the plain semigroups. The proof of those fundamental results depends on little more than the associativity of the Γ-operation defined in Γ-semigroups. However, when we remove this property, we find ourselves faced with the problem of obtaining similar results in Γ-groupoids. For this, we will study only Γ-groupoids satisfying some additional conditions and we show that these are sufficient for the statement of Green's equivalences in case of Γ-groupoids. Additional condition sufficient to prove Green's Lemma for Γ-groupoids is provided and some illustrative examples are presented. The main purpose of this paper is to extend Green's equivalences and Green's Lemma to suitably restricted Γ-groupoids and to obtain some results which are parallel to those obtained for groupoids and semigroups [5, 14].

We introduce below necessary notions and present a few auxiliary results that will be used throughout the paper.

Definition 1. Let M and Γ be two non-empty sets. Any map from $M \times \Gamma \times M \to M$ will be called a *Γ-multiplication* in M and denoted by $(\cdot)_\Gamma$. The result of this multiplication for $a, b \in M$ and $\alpha \in \Gamma$ is denoted by $a\alpha b$. A *Γ-groupoid* M is an ordered pair $(M, (\cdot)_\Gamma)$ where M and Γ are non-empty sets and $(\cdot)_\Gamma$ is a Γ-multiplication on M.

M is called a Γ-semigroup, if in addition, the following assertion is satisfied:

$$\forall (a, b, c, \alpha, \beta) \in M^3 \times \Gamma^2, (a\alpha b)\beta c = a\alpha(b\beta c).$$

Example 1. Let M be a semigroup and Γ be any nonempty set. If we define $a\gamma b = ab$ for all $a, b \in M$ and $\gamma \in \Gamma$. Then M is a Γ-semigroup.

Example 2. Let M be a set of all negative rational numbers. Obviously M is not a semigroup under usual product of rational numbers. Let $\Gamma = \{-\frac{1}{p} : p \text{ is prime}\}$. Let $a, b, c \in M$ and $\alpha \in \Gamma$. Now if $a\alpha b$ is equal to the usual product of rational numbers a, α, b, then $a\alpha b \in M$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$. Hence M is a Γ-semigroup.

Example 3. Let $M = \{-i, 0, i\}$ and $\Gamma = M$. Then M is a Γ-semigroup under the multiplication over complex numbers while M is not a semigroup under complex number multiplication.

Notice that every semigroup is a Γ-semigroup and Γ-semigroups are a generalization of semigroups. The same holds for Γ-groupoids.

A Γ-groupoid M is said to be *commutative* if for all $a, b \in M, \gamma \in \Gamma, a\gamma b = b\gamma a$.

2. On Green's relations in Γ-groupoids

Let M be a Γ-groupoid. If $\mathcal{E}$ is any binary relation on the set M and $a, b \in M$, then let $a\mathcal{E}b$ mean that a is $\mathcal{E}$-related to b and, whenever $\mathcal{E}$ is an equivalence relation on M, let the $\mathcal{E}$-equivalence class containing a be denoted by E_a, i.e., $E_a = \{x \in M | x\mathcal{E}a\}$. We define now two relations, the so-called Green's relations on a Γ-groupoid M.

Definition 2. Let M be a Γ-groupoid and $a, b \in M$. We define $a\mathcal{R}b$ if and only if either $a = b$ or there exist $x, y \in M$ and $\alpha, \beta \in \Gamma$, such that $a\alpha x = b$ and $b\beta y = a$. Dually, we define $c\mathcal{L}d$ if and only if either $c = d$ or there exist $u, v \in M$ and $\gamma, \delta \in \Gamma$ such that $u\gamma c = d$ and $v\delta d = c$.

When these are equivalence relations we will write R_a for the $\mathcal{R}$-class of a, and L_c for the $\mathcal{L}$-class of c.

When M is associative, that is, a Γ-semigroup, then it is known that the relations $\mathcal{R}$ and $\mathcal{L}$ are equivalence relations [6]. In this case we have $a\mathcal{R}b$ iff $a\Gamma M \cup \{a\} = b\Gamma M \cup \{b\}$.

For arbitrary Γ-groupoids, these two subsets of M need not have any particular relationship even though $a\mathcal{R}b$. For this, we give the following definition.

Definition 3. Let M be a Γ-groupoid. M is said to be *left consistent* if $H\gamma(x\alpha y) = (H\gamma x)\alpha y$ for any $x, y \in M$, $\gamma, \alpha \in \Gamma$ and any Γ-subgroupoid H of M. M is said to be *weakly left consistent* if the above holds just for $H = M$.

Definition 4. Let M be a Γ-groupoid. M is said to be *right consistent* if $(x\alpha y)\gamma H = x\alpha(y\gamma H)$ for any $x, y \in M$, $\alpha, \gamma \in \Gamma$ and any Γ-subgroupoid H of M. M is said to be *weakly right consistent* if the above holds just for $H = M$.

Definition 5. Let M be a Γ-groupoid. M is said to be [*weakly*]*consistent* if it is both [weakly] left and [weakly] right consistent.

Definition 6. Let M be a Γ-groupoid. M is said to be *intra-consistent* if $(x\gamma H)\alpha y = x\gamma(H\alpha y)$ for any $x, y \in M, \gamma, \alpha \in \Gamma$ and any Γ-subgroupoid H of M. M is said to be *weakly intra-consistent* if the above holds just for $H = M$.

Proposition 1. *Let M be a weakly right consistent or a weakly intra-consistent Γ-groupoid. Then $a\mathcal{R}b$ if and only if $a\Gamma M \cup \{a\} = b\Gamma M \cup \{b\}$ for $a, b \in M$.*

Proof. " $\Rightarrow$ ". Let M be weakly right consistent and assume $a\mathcal{R}b$. If $a = b$ the result is evident. Otherwise there exist $x, y \in M$ and $\alpha, \beta \in \Gamma$ such that $a\alpha x = b$ and $b\beta y = a$. Let $\gamma \in \Gamma$ and so $a\gamma M \subseteq a\Gamma M$. Then we have: $a\gamma M = (b\beta y)\gamma M = b\beta(y\gamma M) \subseteq b\beta M \subseteq b\Gamma M$, that is, $a\Gamma M \subseteq b\Gamma M$. On the other side, we have: $b\gamma M = (a\alpha x)\gamma M = a\alpha(x\gamma M) \subseteq a\alpha M \subseteq a\Gamma M$, that is $b\Gamma M \subseteq a\Gamma M$. Hence $a\Gamma M = b\Gamma M$. Since $a \in b\Gamma M$ and $b \in a\Gamma M$, the requested result follows. If M is weakly intra-consistent and $a\mathcal{R}b$ and $a \neq b$ we can show in a similar way that $a \in a\Gamma M = b\Gamma M$, and $b \in b\Gamma M$, and the result follows immediately.

" $\Leftarrow$ ". The converse is trivial. □

An immediate corollary of the above proposition is the following.

Corollary 1. *If M is either a weakly consistent or a weakly intra-consistent Γ-groupoids, $\mathcal{R}$ and $\mathcal{L}$ are equivalence relations. Indeed if M is weakly consistent, then $\mathcal{R}$ is a left congruence and $\mathcal{L}$ is a right congruence.*

Problem 1. *In general, $\mathcal{R}$ need not be a left congruence on a weakly intra-consistent Γ-groupoid. Can one finds an example of a non-trivial weakly intra-consistent Γ-groupoid in which $\mathcal{R}$ is not a left congruence?*

Let we consider now the case M is a commutative Γ-groupoid.

Proposition 2. *Let M be a commutative Γ-groupoid. Then M is [weakly] left consistent if and only if M is [weakly] right consistent and therefore [weakly] consistent.*

Proof. Let $a, b \in M, \alpha, \beta \in \Gamma$ and H be any Γ-subgroupoid of M. By the commutativity of M we have $(a\alpha b)\beta H = H\beta(b\alpha a)$ and $a\alpha(b\beta H) = (b\beta H)\alpha a = = (H\beta b)\alpha a$. The two equalities are linked if M is either [weakly] left or [weakly] right consistent and hence the conditions [with $H = M$] are equivalent. □

Remark 1. For the commutative groupoids, if they are [weakly] right consistent and therefore [weakly] consistent, in either case they are [weakly] intra-consistent. When we pass to Γ-groupoids, this property does not hold. In fact, if M is [weakly] right (or left) consistent Γ-groupoid, by the commutativity of M, we have: $(a\alpha H)\beta b = b\beta(a\alpha H) = (b\beta a)\alpha H = (a\beta b)\alpha H = a\beta(b\alpha H) = a\beta(H\alpha b)$, for any $a, b \in M, \alpha, \beta\Gamma$ and H is any Γ-subgroupoid of M [with $H = M$], which shows that M is not [weakly] intra-consistent in general.

Example 4. Let $M = \{x, y, z, t\}$ and $\Gamma = \{\alpha, \beta\}$ with the Γ-multiplication defined by

α	x	y	z	t
x	x	x	y	y
y	y	y	x	x
z	z	z	t	t
t	t	t	z	z

α	x	y	z	t
x	x	x	y	y
y	y	y	x	x
z	t	t	z	z
t	z	z	t	t

It can be easily verified that M is weakly left consistent and weakly intra-consistent but not weakly right consistent.

Problem 2. *Can one find an example of a non-trivial Γ-groupoid which is both weakly left and right consistent but not weakly intra-consistent or to prove that a weakly consistent Γ-groupoid is weakly intra-consistent?*

In order to prove Green's Lemma for Γ-groupoids we will need the following result, whose proof is straightforward.

Lemma 1. *Let M be a Γ-groupoid. If M is weakly right consistent, then for all $a \in M$ and $\rho \in \Gamma$, $a\rho M$ is a Γ-subgroupoid. Also, $a\Gamma M$ is a Γ-subgroupoid.*

Corollary 2. *Let M be a Γ-groupoid. If M is weakly right consistent, then for all $a \in M, \rho \in \Gamma$, $a\rho M \cup \{a\}$ is a Γ-subgroupoid. Also, $a\Gamma M \cup \{a\}$ is a Γ-subgroupoid.*

Let M be a Γ-groupoid. The mapping $\lambda_\alpha^s : M \to M$ defined by $\lambda_\alpha^s a = s\alpha a$ for all $s, a \in M$ and $\alpha \in \Gamma$ is called *left translation* of Γ-semigroup M. The mapping $\rho_{s'}^\beta : M \to M$ defined by $b\rho_{s'}^\beta = b\beta s'$ for all $s', b \in M$ and $\beta \in \Gamma$ is called *right translation* of Γ-semigroup M.

Theorem 1. *Let M be a consistent Γ-groupoid and suppose $c\mathcal{R}b$ for some $c \neq b$. Then there are $s, s' \in M, \alpha, \beta \in \Gamma$ such that $c\alpha s = b, b\beta s' = c$ and the right translations $\rho_s^\alpha, \rho_{s'}^\beta$, are, respectively, mappings from L_c into L_b and L_b into L_c, which are $\mathcal{R}$-class preserving, that is, for $x \in L_c, x\mathcal{R}x\rho_s^\alpha$ and for $y \in L_b, y\mathcal{R}y\rho_{s'}^\beta$.*

Proof. Let $c\mathcal{R}b$ for some $c \neq b$. Since $c \neq b$, the existence of s, s' follows from the Definition 2. Now let $a\mathcal{L}c$ and $d\mathcal{L}b$. By Corollary 1, $\mathcal{L}$ is a right congruence, thus we have $a\alpha s\mathcal{L}c\alpha s = b$ and $d\beta s'\mathcal{L}b\beta s' = c$. Thus we have $L_c\rho_s^\alpha \subseteq L_b$ and $L_b\rho_{s'}^\beta \subseteq L_c$.

Now, if $a \neq c$, then for any $\rho \in \Gamma$, $a\Gamma M \supseteq a\alpha(s\rho M) = (t\gamma c)\alpha(s\rho M)$ where $t\gamma c = a$ for some $t \in M, \gamma \in \Gamma$ by the Definition 2. Since $s\rho M$ is a Γ-subgroupoid by Lemma 1, we have $(t\gamma c)\alpha(s\rho M) = t\gamma[c\alpha(s\rho M)] = t\gamma[(c\alpha s)\rho M] = t\gamma(b\rho M)$. Thus $a = t\gamma c = t\gamma(b\beta s') \in t\gamma(b\Gamma M) \subseteq a\Gamma M$. Continuing we have $a\Gamma M \supseteq (a\alpha s)\rho M = t\gamma(b\rho M) \supseteq t\gamma[b\beta(s'\rho M)] = t\gamma[(b\beta s')\rho M] = t\gamma(c\rho M) = (t\gamma c)\rho M = a\rho M$, whence $a\Gamma M = (a\alpha s)\Gamma M$. But $a \in a\Gamma M$ and $a\alpha s \in a\Gamma M = (a\alpha s)\Gamma M$ so that we can conclude by Proposition 1 that $a\mathcal{R}a\alpha s$. If $a = c$, the preceding argument can be simplified to show that $c\mathcal{R}c\alpha s$. In a similar way, it can be shown $d\mathcal{R}d\beta s'$. □

Corollary 3. *If M is a consistent Γ-groupoid, then $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$ on M.*

Proof. If $a\mathcal{L}c\mathcal{R}b$, then the above Theorem yields an $s \in M$ such that $a\mathcal{R}a\gamma s\mathcal{L}b$ for some $\gamma \in \Gamma$, and so $\mathcal{L} \circ \mathcal{R} \subseteq \mathcal{R} \circ \mathcal{L}$. The reverse inclusion is proven dually. □

Now we give the following definition.

Definition 7. A Γ-groupoid M is said to be $\mathcal{D}$-Γ-groupoid if $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ on M.

In such cases we define $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$ and $\mathcal{D}$ is then clearly an equivalence relation. Note that the consistent Γ-groupoids are $\mathcal{D} - \Gamma$-groupoid, while the converse is not necessarily true. The following example shows this.

Example 5. Let $M = \{x, y\}$ and $\Gamma = \{\alpha, \beta\}$ with the Γ-multiplication defined by

α	x	y
x	y	x
y	y	x

β	x	y
x	x	x
y	x	x

Here we have that $\mathcal{L} = \mathcal{R} = \omega$, the universal relation on M, so that M is certainly $\mathcal{D} - \Gamma$-groupoid, while $M\alpha(x\beta x) = y \neq x = (M\alpha y)\beta y$, that is, M is not a consistent Γ-groupoid.

3. Green's Lemma for Γ-groupoids

Theorem 1 shows us that for a consistent Γ-groupoid M certain right translations define maps between two $\mathcal{L}$-classes which are $\mathcal{R}$-class preserving. In general, it is not known if ρ_s^α and $\rho_{s'}^\beta$ are mutually inverse maps between L_c and L_b, as is the case of Γ-semigroups [6].

In the following results are provided additional conditions which suffice to guarantee this result.

Definition 8. Let M be a $\mathcal{D}$-Γ-groupoid. A $\mathcal{D}$-class, D of M is said to be *regular* if there is an γ-idempotent element ($x\gamma x = x$) in each $\mathcal{L}$ and $\mathcal{R}$-class of D.

Lemma 2. *Let M be a consistent Γ-groupoid. If $e\gamma e = e$ for some $\gamma \in \Gamma$, then $x\gamma e = x$ for all $x \in \mathcal{L}$-class, L_e, and $e\gamma y = y$ for all $y \in \mathcal{R}$-class R_e.*

Proof. Let $x \in L_e$. Then $x = t\alpha e$ for some $t \in M$ and $\alpha \in \Gamma$. Now $x\gamma e = (t\alpha e)\gamma e = t\alpha(e\gamma e) = t\alpha e = x$ since $\{e\}$ is a Γ-subgroupoid of M. The other result is dual. □

Proposition 3. *Let D be a regular $\mathcal{D}$-class of a consistent Γ-groupoid M. Then for any $a \in D$, there exist $t, t' \in M$ and $\gamma, \gamma', \alpha, \beta \in \Gamma$ such that $a = a\gamma(t\alpha a) = (a\beta t')\gamma' a$.*

Proof. Let $a \in D$. Since D is regular, there is an γ-idempotent $e \in L_a = L_e$. By Lemma 2, $a\gamma e = a$. Since $e\mathcal{L}a$, there is a $t \in M, \alpha \in \Gamma$ such that $t\alpha a = e$. Then $a = a\gamma e = a\gamma(t\alpha a)$. Dually one obtains $a = (a\beta t')\gamma' a$. □

Remark 2. A converse of Proposition 3 is false: a Γ-groupoid M may be consistent, and for every $a \in M$ may have $t, t' \in M$ and $\gamma, \gamma', \alpha, \beta \in \Gamma$ such that $a = a\gamma(t\alpha a) = (a\beta t')\gamma' a$, and yet M may have no idempotents. We have the following example.

Example 6. Let $M = \{x, y\}$ and $\Gamma = \{\alpha, \beta\}$ with the Γ-multiplication defined by

α	x	y
x	y	x
y	y	x

β	x	y
x	y	y
y	x	x

In the Γ-groupoid M there no idempotents and further, for example, $x = x\alpha(x\beta x) = (x\alpha x)\beta x$. Moreover, here we have $\mathcal{L} = \mathcal{R} = \omega$.

Definition 9. Let M be a Γ-groupoid. M is said to be *almost associative* if whenever H is a Γ-subgroupoid of M and $a, b, c \in M, \alpha, \beta, \gamma \in \Gamma$, we have $H\gamma[(a\alpha b)\beta c] = H\gamma[a\alpha(b\beta c)]$ and $[a\alpha(b\beta c)]\gamma H = [(a\alpha b)\beta c]\gamma H$.

Theorem 2. *A regular, consistent, almost associative Γ-groupoid is associative.*

Proof. Let $a,b,c \in M$. Then $(a\alpha b)\beta c = (a\alpha b)\beta(c\gamma e)$ for some $e\gamma e = e \in L_c, \gamma \in \Gamma$. But $(a\alpha b)\beta(c\gamma e) = [(a\alpha b)\beta c]\gamma e = [a\alpha(b\beta c)]\gamma e = a\alpha[(b\beta c)\gamma e] = a\alpha[b\beta(c\gamma e)] = a\alpha(b\beta c)$ since M is consistent and almost associative and $\{e\}$ is a Γ-subgroupoid. We thus have that $(a\alpha b)\beta c = a\alpha(b\beta c)$ for any $a,b,c \in M$ and $\alpha,\beta \in \Gamma$, i.e., M is associative. □

Now, based on the above results, we are ready to state and to prove the so-called Green's Lemma for Γ-groupoids.

Corollary 4. *(Green's Lemma). Let M be a consistent almost associative Γ-groupoid. If D is a regular $\mathcal{D}$-class of M and $c\mathcal{R}b$ for $c,b \in D$, then there exist $s,s' \in M, \alpha,\beta \in \Gamma$ such that $c\alpha s = b, b\beta s' = c$ and the right translations $\rho_s^\alpha, \rho_{s'}^\beta$, are mutually inverse bijections between L_c and L_b and are $\mathcal{R}$-class preserving.*

Proof. By Theorem 1 we need only to show that ρ_s^α and $\rho_{s'}^\beta$ are mutually inverse bijections. Let $f\gamma f = f \in R_c$ for some $\gamma \in \Gamma$, and $f\gamma u = c$ for some $u \in M$. Then $c = b\beta s' = (c\alpha s)\beta s' = ((f\gamma u)\alpha s)\beta s' = (f\gamma(u\alpha s))\beta s' = f\gamma((u\alpha s)\beta s') = f\gamma(u\alpha(s\beta s')) = (f\gamma u)\alpha(s\beta s') = c\alpha(s\beta s')$.

Now let $a \in L_c$ and $e\delta e = e \in L_c$ for some $\delta \in \Gamma$. There exist $t \in M$ and $\mu \in \Gamma$ such that $t\mu c = a$. Then $(a)\rho_s^\alpha\rho_{s'}^\beta = (a\alpha s)\beta s'$. Since $(a\alpha s)\beta s' \in L_a \cap R_a$ by Theorem 1, $(a\alpha s)\beta s' = [(a\alpha s)\beta s']\delta e = [a\alpha(s\beta s')]\delta e = [(t\mu c)\alpha(s\beta s')]\delta e = [t\mu(c\alpha(s\beta s'))]\delta e = [t\mu c]\delta e = a\delta e = a$ since $c = c\alpha(s\beta s')$ from above. Thus $\rho_s^\alpha\rho_{s'}^\beta$ is the identity map on L_c. Similarly $\rho_{s'}^\beta\rho_s^\alpha$ is the identity map on L_b. The result now follows. □

It is clear that, as it is shown in this paper, a consistent, almost associative Γ-groupoid with every $(\mathcal{R}\circ\mathcal{L})$-equivalence class regular is necessarily a Γ-semigroup and so the Green's Lemma takes its familiar form in Γ-semigroups [3,6].

From all the above, the following problems arise:

Problem 3. *What we can say about the validation of Green's Lemma for infinite Γ-groupoids?*

Problem 4. *How necessary is the regularity in the Corollary 4?*

Problem 5. *Is there any natural condition for Γ-groupoids weaker than that of almost associativity which can replace it leaving Corollary 4 true but Theorem 2 false?*

Problem 6. *Is there a regular consistent Γ-groupoid in which the right translations of Theorem 1 are not mutually inverse?*

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