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Characterization of $V(\mathbb{Z}C_n^+)$ of rank ≤ 4

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CHARACTERIZATION OF $V(\mathbb{Z}\mathbf{C}_n^+)$ OF RANK ≤ 4

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Abstract. There are only a few cases for cyclic groups $\mathbf{C}_n$, where the unit group of their integral group rings are determined. In this article, we have completed the characterization of torsion-free part $V(\mathbb{Z}\mathbf{C}_n^+)$ of the integral group ring of cyclic group $\mathbf{C}_n$, for $n = 11, 15, 16, 20, 24$ and 30 where $\rho \leq 4$. We explicitly find all the generators.

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Keywords: group ring, unit group, normalizer, generator

1. INTRODUCTION

Let G be a finite group and $\mathcal{U}(\mathbb{Z}G)$ be the unit group of integral group ring $\mathbb{Z}G$. We denote the group of normalized units of $\mathbb{Z}G$ by $\mathcal{U} = V(\mathbb{Z}G)$ and the subgroup of the central units of $\mathcal{U}$ by $\mathcal{Z}(\mathcal{U})$. By $\mathcal{N}_{\mathcal{U}}(G)$ we mean the normalizer of G in $\mathcal{U}$. The structure of the unit group $\mathcal{U}(\mathbb{Z}G)$ has been of a fundamental interest after the G. Higman's thesis written in 1939. Problem 43 in [17] asks whether the normalizer property; $\mathcal{N}_{\mathcal{U}}(G) = G\mathcal{Z}(\mathcal{U})$, holds for any finite group G.

In [5] it is shown by Coleman in particular that, this property holds for any finite nilpotent group. Jackowski and Marciniak [8] extended this to finite groups of odd order and the groups having normal Sylow 2-subgroup. Later in [14] Li et al. have shown that if the intersection of non-normal subgroups of G is non-trivial then, G satisfies the normalizer property. Meanwhile, the normalizer property is verified by Li [13] for some metabelian groups and by Marciniak and Roggenkamp [15] for finite metabelian groups with an abelian Sylow 2-subgroup. We refer the reader to [2] for a general information about the theory of group rings and to [6,7,9,11,16] for a survey on results on the unit group of integral group rings.

An explicit characterization of the normalizer of $D_n = \langle a, b : a^n = b^2 = 1, b^{-1}ab = a^{-1}\rangle$, dihedral group of order $2n$, in the normalized units of $\mathbb{Z}D_n$ has recently been given by Bilgin [4]. In the present paper, we give the characterization of torsion-free part of the integral group ring of some cyclic groups. Before we give the main theorems we obtained, we introduce some notations and give some basic facts.

Let $\mathbf{C}_n =< a : a^n = 1 >$ be a cyclic group of order n, $\mathbb{Z}\mathbf{C}_n$ be its integral group ring and $V(\mathbb{Z}\mathbf{C}_n)$ be its normalized units. Now, let us consider the following subring of symmetric elements of $\mathbb{Z}\mathbf{C}_n$

$$\mathbb{Z}\mathbf{C}_n^+ = \left\{ \gamma = \sum_{i=0}^{n-1} \gamma_i a^i : \gamma_i = \gamma_{n-i} \right\}.$$

If we denote the unit group of $\mathbb{Z}\mathbf{C}_n^+$ by $\mathcal{U}(\mathbb{Z}\mathbf{C}_n^+)$ and its normalized units by $V(\mathbb{Z}\mathbf{C}_n^+)$ then we can state Higman's following result :

Theorem 1. $\mathcal{N}_{\mathcal{U}}(D_n) = D_n \times V(\mathbb{Z}\mathbf{C}_n^+)$.

When we modify Higman's result [17] for a cyclic group of order n instead of a finite abelian group, we obtain the following corollary:

Corollary 1. $V(\mathbb{Z}\mathbf{C}_n^+) = \mathcal{Z}(D_n) \times F$, *where F is a free abelian group of rank*

$$\rho = \frac{1}{2}\varphi(n) - 1.$$

Corollary 1 reduces the characterization of $V(\mathbb{Z}\mathbf{C}_n)$ to the construction of generator(s) of F. So, $V(\mathbb{Z}\mathbf{C}_n)$ is trivial when $n = 1, 2, 3, 4$ or 6. Torsion free part of $V(\mathbb{Z}\mathbf{C}_n)$ has a unique generator if $n = 5, 8, 10$ or 12, and it has been characterized for $n = 5, 8$ and 12 in [3, 10] as follows

$$V(\mathbb{Z}\mathbf{C}_5) = \mathbf{C}_5 \times \left\langle -1 + a + a^4 \right\rangle, \tag{1.1}$$

$$V(\mathbb{Z}\mathbf{C}_8) = \mathbf{C}_8 \times \left\langle -1 - (a + a^{-1}) + (a^3 + a^{-3}) + 2a^4 \right\rangle, \tag{1.2}$$

$$\begin{aligned} V(\mathbb{Z}\mathbf{C}_{12}) = \mathbf{C}_{12} & \\ \times & \left\langle 3 + 2(a + a^{-1}) + (a^2 + a^{-2}) - (a^4 + a^{-4}) - 2(a^5 + a^{-5}) - 2a^6 \right\rangle. \end{aligned} \tag{1.3}$$

$\rho = 2$ when $n = 7, 9, 14$ or 18. For $n = 7$, F was characterized by Karpilovsky [10] and for the cases $n = 7$ and 9, it was characterized by Aleev [1]. A different characterization for $n = 7$ and 9 was also given by Kokluce and Kelebek [12] as

$$V(\mathbb{Z}\mathbf{C}_7) = \mathbf{C}_7 \times \left\langle -1 + (a + a^{-1}) \right\rangle \times \left\langle -1 + 2(a + a^{-1}) - (a^2 + a^{-2}) \right\rangle, \tag{1.4}$$

$$\begin{aligned} V(\mathbb{Z}\mathbf{C}_9) = \mathbf{C}_9 \times & \left\langle -1 + (a + a^{-1}) - (a^4 + a^{-4}) \right\rangle \\ & \times \left\langle -1 + (a + a^{-1}) + (a^2 + a^{-2}) + (a^3 + a^{-3}) - 2(a^4 + a^{-4}) \right\rangle. \end{aligned} \tag{1.5}$$

There is a strong relationship between F and $V(\mathbb{Z}\mathbf{C}_n^+)$. It is clear that $V(\mathbb{Z}\mathbf{C}_n^+)$ and F have the same rank. In some cases they are the same. Thus the characterization of $V(\mathbb{Z}\mathbf{C}_n^+)$ encourages one to make the characterization of $V(\mathbb{Z}\mathbf{C}_n)$.

In this study, we have completed the characterization of normalizers of dihedral groups where the rank $\rho \leq 4$. The following table lists the orders of dihedral groups corresponding to rank $\rho \leq 4$.

TABLE 1.

rank(ρ)	order(n)
0	1,2,3,4,6
1	5,8,10,12
2	7,9,14,18
3	15,16,20,24,30
4	11,22

Let $\omega = e^{\frac{2\pi i}{n}}$ and $\alpha = \omega + \omega^{-1}$. The minimal polynomial $\min_Q(\alpha, x)$ of α over Q can be obtained by a simple calculation. The image of $\mathbb{Z}\mathbf{C}_n^+$ under the ring homomorphism

$$\begin{array}{rccc} \psi: & \mathbb{Z}\mathbf{C}_n & \rightarrow & \mathbb{Z}[] \\ & \sum \gamma_i a^i & \mapsto & \sum \gamma_i \omega^i \end{array}$$

gives the ring of integers $\mathbb{Z}[]$. In determination of the normalized units $V(\mathbb{Z}\mathbf{C}_n^+)$ of $\mathbb{Z}\mathbf{C}_n^+$ we firstly need to find the fundamental units of $\mathbb{Z}[]$ whose minimal polynomial is $\min_Q(\alpha, x)$. The following Table 2 list the all fundamental units of $\mathbb{Z}[]$ for necessary cases.

TABLE 2.

ρ	n	$\min_Q(\alpha, x)$	Fundamental Units
3	15	$x^4 - x^3 - 4x^2 + 4x + 1$	$\varepsilon_1 = \alpha - 1$ $\varepsilon_2 = \alpha^2 - 3$ $\varepsilon_3 = \alpha^3 - 3\alpha$
3	16	$x^4 - 4x^2 + 2$	$\varepsilon_1 = \alpha - 1$ $\varepsilon_2 = \alpha^2 - 1$ $\varepsilon_3 = \alpha^2 + \alpha - 1$
3	20	$x^4 - 5x^2 + 5$	$\varepsilon_1 = \alpha^2 - 2$ $\varepsilon_2 = \alpha^2 + \alpha - 2$ $\varepsilon_3 = \alpha^3 - \alpha^2 - 3\alpha + 3$
4	11	$x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$	$\varepsilon_1 = \alpha$ $\varepsilon_2 = \alpha + 1$ $\varepsilon_3 = \alpha^2 - 2$ $\varepsilon_4 = \alpha^4 + \alpha^3 - 3\alpha^2 - 3\alpha$

2. Motivation for Contruction of $V(\mathbb{Z}\mathbf{C}_n^+)$.

Lemma 1. *Any $\gamma \in V(\mathbb{Z}\mathbf{C}_n^+)$ can be written as*

$$\gamma = \gamma_0 + \sum_{i=1}^{k} \gamma_i C_i.$$

where $C_i = a^i + a^{-i}$.

Proof. Let us denote $C_i = a^i + a^{-i}$ and $\gamma \in V(\mathbb{Z}\mathbf{C}_n^+)$. If $n = 2k+1$ then $\gamma = \gamma_0 + \sum_{i=1}^{k} \gamma_i C_i$. If $n = 2k$ then $\gamma = \gamma_0 + \gamma_k' a^k + \sum_{i=1}^{k-1} \gamma_i C_i$. By considering their augmentations, we have

$$\varepsilon(\gamma) = \begin{cases} \gamma_0 + 2\sum_{i=1}^{k} \gamma_i \quad , & n = 2k+1 \\ \gamma_0 + \gamma_k' + 2\sum_{i=1}^{k-1} \gamma_i, & n = 2k. \end{cases}$$

by modulo 2 ,we obtain

$$\varepsilon(\gamma) \equiv 1(mod 2) \Rightarrow \begin{cases} \gamma_0 \equiv 1(mod 2), & n = 2k+1 \\ \gamma_0 + \gamma_k' \equiv 1(mod 2), & n = 2k. \end{cases}$$

By choosing γ_0 as an odd integer in both cases we obtain $\gamma_k' = 2\gamma_k$, for some $\gamma_k \in \mathbb{Z}$ we have

$$\gamma_k' a^k = 2\gamma_k a^k = \gamma_k(a^k + a^{-k}) = \gamma_k C_k.$$

So, $\gamma \in V(\mathbb{Z}\mathbf{C}_n^+)$ can be written as

$$\gamma = \gamma_0 + \sum_{i=1}^{k} \gamma_i C_i$$

in both cases. □

Proposition 1. *Let H be a subgroup of a finite abelian group G. We can define a group epimorphism:*

$$\varphi: \quad G \to G/H$$
$$g \mapsto gH.$$

If we extend φ linearly over $\mathbb{Z}$, then we can get the natural ring homomorphism

$$\overline{\varphi}: \quad \mathbb{Z}G \to \mathbb{Z}(G/H)$$
$$\sum \gamma_g g \mapsto \sum \gamma_g (gH).$$

If $G/H \cong \mathbf{C}_2, \mathbf{C}_3, \mathbf{C}_4$ or $\mathbf{C}_6$, then for any torsion-free unit $\gamma \in V(\mathbb{Z}G)$, $\overline{\varphi}(\gamma) = H$.

Remark 1. If $\gamma \in V(\mathbb{Z}\mathbf{C}_n^+)$ and $n = 2k$ then $a^k\gamma \in V(\mathbb{Z}\mathbf{C}_n^+)$. The coefficient of identity of either γ or $a^k\gamma$ is odd. $\gamma \in V(\mathbb{Z}\mathbf{C}_n^+)$ can be chosen as a generator if the coefficient of its identity is odd.

Proposition 2. *Let n be an odd integer then $\mathbf{C}_{2n} =< a : a^{2n} = 1 >$ and $H =< a^2 >$. Then $V(\mathbb{Z}\mathbf{C}_n^+)$ and $V(\mathbb{Z}H^+)$ have the same rank.*

Proof. If we extend group epimorphism linearly over $\mathbb{Z}$

$$f: \begin{array}{l} \mathbf{C}_{2n} \to H \\ a^i \mapsto a^{2i}. \end{array}$$

then, we obtain the following ring epimorphism:

$$\overline{f}: \begin{array}{l} \mathbb{Z}\mathbf{C}_{2n} \to \mathbb{Z}H \\ \sum_{i=0}^{2n-1} \gamma_i a^i \mapsto \sum_{i=0}^{2n-1} \gamma_i a^{2i}. \end{array} \tag{2.1}$$

If we restrict $\overline{f}$ to multiplicative torsion-free group we have

$$\widetilde{f}: \quad V(\mathbb{Z}\mathbf{C}_n^+) \to V(\mathbb{Z}H^+).$$

Since

$$\begin{aligned} \rho_{2n} &= \frac{1}{2}\varphi(2n) - 1 \\ &= \frac{1}{2}\varphi(2)\varphi(n) - 1 \\ &= \frac{1}{2}\varphi(n) - 1 \\ &= \rho_n, \end{aligned}$$

$V(\mathbb{Z}\mathbf{C}_n^+)$ and $V(\mathbb{Z}H^+)$ have the same rank. □

Remark 2. Let be n be an odd integer. If $\gamma = \gamma_0 + \sum_{i=0}^{k} \gamma_i C_i$ is a generator of $V(\mathbb{Z}H^+) \subset \mathbb{Z}\mathbf{C}_{2n}$, then by Proposition 2 $\gamma = \gamma_0 + \sum_{i=0}^{k} \gamma_i a^{2i}$ is a generator of $V(\mathbb{Z}\mathbf{C}_n^+)$.

By using Remark 2 and considering (1.1), (1.4) and (1.5), the unit groups $V(\mathbb{Z}\mathbf{C}_{10})$, $V(\mathbb{Z}\mathbf{C}_{14})$ and $V(\mathbb{Z}\mathbf{C}_{18})$ can as follows be given as follows:

$$V(\mathbb{Z}\mathbf{C}_{10}) = \mathbf{C}_{10} \times \langle -1 + a^2 + a^8 \rangle,$$

$$V(\mathbb{Z}\mathbf{C}_{14}) = \mathbf{C}_{14} \times \langle -1 + (a^2 + a^{-2}) \rangle \times \langle -1 + 2(a^2 + a^{-2}) - (a^4 + a^{-4}) \rangle,$$

$$\begin{aligned} V(\mathbb{Z}\mathbf{C}_{18}) = \mathbf{C}_{18} &\times \langle -1 + (a^2 + a^{-2}) - (a^8 + a^{-8}) \rangle \\ &\times \langle -1 + (a^2 + a^{-2}) + (a^4 + a^{-4}) + (a^6 + a^{-6}) - 2(a^8 + a^{-8}) \rangle. \end{aligned}$$

3. CHARACTERIZATION OF $V(\mathbb{Z}\mathbf{C}_n^+)$.

Characterization of $V(\mathbb{Z}\mathbf{C}_{15}^+)$ and $V(\mathbb{Z}\mathbf{C}_{30}^+)$

Theorem 2.

$V(\mathbb{Z}\mathbf{C}_{15}^+) = <-1+C_1-C_2+C_3-C_4+C_5, -1+C_2-C_4+C_5+C_6-C_7, -1+C_3>.$

Proof. Let $\gamma \in V(\mathbb{Z}\mathbf{C}_{15}^+)$ be a generator of torsion-free unit. Then, by Lemma 1 we have

$$\gamma = \gamma_0 + \sum_{i=1}^{7} \gamma_i C_i.$$

Let us consider the subgroups $H_1 = \langle a^3 \rangle$ and $H_2 = \langle a^5 \rangle$ of prime order. Since $\mathbf{C}_{15}/H_1 \cong \mathbf{C}_3$ and $\mathbf{C}_{15}/H_2 \cong \mathbf{C}_5$ by Propositon 1, we have $\overline{\varphi}_j(\gamma) = H_j (j = 1,2)$. For $\overline{\varphi}_1(\gamma) = H_1$, we obtain

$$\begin{aligned} \gamma_0 + 2\gamma_3 + 2\gamma_6 &= 1, \\ \gamma_1 + \gamma_2 + \gamma_4 + \gamma_5 + \gamma_7 &= 0. \end{aligned} \tag{3.1}$$

and for $\overline{\varphi}_2(\gamma) = H_2$, we obtain

$$\begin{aligned} \gamma_0 + 2\gamma_5 &= 1, \\ \gamma_1 + \gamma_4 + \gamma_6 &= 0, \\ \gamma_2 + \gamma_3 + \gamma_7 &= 0. \end{aligned} \tag{3.2}$$

Substituting $\gamma_1 = p, \gamma_2 = q, \gamma_4 = r, \gamma_7 = s$ in (3.1) and (3.2), we obtain

$$\begin{aligned} \gamma_0 &= 1 + 2(p+q+r+s), \\ \gamma_3 &= -q-s, \\ \gamma_5 &= -(p+q+r+s), \\ \gamma_6 &= -p-r. \end{aligned} \tag{3.3}$$

Denoting $\omega = e^{\frac{2\pi i}{15}}$ and $\alpha = \omega + \omega^{-1}$ we get the minimal polynomial of α over $\mathbb{Q}$ as follows:

$$\min_{\mathbb{Q}}(\alpha, x) = x^4 - x^3 - 4x^2 + 4x + 1. \tag{3.4}$$

Now, let us consider the following ring homomorphism:

$$\begin{aligned} \psi : \quad \mathbb{Z}\mathbf{C}_{15} &\to \mathbb{Z}[] \\ \sum \gamma_i a^i &\mapsto \sum \gamma_i \omega^i. \end{aligned}$$

By (3.3) and (3.4), the image of the unit is

$$\begin{aligned} \psi(\gamma) &= \psi(\gamma_0 + \sum_{i=1}^{7} \gamma_i C_i) \\ &= \gamma_0 + \gamma_1(\alpha) + \gamma_2(\alpha^2 - 2) + \gamma_3(\alpha^3 - 3\alpha) + \gamma_4(\alpha^3 - 4\alpha + 1) \end{aligned}$$

$$+\gamma_5(-1)+\gamma_6(-\alpha^3+3\alpha-1)+\gamma_7(-\alpha^3-\alpha^2+3\alpha+2)$$
$$=(1+4p+q+5r+5s)+(-2p+3q-7r+6s)\alpha+(q-s)\alpha^2$$
$$+(p-q+2r-2s)\alpha^3.$$

Since $\psi(\gamma)\in\mathcal{U}(\mathbb{Z}[\alpha])=\{\sum_{i=0}^{3}a_i\alpha^i:\alpha^4-\alpha^3-4\alpha^2+4\alpha+1=0\}$ by Table 2,

$$\mathcal{U}(\mathbb{Z}[\alpha])=\langle\alpha-1,\alpha^2-3,\alpha^3-3\alpha\rangle.$$

As a consequence of the calculations we see that $\gamma\in U(\mathbb{Z}[a+a^{-1}])$ can be obtained in the following single product of the fundamental units or their inverses;

$$\varepsilon_1^2,\varepsilon_2^2,\varepsilon_1^{-2},\varepsilon_2^{-2}.$$

One can easily see that

$$\psi(\gamma)=\ \varepsilon_1^2\Rightarrow\gamma=-1+C_1-C_2+C_3-C_4+C_5,$$
$$\psi(\gamma)=\varepsilon_2^2\ \Rightarrow\gamma=-1+C_2-C_4+C_5+C_6-C_7.$$

Since $H_1=\langle a^3\rangle$ is a cyclic group of order 5, by (1.1), its unit group of integral group ring $V(\mathbb{Z}H_1)=H_1\times\langle-1+(a^3+a^{-3})\rangle$, so the third unit is $-1+C_3$. □

Corollary 2.

$$V(\mathbb{Z}\mathbf{C}_{30}^+)$$
$$=<-1+C_2-C_4+C_5+C_6-C_7,-1-C_1+C_3+C_4+C_5-C_7,-1+C_6>.$$

Proof. By considering Theorem 2 with Remark 2 we get the required result. □

Characterization of $V(\mathbb{Z}\mathbf{C}_{16}^+)$

Theorem 3.

$$V(\mathbb{Z}\mathbf{C}_{16}^+)=<1-C_2-C_3+C_5+C_6,1-C_1+C_2-C_6+C_7,-1-C_2+C_6+C_8>$$

Proof. Let $\gamma\in V(\mathbb{Z}\mathbf{C}_{16}^+)$ be a generator of torsion-free unit. Then, by Lemma 1 we have

$$\gamma=\gamma_0+\sum_{i=1}^{8}\gamma_iC_i.$$

Let us consider the prime subgroup $H=\langle a^8\rangle$. Since $\mathbf{C}_{16}/H\cong\mathbf{C}_8$ by Propositon 1 $\mathcal{U}(\mathbb{Z}\ (\mathbf{C}_{16}/H))$ is trivial. For any $\gamma\in V(\mathbb{Z}\mathbf{C}_{16}^+)$ we get

$$\overline{\varphi}(\gamma)=H\Rightarrow\gamma_0+\gamma_8=1,\tag{3.5}$$
$$\gamma_i+\gamma_{8-i}=0\text{ for }i=1,2,3,4.$$

Substituting $\gamma_0=p,\gamma_1=q,\gamma_2=r,\gamma$ in (3.5), we have

$$\gamma_4=0,\gamma_5=-s,\gamma_6=-r,\gamma_7=-q,\gamma_8=1-p.$$

Denote $\omega = e^{\frac{2\pi i}{16}}$ and $\alpha = \omega + \omega^{-1}$ then the minimal polynomial of α over $\mathbb{Q}$ can be obtained as follows

$$\min_{\mathbb{Q}}(\alpha, x) = x^4 - 4x^2 + 2. \tag{3.6}$$

By considering the following ring homomorphism

$$\begin{aligned} \psi : \quad \mathbb{Z}\mathbf{C}_{16} &\to \mathbb{Z}[] \\ \sum \gamma_i a^i &\mapsto \sum \gamma_i \omega^i, \end{aligned}$$

and (3.6), the image of the unit can be given as

$$\begin{aligned} \psi(\gamma) &= \psi(\gamma_0 + \sum_{i=1}^{8} \gamma_i C_i) \\ &= \gamma_0 + \gamma_1(\alpha) + \gamma_2(\alpha^2 - 2) + \gamma_3(\alpha^3 - 3\alpha) + \gamma_4(0) \\ &+ \gamma_5(-\alpha^3 + 3\alpha) + \gamma_6(-\alpha^2 + 2) + \gamma_7(-\alpha) - \gamma_8 \\ &= (-1 + 2p - 4r) + (2q - 6s)\alpha + (2r)\alpha^2 + (2s)\alpha^3. \end{aligned}$$

Since $\psi(\gamma) \in \mathcal{U}(\mathbb{Z}[\alpha]) = \{\sum_{i=0}^{4} a_i\alpha^i : \alpha^4 - 4\alpha^2 + 2 = 0\}$, by Table 2 we have

$$\mathcal{U}(\mathbb{Z}[\alpha]) = \langle \alpha - 1, \alpha^2 - 1, \alpha^2 + \alpha - 1 \rangle.$$

The calculations shows that $\gamma \in U(\mathbb{Z}[a + a^{-1}])$ can be obtained in the following double products of the fundamental units or their inverses;

$$\varepsilon_1^2\varepsilon_2^{-1}, \varepsilon_1^2\varepsilon_2, \varepsilon_1^{-2}\varepsilon_2, \varepsilon_1^{-2}\varepsilon_2^{-1}, \varepsilon_3^2\varepsilon_2, \varepsilon_3^{-2}\varepsilon_2, \varepsilon_3^{-2}\varepsilon_2^{-1}, \varepsilon_3^2\varepsilon_2^{-1}.$$

One can easily see that these units can be generated three units. The first one can be chosen as

$$\psi(\gamma) = \varepsilon_2\varepsilon_1^{-2} \Rightarrow \gamma = C_2 - C_3 + C_5 - C_6 + a^8.$$

By considering Remark 1 we may write the first unit as follows

$$a^8\gamma = 1 - C_2 - C_3 + C_5 + C_6$$

The second generator is

$$\psi(\gamma) = \varepsilon_2\varepsilon_3^{-2} \Rightarrow \gamma = 1 - C_1 + C_2 - C_6 + C_7.$$

Since $K = \langle a^2 \rangle$ is a cyclic group of order 8, by (1.2), its unit group of integral group ring $V(\mathbb{Z}K) = K \times \langle -1 - (a^2 + a^{-2}) + (a^6 + a^{-6}) + 2a^8 \rangle$, so the third generator is $-1 - C_2 + C_6 + 2a^8$. By Lemma 1, the third generator can be written as $-1 - C_2 + C_6 + C_8$ which finishes the proof. □

Characterization of $V(\mathbb{Z}\mathbf{C}_{20}^+)$

Theorem 4.

$$V(\mathbb{Z}\mathbf{C}_{20}^+) =< 1+C_1+C_2+C_3-C_7-C_8-C_9-C_{10},$$
$$1-C_1+C_3-C_4+C_6-C_7+C_9-C_{10}, -1+C_4 >$$

Proof. Let $\gamma \in V(\mathbb{Z}\mathbf{C}_{20}^+)$ be a generator of torsion-free unit. By Lemma 1 we have

$$\gamma = \gamma_0 + \sum_{i=1}^{10} \gamma_i C_i.$$

Let us consider subgroups $H_1 = \langle a^{10} \rangle$ and $H_2 = \langle a^4 \rangle$ of prime indices. Since $\mathbf{C}_{20}\ /H_1 \cong \mathbf{C}_{10}$ and $\mathbf{C}_{20}\ /H_2 \cong \mathbf{C}_4$ by Propositon 1 we have $\overline{\varphi}_j(\gamma) = H_j$ $(j = 1,2)$.For $\overline{\varphi}_1(\gamma) = H_1$, we obtain

$$\begin{aligned} \gamma_0 + \gamma_{10} &= 1, \\ \gamma_i + \gamma_{10-i} &= 0, \qquad (i = 1,2,3,4,5). \end{aligned} \tag{3.7}$$

and for $\overline{\varphi}_2(\gamma) = H_2$, we get

$$\begin{aligned} \gamma_0 + 2\gamma_4 + 2\gamma_8 &= 1, \\ \gamma_1 + \gamma_3 + \gamma_5 + \gamma_7 + \gamma_9 &= 0, \\ 2\gamma_2 + 2\gamma_6 + \gamma_{10} &= 0. \end{aligned} \tag{3.8}$$

If we substitude $\gamma_1 = p, \gamma_2 = q, \gamma_3 = r, \gamma_4 = s$ in (3.7) and (3.8), we can write

$$\begin{aligned} \gamma_0 &= 1+2q-2s, \gamma_5 = 0, \\ \gamma_6 &= -s, \gamma_7 = -r, \gamma_8 = -q, \\ \gamma_9 &= -p, \gamma_{10} = -2q+2s. \end{aligned} \tag{3.9}$$

By taking $\omega = e^{\frac{2\pi i}{20}}$ and $\alpha = \omega + \omega^{-1}$ we can write the minimal polynomial of α over $\mathbb{Q}$ as follows;

$$\min_{\mathbb{Q}}(\alpha, x) = x^4 - 5x^2 + 5. \tag{3.10}$$

Now consider the following ring homomorphism,

$$\begin{aligned} \psi : \quad \mathbb{Z}\mathbf{C}_{20} &\rightarrow \mathbb{Z}[] \\ \sum \gamma_i a^i &\mapsto \sum \gamma_i \omega^i \end{aligned}$$

with equations (3.9) and (3.10). This gives the image of the unit as

$$\psi(\gamma) = \psi(\gamma_0 + \sum_{i=1}^{10} \gamma_i C_i), (C_{10} = 2a^3)$$
$$= \gamma_0 + \gamma_1(\alpha) + \gamma_2(\alpha^2-2) + \gamma_3(\alpha^3-3\alpha) + \gamma_4(\alpha^4-4\alpha^2+2) + \gamma_5(0),$$
$$+\gamma_6(-\alpha^4+4\alpha^2-2) + \gamma_7(-\alpha^3+3\alpha) + \gamma_8(-\alpha^2+2) + \gamma_9(-\alpha) - \gamma_{10}$$

$$= (1-10s)+(2p-6r)\alpha+(2q+2s)\alpha^2+(2r)\alpha^3.$$

Since $\psi(\gamma) \in U(\mathbb{Z}[\alpha]) = \{\sum_{i=0}^{4} a_i\alpha^i : \alpha^4-5\alpha^2+5=0\}$. By using Table 2

$$U(\mathbb{Z}[\alpha]) = \langle \alpha^2-2, \alpha^2+\alpha-2, \alpha^3-\alpha^2-3\alpha+3\rangle.$$

$\gamma \in U(\mathbb{Z}[a+a^{-1}])$ can be obtained in the following single product of the fundamental units or their inverses;

$$\varepsilon_2^2, \varepsilon_3^2, \varepsilon_2^{-2}, \varepsilon_3^{-2}.$$

The two generators are obtained as follows:

$$\psi(\gamma) = \varepsilon_2^2 \Rightarrow \gamma = -1-C_1-C_2-C_3+C_7+C_8+C_9+2C_{10}$$
$$\psi(\gamma) = \varepsilon_3^2 \Rightarrow \gamma = -1+C_1-C_3+C_4-C_6+C_7-C_9+2C_{10}.$$

Their augmentations are negative, to make normalized units we must multiply by -1 and also by regarding Lemma 1 we get

$$\psi(\gamma) = \varepsilon_2^2 \Rightarrow \gamma = 1+C_1+C_2+C_3-C_7-C_8-C_9-C_{10}$$
$$\psi(\gamma) = \varepsilon_3^2 \Rightarrow \gamma = 1-C_1+C_3-C_4+C_6-C_7+C_9-C_{10}.$$

Since $H_2 = \langle a^4\rangle$ is a cyclic group of order 5, by (1.1), its unit group of integral group ring $V(\mathbb{Z}H_2) = H_2 \times \langle -1+(a^4+a^{-4})\rangle$, so the third generator is clearly $-1+C_4$. □

Characterization of $V(\mathbb{Z}\mathbf{C}_{24}^{+})$

Theorem 5.

$V(\mathbb{Z}\mathbf{C}_{24}^{+}) =< -5-2C_1-4C_3-3C_4+2C_5-2C_7+3C_8+4C_9+2C_{10}+3C_{12},$
$3+2C_2+C_4-C_8-2C_{10}-C_{12}, -1+C_1-C_4+C_5-C_7+C_8-C_{11}+C_{12} >$

Proof. Let $\gamma \in V(\mathbb{Z}\mathbf{C}_{24}^{+})$ be a generator of torsion-free unit. Then, by Lemma 1 we have

$$\gamma = \gamma_0 + \sum_{i=1}^{12} \gamma_i C_i$$

Consider the subgroups $H_1 = \langle a^{12}\rangle$ and $H_2 = \langle a^8\rangle$ of prime orders. Since $\mathbf{C}_{24}/H_1 \cong \mathbf{C}_{12}$ and $\mathbf{C}_{24}/H_2 \cong \mathbf{C}_8$, by Propositon 1 we have, $\overline{\varphi}_j(\gamma) = H_j$ $(j = 1,2)$.

For $\overline{\varphi}_1(\gamma) = H_1$, we obtain

$$\gamma_0+\gamma_{12} = 1, \tag{3.11}$$
$$\gamma_i+\gamma_{12-i} = 0, \qquad (i = 1,2,3,4,5,6).$$

and for $\overline{\varphi}_2(\gamma) = H_2$, we get

$$\begin{aligned} \gamma_0 + 2\gamma_8 &= 1, \\ \gamma_1 + \gamma_7 + \gamma_9 &= 0, \\ \gamma_2 + \gamma_6 + \gamma_{10} &= 0, \\ \gamma_3 + \gamma_5 + \gamma_{11} &= 0, \\ 2\gamma_4 + \gamma_{12} &= 0. \end{aligned} \tag{3.12}$$

Substituting $\gamma_2 = p, \gamma_3 = q, \gamma_4 = r, \gamma_5 = s$ in (3.11) and (3.12), we have

$$\begin{array}{lll} \gamma_0 = 1 + 2r, & \gamma_1 = q + s, & \gamma_6 = 0 \\ \gamma_7 = -s, & \gamma_8 = -r, & \gamma_9 = -q, \\ \gamma_{10} = -p, & \gamma_{11} = -q - s, & \gamma_{12} = -2r \end{array} \tag{3.13}$$

On the other hand, if we denote $\omega = e^{\frac{2\pi i}{24}}$ and $\alpha = \omega + \omega^{-1} = 2\cos(\frac{\pi}{12})$ then we get the minimal polynomial of α over $\mathbb{Q}$ as

$$\min_{\mathbb{Q}}(\alpha, x) = x^4 - 4x^2 + 1. \tag{3.14}$$

Considering the ring homomorphism

$$\begin{array}{rl} \psi : \quad \mathbb{Z}\mathbf{C}_{24} & \to \mathbb{Z}[] \\ \sum \gamma_i a^i & \mapsto \sum \gamma_i \omega^i \end{array}$$

with (3.13) and (3.14), the image of the unit can be obtained as

$$\begin{aligned} \psi(\gamma) &= \psi(\gamma_0 + \sum_{i=1}^{11} \gamma_i C_i) \\ &= (\gamma_0 - \gamma_{12}) + (\gamma_1 - \gamma_{11})(\frac{\sqrt{6}+\sqrt{2}}{2}) + (\gamma_2 - \gamma_{10})(\sqrt{3}) \\ &+ (\gamma_3 - \gamma_9)(\sqrt{2}) + (\gamma_4 - \gamma_8)(1) + (\gamma_5 - \gamma_7)(\frac{\sqrt{6}-\sqrt{2}}{2}) \\ &= (1 + 6r) + 3q\sqrt{2} + 2p\sqrt{3} + (q + 2s)\sqrt{6}. \end{aligned}$$

Since $\psi(\gamma) \in \mathcal{U}(\mathbb{Z}[\sqrt{2}]) \times \mathcal{U}(\mathbb{Z}[\sqrt{3}]) \times \mathcal{U}(\mathbb{Z}[\sqrt{6}])$,

$$\begin{aligned} i)\psi(\gamma) \in \mathcal{U}(\mathbb{Z}[\sqrt{2}]) \quad &\Rightarrow p = 0, q = -2s \\ &\Rightarrow \psi(\gamma) = (1 + 6r) - 6s\sqrt{2} = \pm(1 \pm \sqrt{2})^k, (k \in \mathbb{Z}) \\ &\Rightarrow (1 + 6r) - 6s\sqrt{2} = \pm(17 \pm 12\sqrt{2}), (k = 4) \\ &\Rightarrow \gamma = -5 - 2C_1 - 4C_3 - 3C_4 + 2C_5 - 2C_7 + 3C_8 + 4C_9 \\ &\qquad +2C_{11} + 6a^{12}, \end{aligned}$$

$$
\begin{aligned}
ii)\psi(\gamma)\in\mathcal{U}(\mathbb{Z}[\sqrt{3}]) \quad &\Rightarrow q=s=0\\
&\Rightarrow \psi(\gamma)=(1+6r)+2p\sqrt{3}=\pm(2\pm\sqrt{3})^k,(k\in\mathbb{Z})\\
&\Rightarrow (1+6r)+2p\sqrt{3}=\pm(7\pm4\sqrt{3}),(k=2)\\
&\Rightarrow \gamma=3+2C_2+C_4-C_8-2C_{10}-2a^{12},
\end{aligned}
$$

$$
\begin{aligned}
iii)\psi(\gamma)\in U(\mathbb{Z}[\sqrt{6}]) \quad &\Rightarrow p=q=0\\
&\Rightarrow \psi(\gamma)=(1+6r)+2s\sqrt{6}=\pm(5\pm2\sqrt{6})^k,(k\in\mathbb{Z})\\
&\Rightarrow \psi(\gamma)=(1+6r)+2s\sqrt{6}=\pm5\pm2\sqrt{6},(k=1)\\
&\Rightarrow \gamma=-1+C_1-C_4+C_5-C_7+C_8-C_{11}+2a^{12}.
\end{aligned},
$$

By considering Lemma 1 the generators can be written respectively as follows

$$
\begin{aligned}
&-5-2C_1-4C_3-3C_4+2C_5-2C_7+3C_8+4C_9+2C_{11}+3C_{12},\\
&3+2C_2+C_4-C_8-2C_{10}-C_{12},\\
&-1+C_1-C_4+C_5-C_7+C_8-C_{11}+C_{12}.
\end{aligned}
$$

□

Characterization of $V(\mathbb{Z}\mathbf{C}_{11}^{+})$ and $V(\mathbb{Z}\mathbf{C}_{22}^{+})$

Theorem 6. *The normalized units of $V(\mathbb{Z}\mathbf{C}_{11}^{+})\subset\mathbb{Z}\mathbf{C}_{11}$ are generated by the set*

$$V(\mathbb{Z}\mathbf{C}_{11}^{+})=<-1+C_1,-1+C_2,-1-C_3,-1+C_4>.$$

Proof. Let $\gamma\in V(\mathbb{Z}\mathbf{C}_{11}^{+})$ be a generator of torsion-free unit. Then, we have

$$\gamma=\gamma=\gamma_0+\sum_{i=1}^{5}\gamma_iC_i$$

and

$$\gamma_0+2\gamma_1+2\gamma_2+2\gamma_3+2\gamma_4+2\gamma_5=1. \tag{3.15}$$

If we substitute $\gamma_1=p,\gamma_2=q,\gamma_3=r,\gamma_4=s,,\gamma_5=t$ in (3.15), we can write

$$\gamma_0=1-2(p+q+r+s+t). \tag{3.16}$$

On the other hand, if we denote $\omega=e^{\frac{2\pi i}{11}}$ and $\alpha=\omega+\omega^{-1}$ then we can get the minimal polynomial of α over $\mathbb{Q}$ as follows;

$$\min{}_{\mathbb{Q}}(\alpha,x)=x^5+x^4-4x^3-3x^2+3x+1. \tag{3.17}$$

By regarding the following ring homomorphism:

$$
\begin{aligned}
\psi:\quad \mathbb{Z}C_{11} &\rightarrow \mathbb{Z}[]\\
\sum\gamma_ia^i &\mapsto \sum\gamma_i\omega^i
\end{aligned}
$$

with equations (3.16) and (3.17), the image of the unit is

$$\begin{aligned}\psi(\gamma) &= \psi(\gamma_0 + \sum_{i=1}^{5} \gamma_i C_i) \\ &= \gamma_0 + \gamma_1\alpha + \gamma_2(\alpha^2 - 2) + \gamma_3(\alpha^3 - 3\alpha) + \gamma_4(\alpha^4 - 4\alpha^2 + 2) \\ &+ \gamma_5(\alpha^5 - 5\alpha^3 + 5\alpha) \\ &= (1 - 2p - 4q - 2r - 3t) + (p - 3r + 2t)\alpha + (q - 4s + 3t)\alpha^2 \\ &+ (r - t)\alpha^3 + (s - t)\alpha^4.\end{aligned}$$

Since $\psi(\gamma) \in U(\mathbb{Z}[\alpha]) = \{\sum_{i=0}^{4} a_i\alpha^i : \alpha^5 + \alpha^4 - 4\alpha^3 - 3\alpha^2 + 3\alpha + 1 = 0\}$. By Table 2

$$U(\mathbb{Z}[\alpha]) = \langle \alpha, \alpha + 1, \alpha^2 - 2, \alpha^4 + \alpha^3 - 3\alpha^2 - 3\alpha \rangle.$$

$\gamma \in U(\mathbb{Z}[a + a^{-1}])$ can be obtained in the following single product of the fundamental units or their inverses;

$$-\varepsilon_1\varepsilon_2^{-1}\varepsilon_3\varepsilon_4^{-1}, \varepsilon_1^{-1}\varepsilon_2^{-1}\varepsilon_4, \varepsilon_1\varepsilon_4, -\varepsilon_1^{-1}\varepsilon_2\varepsilon_3^{-2}.$$

Here

$$\begin{aligned}\psi(\gamma) &= -\varepsilon_1\varepsilon_2^{-1}\varepsilon_3\varepsilon_4^{-1} &\Rightarrow \gamma = -1 + C_1 \\ \psi(\gamma) &= \varepsilon_1^{-1}\varepsilon_2^{-1}\varepsilon_4 &\Rightarrow \gamma = -1 + C_2 \\ \psi(\gamma) &= \varepsilon_1\varepsilon_4 &\Rightarrow \gamma = -1 + C_3 \\ \psi(\gamma) &= -\varepsilon_1^{-1}\varepsilon_2\varepsilon_3^{-2} &\Rightarrow \gamma = -1 + C_4\end{aligned}$$

□

Corollary 3.

$$V(\mathbb{Z}C_{22}^+) = < -1 + C_2, -1 + C_4, -1 + C_6, -1 + C_8 > .$$

Proof. We obtain desired result by considering Theorem 6 with Remark 2. □

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