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SOME GENERAL BASKAKOV TYPE OPERATORS

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Abstract. A general class of linear positive operators which generalizes Baskakov's operator is consrtucted. The operators of this type which preserve exactly two test functions from the set $\{e_0, e_1, e_2\}$ are determined in each case, and for the operators obtained, we give their approximation theorem, convergence theorem and Voronovskaja-type theorem.

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1. Introduction

Let $\mathbb{N}$ be a set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

In [6], J. P. King constructed and studied general operators which generalizes the classical Berstein operators. Some King-type operators were studied in [3–6], [8, 9].

In 1957, V. A. Baskakov [2], for $m \in \mathbb{N}$ has introduced the linear positive operator

$$(V_m f)(x) = (1+x)^{-m} \sum_{k=0}^{\infty} \binom{m+k-1}{k} \left(\frac{x}{1+x}\right)^k f\left(\frac{k}{m}\right) \tag{1.1}$$

defined for any $f \in C_2([0,+\infty)) = \left\{ f \in C([0,+\infty)) \middle| \lim_{x \longrightarrow \infty} \frac{f(x)}{1+x^2} < +\infty \right\}$ and $x \in [0,+\infty)$. He proved that if $f \in C_2([0,+\infty))$ then $V_m f \longrightarrow f$ uniform on any compact $[a,b] \subset [0,+\infty)$. Note that the operators (1.1) preserve the test functions e_0 and e_1. Generalizations of the operators (1.1) were introduced by M.A.Özarslan, G.Duman and N.I.Mahmudov in [10] by the form

$$(T_m f)(x) = \sum_{k=0}^{\infty} \binom{m+k-1}{k} (u_m(x))^k (1+u_m(x))^{-m-k} f\left(\frac{k}{m}\right) \tag{1.2}$$

for $m \in \mathbb{N}, x \in [0,+\infty)$, and they show that if $u_m(x) \longrightarrow x$ on a compact $[a,b] \subset [0,+\infty)$, then $T_m f \longrightarrow f$ uniform on $[a,b]$ for all $f \in C_2([0,+\infty))$.

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A similar result was obtained in [9] by L. Rempulska and K. Tomczak for the case in which the modified operators of Baskakov type preserve the test functions e_0 and e_2.

In this paper, we introduce a general class of linear positive operators. We determine the operators of the general class which preserve only two test functions e_0 and e_1 or e_0 and e_2 or e_1 and e_2.

In all these cases we give approximation properties, convergence theorems and Voronovskaja-type theorems.

The paper is organized as follows. In Section 2 we recall some results obtained by O.T.Pop in [7] which are essentially used for obtaining the main results of the paper. Section 3 is devoted to the construction of the general class of linear and positive operators defined by infinite sum, which we announced in the start. For the constructed class we establish a convergence theorem and Voronovskaja type theorem. In Section 4 we prove that in the general class constructed in Section 3 exists a unique operator which preserve the test functions e_0 and e_1, the classical Baskakov operator. In Section 5 we obtain a King type operator, which is an operator that preserves the test functions e_0 and e_2 defined on semiaxis $[0,+\infty)$. We find here a result due the L. Rempulska and K. Tomczak [9].

Finally, in Section 6, we determine the operators from the general class which preserve the test function e_1 and e_2.

2. Preliminaries

In this section we recall some results from [7], which we shall use in the present paper. Let I, J be real intervals with the property $I \cap J$ is a nonempty interval. For any $m,k \in \mathbb{N}_0, m \neq 0$, we consider the functions $\varphi_{m,k} : J \longrightarrow \mathbb{R}$, with the property that $\varphi_{m,k}(x) \geq 0$, for any $x \in J$ and the linear positive functionals $A_{m,k} : E(I) \longrightarrow \mathbb{R}$.

For any $m \in \mathbb{N}$ we define the operator $L_m : E(I) \longrightarrow F(J)$, by

$$(L_m f)(x) = \sum_{k=0}^{\infty} \varphi_{m,k}(x) A_{m,k}(f), \tag{2.1}$$

where $E(I)$ is a linear space of real valued functions defined on I, for which the operators (2.1) are convergent and $F(J)$ is a subset of real valued functions defined on J.

Remark 1. The operators $(L_m)_{m\in\mathbb{N}}$ are linear and positive on $E(I \cap J)$.

For $m \in \mathbb{N}$ and $i \in \mathbb{N}_0$, we define $T_{m,i}$ by

$$(T_{m,i} L_m)(x) = m^i (L_m \psi_x^i)(x) = m^i \sum_{k=0}^{\infty} \varphi_{m,k}(x) A_{m,k}(\psi_x^i) \tag{2.2}$$

for any $x \in I \cap J$, where $\psi_x : I \longrightarrow \mathbb{R}, \psi_x(t) = t - x$.

In what follows $s \in \mathbb{N}_0$ is even and we assume that the following condition: there exist the smallest $\alpha_s, \alpha_{s+2} \in [0, +\infty)$, so that

$$\lim_{m \longrightarrow \infty} \frac{(T_{m,j} L_m)(x)}{m^{\alpha_j}} = B_j(x) \in \mathbb{R} \tag{2.3}$$

for any $x \in I \cap J$ and $j \in \{s, s+2\}$,

$$\alpha_{s+2} < \alpha_s + 2 \tag{2.4}$$

hold.

Theorem 1 ([7]). *Let $f \in E(I)$ be a function. If $x \in I \cap J$ and f is s times differentiable in a neighborhood of x, $f^{(s)}$ is continuous on x, then*

$$\lim_{m \longrightarrow \infty} m^{s-\alpha_s} \left((L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!} (T_{m,i} L_m)(x) \right) = 0. \tag{2.5}$$

Assume that f is s times differentiable on I. Let $K \subset I \cap J$ be a compact interval. For there one we assume that exist $m(s) \in \mathbb{N}$ and constant $k_j \in \mathbb{R}$ depending on K, such that for $m \geq m(s)$ and $x \in K$ the following relation

$$\frac{(T_{m,j} L_m)(x)}{m^{\alpha_j}} \leq k_j, \ j \in \{s, s+2\} \tag{2.6}$$

holds.

Following [7], the convergence expressed by (2.5) is uniform on K and

$$m^{s-\alpha_s} \left| (L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!} (T_{m,i} L_m)(x) \right| \leq \tag{2.7}$$

$$\leq \frac{1}{s!} (k_s + k_{s+2}) \omega \left(f^{(s)}; \frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}} \right),$$

for any $x \in K, m \geq m(s)$, where $\omega(f;\delta)$ denotes the modulus of continuity of the function f.

In the following, we use the identity

$$(1+x)^{-m} = \sum_{k=0}^{\infty} (-1)^k \binom{m+k-1}{k} x^k \tag{2.8}$$

where $x \geq 0$ and $m \in \mathbb{N}$.

By differentiating the relation (2.8) and multiplying with $\frac{x}{m}$, we obtain

$$-x(1+x)^{-m-1} = \sum_{k=0}^{\infty} (-1)^k \binom{m+k-1}{k} x^k \frac{k}{m}. \tag{2.9}$$

Similarly, differentiating the relation (2.9) and multiplying with $\frac{x}{m}$ we get

$$\frac{x}{m}(mx-1)(1+x)^{-m-2}=\sum_{k=0}^{\infty}(-1)^k\binom{m+k-1}{k}x^k\left(\frac{k}{m}\right)^2, \tag{2.10}$$

where $x \geq 0$ and $m \in \mathbb{N}$.

3. The construction of a general linear and positive operators defined by infinite sum

Let $m_0 \in \mathbb{N}$ be given, $\mathbb{N}_1=\{m \in \mathbb{N} | m \geq m_0\}$, the functions $\alpha_m : J \longrightarrow \mathbb{R}$ and $\beta_m : J \longrightarrow \mathbb{R}$ such that $\alpha_m(x)>0, \beta_m(x)>0, \beta_m(x)-\alpha_m(x)>0$ for any $x \in J$ and any $m \in \mathbb{N}_1$.

We define the operators of the following form

$$(P_m f)(x)=\sum_{k=0}^{\infty}\binom{m+k-1}{k}\alpha_m^k(x)\beta_m^{-m-k}(x) f\left(\frac{k}{m}\right), \tag{3.1}$$

for any $m \in \mathbb{N}_1, x \in J$ and $f \in E([0,+\infty))$, where $E([0,\infty))$ is a linear space of real valued functions defined on $[0,\infty)$, for which the operators defined by (3.1) are convergent.

If in (2.8)-(2.10), we substitute x by $-\frac{\alpha_m(x)}{\beta_m(x)}$, we obtain

$$(\beta_m(x)-\alpha_m(x))^{-m}=\sum_{k=0}^{\infty}\binom{m+k-1}{k}(\alpha_m(x))^k(\beta_m(x))^{-m-k} \tag{3.2}$$

$$\alpha_m(x)(\beta_m(x)-\alpha_m(x))^{-m-1}=\sum_{k=0}^{\infty}\binom{m+k-1}{k}(\alpha_m(x))^k(\beta_m(x))^{-m-k}\frac{k}{m} \tag{3.3}$$

$$\frac{1}{m}\alpha_m(x)(m\alpha_m(x)+\beta_m(x))(\beta_m(x)-\alpha_m(x))^{-m-2}= \tag{3.4}$$

$$=\sum_{k=0}^{\infty}\binom{m+k-1}{k}(\alpha_m(x))^k(\beta_m(x))^{-m-k}\left(\frac{k}{m}\right)^2, x \in J, m \in \mathbb{N}.$$

We impose the condition

$$(P_m e_0)(x)=1+u_m(x), \tag{3.5}$$

for any $m \in \mathbb{N}_1$ and any $x \in J$, where $u_m : J \longrightarrow \mathbb{R}, u_m(x)>-1$.

From (3.1), (3.2) and (3.5) follows the equality

$$\beta_m(x)-\alpha_m(x)=(1+u_m(x))^{-\frac{1}{m}} \tag{3.6}$$

for any $m \in \mathbb{N}_1$ and any $x \in J$.

Let us to impose the condition

$$(P_m e_1)(x) = x + v_m(x), \tag{3.7}$$

for any $m \in \mathbb{N}_1$ and any $x \in J$, where $v_m : J \longrightarrow \mathbb{R}, v_m(x) > -x$.

Taking (3.1), (3.3) and (3.7) into account, we get

$$\alpha_m(x)(\beta_m(x) - \alpha_m(x))^{-m-1} = x + v_m(x), m \in \mathbb{N}_1, x \in J. \tag{3.8}$$

From (3.6) and (3.8) it follows

$$\alpha_m(x) = \frac{x + v_m(x)}{1 + u_m(x)}(1 + u_m(x))^{-\frac{1}{m}} \tag{3.9}$$

and

$$\beta_m(x) = \left(1 + \frac{x + v_m(x)}{1 + u_m(x)}\right)(1 + u_m(x))^{-\frac{1}{m}}, \tag{3.10}$$

$m \in \mathbb{N}_1, x \in J$.

Taking (3.9) and (3.10) into account, the operator (3.1) becomes

$$(P_m f)(x) = (1 + u_m(x)) \sum_{k=0}^{\infty} \binom{m+k-1}{k} \left(\frac{x + v_m(x)}{1 + u_m(x)}\right)^k \cdot \tag{3.11}$$

$$\cdot \left(1 + \frac{x + v_m(x)}{1 + u_m(x)}\right)^{-m-k} f\left(\frac{k}{m}\right),$$

$m \in \mathbb{N}_1, x \in J, f \in E([0, +\infty))$.

From (3.1) and (3.4), we have

$$(P_m e_2)(x) = \frac{x + v_m(x)}{m}\left((m+1)\frac{x + v_m(x)}{1 + u_m(x)} + 1\right), \tag{3.12}$$

for any $m \in \mathbb{N}_1$ and any $x \in J$.

Next $(P_m \psi_x^2)(x) = (P_m e_2)(x) - 2x(P_m e_1)(x) + x^2(P_m e_0)(x)$ and taking (3.5), (3.7) and (3.12) into account we get

$$(P_m \psi_x^2)(x) = \frac{m(v_m(x) - x u_m(x))^2 + (x + v_m(x))^2 + (1 + u_m(x))(x + v_m(x))}{m(1 + u_m(x))} \tag{3.13}$$

for any $m \in \mathbb{N}_1$ and any $x \in J$.

Coming back to Theorem 1, for the operators (3.1), we have $I = [0, +\infty)$, $E(I) = C_2([0, +\infty))$

$$\varphi_{m,k}(x) = (1 + u_m(x))\binom{m+k-1}{k}\left(\frac{x + v_m(x)}{1 + u_m(x)}\right)^k \left(1 + \frac{x + v_m(x)}{1 + u_m(x)}\right)^{-m-k} \tag{3.14}$$

and

$$A_{m,k}(f) = f\left(\frac{k}{m}\right), \tag{3.15}$$

for any $m \in \mathbb{N}_1, x \in J$ and $f \in C_2([0,+\infty))$.

In the following, let $K \subset I \cap J$ be a compact interval.

We suppose that there exists the sequences $\big(a_m(K)\big)_{m\in\mathbb{N}_1}, \big(b_m(K)\big)_{m\in\mathbb{N}_1}$, so that

$$\lim_{m\longrightarrow\infty} a_m(K) = \lim_{m\longrightarrow\infty} b_m(K) = 0, \tag{3.16}$$

$$|u_m(x)| \leq a_m(K), \tag{3.17}$$

$$|v_m(x)| \leq b_m(K), \tag{3.18}$$

for any $m \in \mathbb{N}_1$ and any $x \in K$.

In what follows, let us suppose that the following equality

$$\lim_{m\longrightarrow\infty} m(v_m(x) - xu_m(x)) = l(x) \tag{3.19}$$

holds for any $x \in J$, where $l : J \longrightarrow \mathbb{R}$ is a bounded function on K.

Remark 2. From (3.16) − (3.18) it results that if

$$\lim_{m\longrightarrow\infty} u_m(x) = \lim_{m\longrightarrow\infty} v_m(x) = 0, x \in K,$$

then

$$\lim_{m\longrightarrow\infty} m(v_m(x) - xu_m(x))^2 = \lim_{m\longrightarrow\infty} m(v_m(x) - xu_m(x))\cdot$$
$$\cdot \lim_{m\longrightarrow\infty} (v_m(x) - xu_m(x)) = 0.$$

This Remark 2 implies that there exist $m_1 \in \mathbb{N}$ such that

$$(m(v_m(x) - xu_m(x))^2 \leq 1, m \in \mathbb{N}_1, m \geq m_1, x \in K. \tag{3.20}$$

Let us denote

$$M_1(K) = sup\{a_m(K) | m \in \mathbb{N}_1\},$$
$$M_2(K) = sup\{b_m(K) | m \in \mathbb{N}_1\}.$$

Now, let $\mathbb{N}_2 = \{m \in \mathbb{N} | m \geq max(m_0, m_1)\}$.

According to Theorem 1 one obtains $\alpha_0 = 0, \alpha_2 = 1$, $(T_{m,0}P_m)(x) = (P_m e_0)(x)$, for any $m \in \mathbb{N}_1$ and any $x \in K$.

From (3.16) one arrives at

$$\lim_{m\longrightarrow\infty} (T_{m,0}P_m)(x) = 1 = B_0(x), x \in K. \tag{3.21}$$

Consequently we get that exists $m(0) \in \mathbb{N}$ such that

$$(T_{m,0}P_m)(x) = 1 + u_m(x) \leq 1 + M_1(K) = k_0(K) \tag{3.22}$$

holds for any $m \geq max(m_0, m(0))$ and $x \in K$.

We have $(T_{m,2}P_m)(x) = m^2(P_m\psi_x^2)(x)$, $m \in \mathbb{N}_1$, $x \in J$. Taking (3.13), (3.19) and (3.20) into account, we get

$$\lim_{m\longrightarrow\infty} \frac{(T_{m,2}P_m)(x)}{m} = x(1+x) + l(x) = B_2(x), x \in K. \tag{3.23}$$

Also there exists $m(2) \in \mathbb{N}$ such that

$$\frac{(T_{m,2}P_m)(x)}{m} \leq b(1+b)+2 = k_2(K) \tag{3.24}$$

for any $m \geq max(m_0, m(2), m_1)$ and $x \in K$, where $maxK = b$.

Theorem 2. *Let $f \in C_2([0,+\infty))$. Then*

$$\lim_{m \longrightarrow \infty} P_m f = f \tag{3.25}$$

uniformly on K. There exists $m(0) \in \mathbb{N}, m(0)$ depending on K, so that the following inequalities

$$|(P_m f)(x) - (1+u_m(x))f(x)| \leq (k_0(K)+k_2(K))\omega\left(f;\frac{1}{\sqrt{m}}\right), \tag{3.26}$$

$$|(P_m f)(x) - f(x)| \leq |u_m(x)| \cdot |f(x)| + (k_0(K)+k_2(K))\omega\left(f;\frac{1}{\sqrt{m}}\right) \tag{3.27}$$

and

$$|(P_m f)(x) - f(x)| \leq a_m(K)M(K) + (k_0(K)+k_2(K))\omega\left(f;\frac{1}{\sqrt{m}}\right) \tag{3.28}$$

hold for any $m \in \mathbb{N}_2, m \geq m(0)$ and $x \in K$, where

$$M(K) = sup\{|f(x)| \mid x \in K\}.$$

Proof of Theorem 2. Applying the Theorem 1 for $\alpha = 0$ yields (3.25) and (3.26). Next, using the inequality $|a-c|-|b-c| \leq |a-b|$, (3.27) follows, and consequently (3.28) holds. □

Remark 3. The equations (3.26)-(3.28) are asymptotic formula for a class of approximation processes of King's type (see [1]).

Theorem 3. *Let $f \in C_2([0,+\infty))$. If $x \in K$, f is two times differentiable in x and $f^{(2)}$ is continuous in x, the following relations*

$$\lim_{m \longrightarrow \infty} m\left((P_m f)(x) - (1+u_m(x))f(x)\right) = l(x))f^{(1)}(x) + \frac{x(1+x)}{2}f^{(2)}(x) \tag{3.29}$$

holds.

Proof of Theorem 3. If $m \in \mathbb{N}_1, x \in K$, according Theorem 1 yields

$$(T_{m,1}P_m)(x) = m(P_m\psi_x)(x) = m((P_m e_1)(x) - x(P_m e_0)(x)).$$

Applying (3.1) and (3.5) it follows

$$(T_{m,1}P_m)(x) = m(v_m(x) - xu_m(x)). \tag{3.30}$$

Using Theorem 1 for $s = 2$, (3.22), (3.23) and (3.30) one arrives at (3.29). □

Remark 4. The relation (3.29) is a Voronovskaja-type theorem (see [11]).

4. $(P_m)_{m\geq m_0}$ OPERATORS PRESERVING TEST FUNCTIONS e_0 AND e_1

In the following, we consider $K=[a,b]$, where $b>0$. In this case $J=[0,+\infty)$ and $m_0=1$, then $\mathbb{N}_1=\mathbb{N}$. If the operators, $(P_m)_{m\in\mathbb{N}}$ preserve e_0 and e_1, we have $P_m e_0=e_0$ and $P_m e_1=e_1$, for any $m\in\mathbb{N}$. Taking (3.5) and (3.7) into account, it results that $u_m(x)=v_m(x)=0$ and $l(x)=0$ for any $m\in\mathbb{N}$ and any $x\in[0,+\infty)$.

In this case, we get again the classical Baskakov operators. One has $a_m([a,b])$ $=b_m([a,b])=0$, for any $m\in\mathbb{N}$, $k_0([a,b])=1$ and $k_2([a,b])=b(1+b)+2$. Our statements turn into well known results.

Theorem 4 ([2]). *Let $f\in C_2([0,+\infty))$ one has*

$$\lim_{m\longrightarrow\infty} P_m f=f \tag{4.1}$$

uniformly on any compact interval $[a,b]\subset\mathbb{R}_+$ and then exists $m(0)\in\mathbb{N}$, $m(0)$ depending on b so that

$$|(P_m f)(x)-f(x)|\leq(3+b+b^2)\omega\left(f;\frac{1}{\sqrt{m}}\right), m\in\mathbb{N}_2, m\geq m(0), x\in[a,b]. \tag{4.2}$$

Theorem 5 ([2]). *Let $f\in C_2([0,+\infty))$. If $x\in[a,b]$, f is two times differentiable in x and $f^{(2)}$ is continuous in x, then*

$$\lim_{m\longrightarrow\infty} m((P_m f)(x)-f(x))=\frac{x(1+x)}{2}f^{(2)}(x). \tag{4.3}$$

5. $(P_m)_{m\geq m_0}$ OPERATORS PRESERVING THE TEST FUNCTIONS e_0 AND e_2

In this case $J=[0,+\infty)$ and $m_0=1$, then $\mathbb{N}_1=\mathbb{N}$. Because $P_m e_0=e_0$ and $P_m e_2=e_2$ for any $m\in\mathbb{N}$, taking (3.5) into account, it follows $u_m(x)=0$, for any $m\in\mathbb{N}$ and any $x\in[0,+\infty)$.

By using (3.12) yields

$$(m+1)(x+v_m(x))^2+(x+v_m(x))-mx^2=0 \tag{5.1}$$

for any $m\in\mathbb{N}$ and any $x\in[0,+\infty)$.

From (5.1) we get $v_m(x)=\frac{\sqrt{4m(m+1)x^2+1}-1}{2(m+1)}-x$, for any $m\in\mathbb{N}$ and any $x\in[0,+\infty)$, and then the operators from (3.8) become

$$(P_m f)(x)=\sum_{k=0}^{\infty}\binom{m+k-1}{k}\left(\frac{\sqrt{4m(m+1)x^2+1}-1}{2(m+1)}\right)^k\cdot \tag{5.2}$$

$$\cdot\left(1+\frac{\sqrt{4m(m+1)x^2+1}-1}{2(m+1)}\right)^{-m-k}f\left(\frac{k}{m}\right),$$

$m\in\mathbb{N}, x\in[0,+\infty), f\in C_2([0,+\infty))$.

So we came across the results obtained by L. Rempulska and K. Tomczak in [9].

Lemma 1. *We have that*

$$v_m(x) \le \frac{\sqrt{4m(m+1)a^2+1}-1}{2(m+1)} - a, m \in \mathbb{N}, x \in K = [a,b] \tag{5.3}$$

and

$$\frac{\sqrt{4m(m+1)a^2+1}-1}{2(m+1)} - a \le \sqrt{\frac{1}{2}a^2 + \frac{1}{16}} - a, m \in \mathbb{N}. \tag{5.4}$$

Proof of Lemma 1. Since the function v_m is decreasing on $[a,b]$, it gets the maximum value in a and (5.3) follows. By direct computation, (5.4) is obtained. □

Lemma 2. *The following relation*

$$\lim_{m\longrightarrow\infty} m v_m(x) = -\frac{1+x}{2} \tag{5.5}$$

holds, where $x \in K$.

Proof of Lemma 2. We have

$$\lim_{m\longrightarrow\infty} m v_m(x) = \lim_{m\longrightarrow\infty} \frac{m}{2(m+1)} \left(-1 + \sqrt{4m(m+1)x^2+1} - 2(m+1)x\right) =$$

$$= \frac{1}{2}\left(-1 + \lim_{m\longrightarrow\infty} \frac{-4mx^2 - 4x^2 + 1}{\sqrt{4m(m+1)x^2+1} + 2(m+1)x}\right)$$

and (5.5) follows. □

According to the notations from Section 3, taking Lemma 1 and Lemma 2 into account we have $a_m([a,b]) = 0$, for any $m \in \mathbb{N}$, $b_m([a,b])$ $= \frac{\sqrt{4m(m+1)a^2+1}-1}{2(m+1)} - a$, $l(x) = -\frac{1+x}{2}$, for any $m \in \mathbb{N}$, any $x \in [a,b]$, $b_m([a,b]) \le \sqrt{\frac{1}{2}a^2 + \frac{1}{16}} - a = M_2([a,b])$, for any $m \in \mathbb{N}$ and then $M_1([a,b]) = 0$, $k_0([a,b]) = 1$, $k_2([a,b]) = b(1+b)+2$.

As consequences of Theorem 2 we get

Theorem 6. *For any $f \in C_2([0,+\infty))$ it follows*

$$\lim_{m\longrightarrow\infty} P_m f = f \tag{5.6}$$

uniformly on compact $[a,b]$ and there exists $m(0) \in \mathbb{N}$, $m(0)$ depending on b, so that

$$|(P_m f)(x) - f(x)| \le (3 + b(1+b))\,\omega\left(f; \frac{1}{\sqrt{m}}\right), m \in \mathbb{N}_2, m \ge m(0), x \in [a,b]. \tag{5.7}$$

Theorem 7. *Let $f \in C_2([0,+\infty))$. If $x \in [a,b]$, f is two times differentiable in x and $f^{(2)}$ is continuous in x, then*

$$\lim_{m\longrightarrow\infty} m((P_m f)(x) - f(x)) = -\frac{1+x}{2} f^{(1)}(x) + \frac{x(1+x)}{2} f^{(2)}(x). \tag{5.8}$$

Proof of Theorem 7. Taking Lemma 2 into account and applying (3.29), (5.8) is obtained. □

6. $(P_m)_{m\geq m_0}$ OPERATORS PRESERVING THE TEST FUNCTIONS e_1 AND e_2

In this case $m_0\in\mathbb{N}$, $m_0\geq 2$ is a fixed number and $J=\left[\frac{1}{m_0-1},+\infty\right)$. If $P_me_1=e_1$, for any $m\in\mathbb{N}_1$, yields $v_m(x)=0$, for any $m\in\mathbb{N}_1$ and any $x\in\left[\frac{1}{m_0-1},+\infty\right)$. For $x\geq\frac{1}{m_0-1}$, we have $\frac{mx-1}{x+1}\geq\frac{m-m_0+1}{m_0}$ because the function $\frac{x+1}{mx-1}$ is decreasing on $\left[\frac{1}{m_0-1},+\infty\right)$, from where $\frac{mx-1}{x+1}>0$ for any $m\in\mathbb{N}_1$ and any $x\in\left[\frac{1}{m_0-1},+\infty\right)$. Taking (3.12) into account, from $P_me_1=e_1$ and $P_me_2=e_2$ for any $m\in\mathbb{N}_1$, we have $\frac{m+1}{m}\frac{x^2}{1+u_m(x)}+\frac{x}{m}=x^2$, for any $x\in\left[\frac{1}{m_0-1},+\infty\right)$, from where

$$u_m(x)=\frac{x+1}{mx-1}, m\in\mathbb{N}_1, x\in\left[\frac{1}{m_0-1},+\infty\right). \tag{6.1}$$

Then the operators from (3.11) become

$$(P_mf)(x) \tag{6.2}$$
$$=\frac{(m+1)x}{mx-1}\sum_{k=0}^{\infty}\binom{m+k-1}{k}\left(\frac{mx-1}{m+1}\right)^k\left(1+\frac{x-1}{m+1}\right)^{-m-k}f\left(\frac{k}{m}\right)$$

for $m\in\mathbb{N}_1$, $x\in\left[\frac{1}{m_0-1},+\infty\right)$ and $f\in C_2([0,+\infty))$.

According to the notations from Section 3, we have $b_m\left(\left[\frac{1}{m_0-1},b\right]\right)=0, l(x)=-1-x$, for any $m\in\mathbb{N}_1$, and because the function $u_m(x)=\frac{x+1}{mx-1}$ is decreasing on $\left[\frac{1}{m_0-1},+\infty\right)$, we get that

$$u_m(x)\leq\frac{m_0}{m-m_0+1}=a_m\left(\left[\frac{1}{m_0-1},b\right]\right)$$

for any $x\in\left[\frac{1}{m_0-1},b\right)$ and $M_2\left(\left[\frac{1}{m_0-1},b\right]\right)=0$. Then $k_0=1+m_0$, $k_2=b(1+b)+2$ and $M_1\left(\left[\frac{1}{m_0-1},b\right]\right)=m_0$.

Theorem 8. *For any $f\in C_2([0,+\infty))$ it follows*

$$\lim_{m\longrightarrow\infty}P_mf=f \tag{6.3}$$

uniformly on the compact $\left[\frac{1}{m_0-1},b\right]$ and there exists $m(0)\in\mathbb{N}$ depending on b, such that

$$|(P_mf)(x)-f(x)|\leq\frac{m_0}{m-m_0+1}M\left(\left[\frac{1}{m_0-1},b\right]\right)+ \tag{6.4}$$

$$+(3+m_0+b(1+b))\omega\left(f;\frac{1}{\sqrt{m}}\right)$$

for any $m \in \mathbb{N}_2$, $m \geq m(0)$ *and* $x \in \left[\frac{1}{m_0-1}, b\right]$, *where*

$$M\left(\left[\frac{1}{m_0-1}, b\right]\right) = sup\left\{|f(x)| \mid x \in \left[\frac{1}{m_0-1}, b\right]\right\}.$$

Proof of Theorem 8. It results immediately from Theorem 2. □

Theorem 9. *Let* $f \in C_2([0,+\infty))$. *If* $x \in \left[\frac{1}{m_0-1}, b\right]$, *f is two times differentiable in x and* $f^{(2)}$ *is continuous in x, then*

$$\lim_{m\longrightarrow\infty} m((P_m f)(x) - f(x)) = \frac{1+x}{x} f(x) - (1+x) f^{(1)}(x) + \frac{x(1+x)}{2} f^{(2)}(x). \tag{6.5}$$

Proof of Theorem 9. We have $\lim\limits_{m\longrightarrow\infty} m u_m(x) = \frac{1+x}{x}, l(x) = -1-x$, for any $x \in \left[\frac{1}{m_0-1}, b\right]$ and taking (3.29) into account, follows (6.5). □

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