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CONTINUITY FOR MULTILINEAR COMMUTATOR OF MARCINKIEWICZ OPERATOR ON BESOV SPACES

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Abstract. In this paper we prove the continuity for the multilinear commutator associated to the Marcinkiewicz operator on Besov spaces.

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1. Introduction

As the development of singular integral operators, their commutators and multilinear operators have been well studied (see [1–4, 14]). From [2, 4, 9, 13], we know that the commutators and multilinear operators generated by singular integral operators and the Lipschitz functions are bounded on the Triebel-Lizorkin and Lebesgue spaces. The purpose of this paper is to introduce the multilinear commutator associated to the Marcinkiewicz operator and prove the continuity properties for the multilinear commutator on Besov spaces.

2. Preliminaries and Theorems

First, let us introduce some notations. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows, $f_Q = |Q|^{-1} \int_Q f(x) dx$. It is well-known that (see [14, 15])

$$f^{\#}(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

For $\beta \geq 0$, the Besov space $\dot{\wedge}_\beta(R^n)$ is the space of functions f such that

$$||f||_{\dot{\wedge}_\beta} = \sup_{\substack{x,h \in R^n \\ h \neq 0}} \left|\Delta_h^{[\beta]+1} f(x)\right| / |h|^\beta < \infty,$$

where Δ_h^k denotes the k-th difference operator (see [12]).

For $b_j \in \dot{\wedge}_\beta(R^n)$ $(j = 1, \cdots, m)$, set

$$||\vec{b}||_{\dot{\wedge}_\beta} = \prod_{j=1}^{m} ||b_j||_{\dot{\wedge}_\beta}.$$

Given some functions b_j $(j = 1, \cdots, m)$ and a positive integer m and $1 \leq j \leq m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{\dot{\wedge}_\beta} = ||b_{\sigma(1)}||_{\dot{\wedge}_\beta} \cdots ||b_{\sigma(j)}||_{\dot{\wedge}_\beta}$.

Definition 1. Let $0 < p, q \leq \infty$, $\alpha \in R$. For $k \in Z$, set $B_k = \{x \in R^n : |x| \leq 2^k\}$ and $C_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of C_k and χ_0 the characteristic function of B_0.

(1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha, p}(R^n) = \{f \in L_{loc}^q(R^n \setminus \{0\}) : ||f||_{\dot{K}_q^{\alpha, p}} < \infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha, p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} ||f\chi_k||_{L^q}^p\right]^{1/p};$$

(2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha, p}(R^n) = \{f \in L_{loc}^q(R^n) : ||f||_{K_q^{\alpha, p}} < \infty\},$$

where

$$||f||_{K_q^{\alpha, p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p} ||f\chi_k||_{L^q}^p + ||f\chi_{B_0}||_{L^q}^p\right]^{1/p};$$

And the usual modification is made when $p = q = \infty$.

Definition 2. Let $1 \leq q < \infty$, $\alpha \in R$. The central Campanato space is defined by (see [17])

$$CL_{\alpha, q}(R^n) = \{f \in L_{loc}^q(R^n) : ||f||_{CL_{\alpha, q}} < \infty\},$$

where

$$||f||_{CL_{\alpha, q}} = \sup_{r>0} |B(0,r)|^{-\alpha} \left(\frac{1}{|B(0,r)|} \int_{B(0,r)} |f(x) - f_{B(0,r)}|^q dx\right)^{1/q}.$$

Definition 3. Let $0 < \delta < n$, $0 < \gamma \le 1$ and Ω be homogeneous of degree zero on R^n such that $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$, that is there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \le M|x-y|^\gamma$. The Marcinkiewicz multilinear commutator is defined by

$$\mu_{\Omega,\delta}^{\vec{b}}(f)(x) = \left(\int_0^\infty |F_{t,\delta}^{\vec{b}}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_{t,\delta}^{\vec{b}}(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} \left[\prod_{j=1}^m (b_j(x) - b_j(y))\right] f(y)dy.$$

Set

$$F_{t,\delta}(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy,$$

we also define that

$$\mu_{\Omega,\delta}(f)(x) = \left(\int_0^\infty |F_{t,\delta}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

which is the Marcinkiewicz operator (see [16]).

Let H be the space $H = \left\{h : ||h|| = \left(\int_0^\infty |h(t)|^2 \frac{dt}{t^3}\right)^{1/2} < \infty\right\}$. Then, it is clear that

$$\mu_{\Omega,\delta}(f)(x) = ||F_{t,\delta}(f)(x)|| \text{ and } \mu_{\Omega,\delta}^{\vec{b}}(f)(x) = ||F_{t,\delta}^{\vec{b}}(f)(x)||.$$

Note that when $b_1 = \cdots = b_m$, $\mu_{\Omega,\delta}^{\vec{b}}$ is just the m order commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1–9, 11–13]). Our main purpose is to study the boundedness properties for the multilinear commutator $\mu_{\Omega,\delta}^{\vec{b}}$ on Besov spaces.

Now we state our theorems as following.

Theorem 1. *Let* $0 < \delta < n$, $1 < r < n/\delta$, $1/s = 1/r - \delta/n$, $0 < \beta < \min(1/2m, \gamma/m)$ *for* $0 < \gamma \le 1$, *and* $b_j \in \dot{\wedge}_\beta(R^n)$ *for* $j = 1, \cdots, m$. *Then* $\mu_{\Omega,\delta}^{\vec{b}}$ *is bounded from* $L^p(R^n)$ *to* $\dot{\wedge}_{\delta+m\beta-n/p}(R^n)$ *for any* $n/(\delta+m\beta) \le p \le n/\delta$.

Theorem 2. *Let* $0 < \delta < n$, $0 < \beta < \min(1/2m, \gamma/m)$ *for* $0 < \gamma \le 1$, $1 < q_1 < n/(\delta + m\beta)$, $1/q_2 = 1/q_1 - (\delta + m\beta)/n$, $max(-n/q_2 - 1/2, -n/q_2 - \gamma) < \alpha \le -n/q_2$ *and* $b_j \in \dot{\wedge}_\beta(R^n)$ *for* $j = 1, \cdots, m$. *Then* $\mu_{\Omega,\delta}^{\vec{b}}$ *is bounded from* $\dot{K}_{q_1}^{\alpha,\infty}(R^n)$ *to* $CL_{-\alpha/n-1/q_2,q_2}(R^n)$.

Remark 1. Theorem 2 also hold for the nonhomogeneous Herz type Hardy space.

3. PROOFS OF THEOREMS

To prove the theorems, we need the following lemmas.

Lemma 1 ([12]). *For $0<\beta<1, 1\le p\le\infty$, we have*

$$\begin{aligned}||b||_{\dot\wedge_\beta} &\approx \sup_Q \frac{1}{|Q|^{1+\beta/n}}\int_Q |b(x)-b_Q|dx \approx \sup_Q \frac{1}{|Q|^{\beta/n}}\left(\frac{1}{|Q|}\int_Q |b(x)-b_Q|^p dx\right)^{1/p}\\ &\approx \sup_Q \inf_c \frac{1}{|Q|^{1+\beta/n}}\int_Q |b(x)-c|dx \approx \sup_Q \inf_c \frac{1}{|Q|^{\beta/n}}\left(\frac{1}{|Q|}\int_Q |b(x)-c|^p dx\right)^{1/p}.\end{aligned}$$

Lemma 2 ([10]). *For $\alpha<0, 0<q<\infty$, we have*

$$||f||_{\dot K_q^{\alpha,\infty}} \approx \sup_{\mu\in Z} 2^{\mu\alpha}||f\chi_{B_\mu}||_{L^q}.$$

Lemma 3 ([10]). *Let $0<\eta<n$, $1<p<n/\eta$. Suppose $b\in\dot\wedge_\beta(R^n)$, then*

$$|b_{2^{k+1}B}-b_B|\le C||b||_{\dot\wedge_\beta} k|2^{k+1}B|^{\beta/n} \text{ for } k\ge 1.$$

Lemma 4 ([16]). *Let $0<\delta<n$, $1<p<n/\delta$ and $1/q=1/p-\delta/n$. Then $\mu_{\Omega,\delta}$ is bounded from $L^p(R^n)$ to $L^q(R^n)$.*

Lemma 5 ([7]). *Let $0\le\eta<n$, $1<r<n/\eta$, $1/r-1/s=\eta/n$ and $b_j\in\dot\wedge_\beta(R^n)$ for $j=1,\cdots,m$. Then $\mu_{\Omega,\delta}^{\vec b}$ is bounded from $L^r(R^n)$ to $L^s(R^n)$.*

Proof of Theorem 1. It is only to prove that there exists a constant C_0 such that

$$\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}}\int_Q |\mu_{\Omega,\delta}^{\vec b}(f)(x)-C_0|dx\le C||f||_{L^p}.$$

Fix a cube Q, $Q=Q(x_0,d)$, we decompose f into $f=f_1+f_2$ with $f_1=f\chi_Q$, $f_2=f\chi_{(R^n\setminus Q)}$.

When $m=1$, for $C_0=\mu_{\Omega,\delta}(((b_1)_Q-b_1)f_2)(x_0)$, we have

$$\begin{aligned}F_{t,\delta}^{b_1}(f)(x) &= (b_1(x)-(b_1)_Q)F_{t,\delta}(f)(x)\\ &\quad -F_{t,\delta}((b_1-(b_1)_Q)f_1)(x)-F_{t,\delta}((b_1-(b_1)_Q)f_2)(x).\end{aligned}$$

Then

$$\begin{aligned}&|\mu_{\Omega,\delta}^{b_1}(f)(x)-\mu_{\Omega,\delta}(((b_1)_Q-b_1)f_2)(x_0)|\\ &=\Big|||F_{t,\delta}^{b_1}(f)(x)||-||F_{t,\delta}(((b_1)_Q-b_1)f_2)(x_0)||\Big|\\ &\le ||(b_1(x)-(b_1)_Q)F_{t,\delta}(f)(x)||+||F_{t,\delta}((b_1-(b_1)_Q)f_1)(x)||\\ &\quad +||F_{t,\delta}((b_1-(b_1)_Q)f_2)(x)-F_{t,\delta}((b_1-(b_1)_Q)f_2)(x_0)||\\ &=A(x)+B(x)+C(x).\end{aligned}$$

For $A(x)$, for $1 < p < q < n/\delta$, $1/q = 1/p - \delta/n$, by the boundness of $\mu_{\Omega,\delta}$ from $L^p(R^n)$ to $L^q(R^n)$ and Hölder's inequality with exponent$1/q + 1/q' = 1$ and Lemma 1, we have

$$\begin{aligned}
&\frac{1}{|Q|^{1+(\delta+\beta)/n-1/p}} \int_Q |A(x)| dx \\
&\leq \frac{1}{|Q|^{1+(\delta+\beta)/n-1/p}} \left(\int_Q |b_1(x) - (b_1)_Q|^{q'} dx \right)^{1/q'} \left(\int_Q |\mu_{\Omega,\delta}(f)(x)|^q dx \right)^{1/q} \\
&\leq C \frac{|Q|^{\beta/n+1/q'}}{|Q|^{1+(\delta+\beta)/n-1/p}} \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |b_1(x) - (b_1)_Q|^{q'} dx \right)^{1/q'} \left(\int_Q |f(x)|^p dx \right)^{1/p} \\
&\leq C ||b_1||_{\dot{\wedge}_\beta} ||f||_{L^p}.
\end{aligned}$$

For $B(x)$, denoting $p = rt$, $1 < r < s < n/\delta$, $1/s = 1/r - \delta/n$, by the boundness of $\mu_{\Omega,\delta}$ from $L^r(R^n)$ to $L^s(R^n)$ and Hölder's inequality with exponent $1/t + 1/t' = 1$ and Lemma 1, we have

$$\begin{aligned}
&\frac{1}{|Q|^{1+(\delta+\beta)/n-1/p}} \int_Q |B(x)| dx \\
&\leq \frac{1}{|Q|^{(\delta+\beta)/n-1/p}} \left(\frac{1}{|Q|} \int_{R^n} |\mu_{\Omega,\delta}((b_1(x) - (b_1)_Q) f \chi_Q)(x)|^s dx \right)^{1/s} \\
&\leq C \frac{1}{|Q|^{(\delta+\beta)/n-1/p+1/s}} \left(\int_Q |(b_1(x) - (b_1)_Q) f(x)|^r dx \right)^{1/r} \\
&\leq C \frac{1}{|Q|^{(\delta+\beta)/n-1/p+1/s}} \left(\int_Q |b_1(x) - (b_1)_Q|^{rt'} dx \right)^{1/rt'} \left(\int_Q |f(x)|^{rt} dx \right)^{1/rt} \\
&\leq C ||b_1||_{\dot{\wedge}_\beta} ||f||_{L^p}.
\end{aligned}$$

For $C(x)$, note that $|x_0 - y| \approx |x - y|$ for $y \in Q^c$, we have

$$\begin{aligned}
C(x) &= ||F_{t,\delta}((b_1 - (b_1)_Q) f_2)(x) - F_{t,\delta}((b_1 - (b_1)_Q) f_2)(x_0)|| \\
&= \left(\int_0^\infty \left| \int_{|x-y| \leq t} \frac{\Omega(x-y) f_2(y)}{|x-y|^{n-1-\delta}} (b_1(y) - (b_1)_Q) dy \right.\right. \\
&\qquad \left.\left. - \int_{|x_0-y| \leq t} \frac{\Omega(x_0-y) f_2(y)}{|x_0-y|^{n-1-\delta}} (b_1(y) - (b_1)_Q) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
&\leq \left(\int_0^\infty \left[\int_{|x-y| \leq t, |x_0-y| > t} \frac{|\Omega(x-y)||f_2(y)|}{|x-y|^{n-1-\delta}} |(b_1(y) - (b_1)_Q)| dy \right]^2 \frac{dt}{t^3} \right)^{1/2} \\
&\quad + \left(\int_0^\infty \left[\int_{|x-y| > t, |x_0-y| \leq t} \frac{|\Omega(x_0-y)||f_2(y)|}{|x_0-y|^{n-1-\delta}} \left| b_1(y) - (b_1)_Q \right| dy \right]^2 \frac{dt}{t^3} \right)^{1/2} \\
&\quad + (\int_0^\infty [\int_{|x-y| \leq t, |x_0-y| \leq t} \left| \frac{|\Omega(x-y)|}{|x-y|^{n-1-\delta}} - \frac{|\Omega(x_0-y)|}{|x_0-y|^{n-1-\delta}} \right|
\end{aligned}$$

$$\times |b_1(y)-(b_1)_Q||f_2(y)|dy]^2\frac{dt}{t^3})^{1/2}$$
$$\equiv I_1+I_2+I_3.$$

By the Minkowski's inequality and Hölder's inequality and Lemmas 1, 3:

$$\begin{aligned}
I_1 &\le C\int_{Q^c}|b_1(y)-(b_1)_Q|\frac{|f(y)|}{|x-y|^{n-1-\delta}}\left(\int_{|x-y|\le t<|x_0-y|}\frac{dt}{t^3}\right)^{1/2}dy\\
&\le C\int_{Q^c}|b_1(y)-(b_1)_Q|\frac{|f(y)|}{|x-y|^{n-1-\delta}}\left|\frac{1}{|x-y|^2}-\frac{1}{|x_0-y|^2}\right|^{1/2}dy\\
&\le C\sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}|b_1(y)-(b_1)_{2Q}||x_0-y|^{\delta}\frac{|Q|^{1/2n}|f(y)|}{|x_0-y|^{n+1/2}}dy\\
&\le C\sum_{k=0}^{\infty}2^{-k/2}\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\left(\int_{2^{k+1}Q}|f(y)|^pdy\right)^{1/p}\\
&\quad\times\left[\int_{2^{k+1}Q}(|b_1(y)-(b_1)_{2^{k+1}Q}|^{p'}+|(b_1)_{2^{k+1}Q}-(b_1)_Q|^{p'})dy\right]^{1/p'}\\
&\le C\sum_{k=0}^{\infty}2^{-k/2}\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\left(\int_{2^{k+1}Q}|f(y)|^pdy\right)^{1/p}\\
&\quad\times\Bigg[|2^{k+1}Q|^{\beta/n+1/p'}\frac{1}{|2^{k+1}Q|^{\beta/n}}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b_1(y)-(b_1)_{2^{k+1}Q}|^{p'}dy\right)^{1/p'}\\
&\quad+|(b_1)_{2^{k+1}Q}-(b_1)_Q||2^{k+1}Q|^{1/p'}\Bigg]\\
&\le C\sum_{k=0}^{\infty}k2^{k(-1/2+\delta+\beta-n/p)}|Q|^{\beta/n+1/p'-1+\delta/n}||b_1||_{\dot{\Lambda}_\beta}||f||_{L^p}\\
&\le C|Q|^{(\delta+\beta)/n-1/p}|||b_1||_{\dot{\Lambda}_\beta}||f||_{L^p}.
\end{aligned}$$

Similarly, we have $I_2\le C|Q|^{(\delta+\beta)/n-1/p}|||b_1||_{\dot{\Lambda}_\beta}||f||_{L^p}$.
We now estimate I_3. By the following inequality:

$$\left|\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n-1-\delta}}\right|\le C\left(\frac{|x-x_0|}{|x_0-y|^{n-\delta}}+\frac{|x-x_0|^{\gamma}}{|x_0-y|^{n-1-\delta+\gamma}}\right),$$

we obtain

$$\begin{aligned}
I_3 &\le C\int_{Q^c}|b_1(y)-(b_1)_Q|\frac{|f(y)||x-x_0|}{|x_0-y|^{n-\delta}}\left(\int_{|x_0-y|\le t,|x-y|\le t}\frac{dt}{t^3}\right)^{1/2}dy\\
&\quad+C\int_{Q^c}|b_1(y)-(b_1)_Q|\frac{|f(y)||x-x_0|^{\gamma}}{|x_0-y|^{n-1-\delta+\gamma}}\left(\int_{|x_0-y|\le t,|x-y|\le t}\frac{dt}{t^3}\right)^{1/2}dy\\
&\le C\sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}|b_1(y)-(b_1)_Q|\left(\frac{|Q|^{1/n}}{|x_0-y|^{n+1-\delta}}+\frac{|Q|^{\gamma/n}}{|x_0-y|^{n+\gamma-\delta}}\right)|f(y)|dy
\end{aligned}$$

$$\le C\sum_{k=0}^{\infty}(2^{-k}+2^{-k\gamma})\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\int_{2^{k+1}Q}|b_1(y)-(b_1)_Q||f(y)|dy$$
$$\le C\sum_{k=0}^{\infty}(2^{-k}+2^{-k\gamma})\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\left(\int_{2^{k+1}Q}|f(y)|^p dy\right)^{1/p}$$
$$\times\left[|2^{k+1}Q|^{\beta/n+1/p'}\frac{1}{|2^{k+1}Q|^{\beta/n}}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b_1(y)-(b_1)_{2^{k+1}Q}|^{p'}dy\right)^{1/p'}\right.$$
$$\left.+|(b_1)_{2^{k+1}Q}-(b_1)_Q||2^{k+1}Q|^{1/p'}\right]$$
$$\le C\sum_{k=0}^{\infty}k(2^{k(-1+\delta+\beta-n/p)}+2^{k(-\gamma+\delta+\beta-n/p)})|Q|^{\beta/n+1/p'-1+\delta/n}||b_1||_{\dot{\wedge}_\beta}||f||_{L^p}$$
$$\le C|Q|^{(\delta+\beta)/n-1/p}|||b_1||_{\dot{\wedge}_\beta}||f||_{L^p}.$$

Thus

$$\frac{1}{|Q|^{1+(\delta+\beta)/n-1/p}}\int_Q|C(x)|dx$$
$$\le C||b_1||_{\dot{\wedge}_\beta}||f||_{L^p}\frac{1}{|Q|^{1+(\delta+\beta)/n-1/p}}\int_Q|Q|^{(\delta+\beta)/n-1/p}dx\le C||b_1||_{\dot{\wedge}_\beta}||f||_{L^p}.$$

This completes the proof of case $m=1$.
Now, we consider the case $m\ge 2$. We write, for $\vec{b}=(b_1,...,b_m)$,

$$F_{t,\delta}^{\vec{b}}(f)(x)=\int_{|x-y|\le t}\left[\prod_{j=1}^{m}((b_j(x)-(b_j)_Q)-(b_j(y)-(b_j)_Q))\right]f(y)\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}dy$$
$$=\sum_{j=0}^{m}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_Q)_\sigma\int_{|x-y|\le t}(b(y)-(b)_Q)_{\sigma^c}f(y)\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}dy$$
$$=\prod_{j=1}^{m}(b_j(x)-(b_j)_Q F_{t,\delta}(f)(x)+(-1)^m F_{t,\delta}(\prod_{j=1}^{m}(b_j(y)-(b_j)_Q)f)(x)$$
$$+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_Q)_\sigma F_{t,\delta}^{\vec{b}_{\sigma^c}}(f)(x),$$

thus, set $C_0=\mu_{\Omega,\delta}(\prod_{j=1}^{m}(b_j-(b_j)_Q)f_2)(x_0)$,

$$|\mu_{\Omega,\delta}^{\vec{b}}(f)(x)-\mu_{\Omega,\delta}(\prod_{j=1}^{m}(b_j-(b_j)_Q)f_2)(x_0)|$$

$$
\begin{aligned}
&\le ||\prod_{j=1}^{m}(b_j(x)-(b_j)_Q)F_{t,\delta}(f)(x)|| + \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||(b(x)-(b)_Q)_\sigma F_{t,\delta}^{\vec{b}_{\sigma^c}}(f)(x)|| \\
&\quad + ||F_{t,\delta}(\prod_{j=1}^{m}(b_j-(b_j)_Q)f_1)(x)|| + ||F_{t,\delta}(\prod_{j=1}^{m}(b_j-(b_j)_Q)f_2)(x) \\
&\quad - F_{t,\delta}(\prod_{j=1}^{m}(b_j-(b_j)_Q)f_2)(x_0)|| \\
&= S_1(x)+S_2(x)+S_3(x)+S_4(x).
\end{aligned}
$$

For $S_1(x)$, for $1<p<q<n/\delta$, $1/q=1/p-\delta/n$, by the boundness of $\mu_{\Omega,\delta}$ from $L^p(R^n)$ to $L^q(R^n)$ and Hölder's inequality with exponent$1/q_1'+\cdots+1/q_m'+1/q=1$ and Lemma 1, we have

$$
\begin{aligned}
&\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}}\int_Q |S_1(x)|dx \\
&\le C\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}}\prod_{j=1}^{m}\left(\int_Q |b_j(x)-(b_j)_Q|^{q_j'}dx\right)^{1/q_j'}\left(\int_Q |\mu_{\Omega,\delta}(f_1)|^q dx\right)^{1/q} \\
&\le C\frac{|Q|^{m\beta/n+1/q_1'+\cdots q_m'}}{|Q|^{1+(\delta+m\beta)/n-1/p}}\prod_{j=1}^{m}\frac{1}{|Q|^{m\beta/n}}\left(\frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_Q|^{q_j'}dx\right)^{1/q_j'} \\
&\quad \times\left(\int_Q |f(x)|^p dx\right)^{1/p} \\
&\le C||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{L^p}.
\end{aligned}
$$

For $S_2(x)$, denoting $p=rt$, $1<r<s<n/\delta$, $1/s=1/r-\delta/n$, by the boundness of $\mu_{\Omega,\delta}$ from $L^r(R^n)$ to $L^s(R^n)$ and Lemma 1, we have

$$
\begin{aligned}
&\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}}\int_Q |S_2(x)|dx \\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}}\left(\int_Q |b(x)-(b_Q)_\sigma|^{s'}dx\right)^{1/s'} \\
&\quad \times\left(\int_Q \mu_{\Omega,\delta}((b-b_Q)_{\sigma^c}f)(x)|^s dx\right)^{1/s} \\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{|Q|^{|\sigma|\beta/n+1/s'}}{|Q|^{1+(\delta+m\beta)/n-1/p}}\frac{1}{|Q|^{|\sigma|\beta/n}}\left(\frac{1}{|Q|}\int_Q |b(x)-(b_Q)_\sigma|^{s'}dx\right)^{1/s'}
\end{aligned}
$$

$$\times \left(\int_Q |(b(x)-b_Q)_{\sigma^c} f(x)|^r dx \right)^{1/r}$$

$$\le C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{|Q|^{m\beta/n+1/s'+1/rt'}}{|Q|^{1+(\delta+m\beta)/n-1/p}} ||b_\sigma||_{\dot{\wedge}_\beta} ||b_{\sigma^c}||_{\dot{\wedge}_\beta} ||f||_{L^p}$$

$$\le C||\vec{b}||_{\dot{\wedge}_\beta} ||f||_{L^p}.$$

For $S_3(x)$, for $1 < r < s < n/\delta$, $1/s = 1/r - \delta/n$, by the boundness of $\mu_{\Omega,\delta}$ from $L^r(R^n)$ to $L^s(R^n)$, taking $p = rt$, $1 < r < p < \infty$, we have

$$\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}} \int_Q |S_3(x)| dx$$

$$\le \frac{1}{|Q|^{(\delta+m\beta)/n-1/p}} \left(\frac{1}{|Q|} \int_{R^n} |\mu_{\Omega,\delta}(\prod_{j=1}^m ((b_j - (b_j)_Q)) f \chi_B)(x)|^s dx \right)^{1/s}$$

$$\le C \frac{1}{|Q|^{(\delta+m\beta)/n-1/p+1/s}} \left(\int_Q |\prod_{j=1}^m (b_j(x) - (b_j)_Q) f(x)|^r dx \right)^{1/r}$$

$$\le C \frac{1}{|Q|^{(\delta+m\beta)/n-1/p+1/s}} \prod_{j=1}^m \left(\int_Q |b_j(x) - (b_j)_Q|^{rt_j} dx \right)^{1/rt_j} \left(\int_Q |f(x)|^{rt} dx \right)^{1/rt}$$

$$\le C||\vec{b}||_{\dot{\wedge}_\beta} ||f||_{L^p}.$$

For $S_4(x)$, similar to the proof of $C(x)$ in the case $m = 1$, we get

$$S_4(x) \le \left(\int_0^\infty \left[\int_{|x-y|\le t, |x_0-y|>t} \frac{|\Omega(x-y)||f_2(y)|}{|x-y|^{n-1-\delta}} \left| \prod_{j=1}^m (b_j(y) - (b_j)_Q) \right| dy \right]^2 \frac{dt}{t^3} \right)^{1/2}$$

$$+ \left(\int_0^\infty \left[\int_{|x-y|>t, |x_0-y|\le t} \frac{|\Omega(x_0-y)||f_2(y)|}{|x_0-y|^{n-1-\delta}} \left| \prod_{j=1}^m (b_j(y) - (b_j)_Q) \right| dy \right]^2 \frac{dt}{t^3} \right)^{1/2}$$

$$+ (\int_0^\infty [\int_{|x-y|\le t, |x_0-y|\le t} \left| \frac{|\Omega(x-y)|}{|x-y|^{n-1-\delta}} - \frac{|\Omega(x_0-y)|}{|x_0-y|^{n-1-\delta}} \right|$$

$$\times \left| \prod_{j=1}^m (b_j(y) - (b_j)_Q) \right| |f_2(y)| dy]^2 \frac{dt}{t^3})^{1/2}$$

$$\equiv V_1 + V_2 + V_3,$$

Thus, we choose $1 < p_j < \infty$, $j = 1, \cdots, m$, $1/p_1 + \cdots + 1/p_m + 1/p = 1$ and get

$$V_1 \le C \sum_{k=0}^\infty \int_{2^{k+1}Q \setminus 2^k Q} \left| \prod_{j=1}^m (b_j(y) - (b_j)_Q) \right| |x_0 - y|^\delta \frac{|Q|^{1/2n} |f(y)|}{|x_0 - y|^{n+1/2}} dy$$

$$\leq C\sum_{k=0}^{\infty}2^{-k/2}\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\prod_{j=1}^{m}\Big[|2^{k+1}Q|^{m\beta/n+1/p'}||b_j||_{\dot{\Lambda}_\beta}$$
$$+k|2^{k+1}Q|^{m\beta/n+1/p'}||b_j||_{\dot{\Lambda}_\beta}\Big]||f||_{L^p}$$
$$\leq C\sum_{k=0}^{\infty}k2^{k(-1/2+\delta+m\beta-n/p)}|Q|^{m\beta/n+1/p'-1+\delta/n}||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{L^p}$$
$$\leq C|Q|^{(\delta+m\beta)/n-1/p}|||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{L^p}.$$

Similarly, we have $V_2\leq C|Q|^{(\delta+m\beta)/n-1/p}||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{L^p}$. For V_3, we obtain

$$V_3\leq C\sum_{k=0}^{\infty}\int_{2^{k+1}Q\backslash 2^kQ}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_Q)\right|\left(\frac{|Q|^{1/n}}{|x_0-y|^{n+1-\delta}}+\frac{|Q|^{\gamma/n}}{|x_0-y|^{n+\gamma-\delta}}\right)|f(y)|dy$$
$$\leq C\sum_{k=0}^{\infty}(2^{-k}+2^{-k\gamma})\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\int_{2^{k+1}Q}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_Q)\right||f(y)|dy$$
$$\leq C\sum_{k=0}^{\infty}k(2^{k(-1+\delta+m\beta-n/p)}+2^{k(-\gamma+\delta+m\beta-n/p)})$$
$$|2^{k+1}Q|^{m\beta/n+1/p'-1+\delta/n}||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{L^p}$$
$$\leq C|Q|^{(\delta+m\beta)/n-1/p}||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{L^p}.$$

Thus

$$\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}}\int_Q|S_4(x)|dx$$
$$\leq C||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{L^p}\frac{1}{|Q|^{1+(\delta+m\beta)/n-1/p}}\int_Q|Q|^{(\delta+m\beta)/n-1/p}dx\leq C||\vec{b}||_{\dot{\Lambda}_\beta}||f||_{L^p}.$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. Fix a ball $B=B(0,l)$, there exists $\epsilon_0\in\mathbf{Z}$ such that $2^{\epsilon_0-1}\leq l<2^{\epsilon_0}$. We choose x_0 such that $2l<|x_0|<3l$. It is only to prove that

$$2^{\epsilon_0(\alpha+n/q_2)}\left(\frac{1}{2^{\epsilon_0 n}}\int_{|x|<2^{\epsilon_0}}|\mu^{\vec{b}}_{\Omega,\delta}(f)(x)-\mu^{\vec{b}}_{\Omega,\delta}(f_2)(x_0)|^{q_2}dx\right)^{1/q_2}\leq C||f||_{\dot{K}^{\alpha,\infty}_{q_1}}.$$

We write, for $f_1=f\chi_{4B_{\epsilon_0}}$ and $f_2=f\chi_{R^n\backslash 4B_{\epsilon_0}}$, then

$$|\mu^{\vec{b}}_{\Omega,\delta}(f)(x)-\mu^{\vec{b}}_{\Omega,\delta}(f_2)(x_0)|\leq|\mu^{\vec{b}}_{\Omega,\delta}(f_1)(x)|+|\mu^{\vec{b}}_{\Omega,\delta}(f_2)(x)-\mu^{\vec{b}}_{\Omega,\delta}(f_2)(x_0)|.$$

So

$$
\begin{aligned}
&2^{\epsilon_0(\alpha+n/q_2)}\left(\frac{1}{2^{\epsilon_0 n}}\int_{|x|<2^{\epsilon_0}}|\mu^{\vec{b}}_{\Omega,\delta}(f)(x)-\mu^{\vec{b}}_{\Omega,\delta}(f_2)(x_0)|^{q_2}dx\right)^{1/q_2}\\
&\le 2^{\epsilon_0(\alpha+n/q_2)}\left(\frac{1}{2^{\epsilon_0 n}}\int_{|x|<2^{\epsilon_0}}|\mu^{\vec{b}}_{\Omega,\delta}(f_1)(x)|^{q_2}dx\right)^{1/q_2}\\
&\quad+2^{\epsilon_0(\alpha+n/q_2)}\left(\frac{1}{2^{\epsilon_0 n}}\int_{|x|<2^{\epsilon_0}}|\mu^{\vec{b}}_{\Omega,\delta}(f_2)(x)-\mu^{\vec{b}}_{\Omega,\delta}(f_2)(x_0)|^{q_2}dx\right)^{1/q_2}\\
&=J_1+J_2.
\end{aligned}
$$

For J_1, by the (L^{q_1},L^{q_2})-boundedness of $\mu^{\vec{b}}_{\Omega,\delta}$ (see Lemma 4) and Lemma 2, we get

$$
\begin{aligned}
J_1&\le C2^{\epsilon_0(\alpha+n/q_2)}2^{-\epsilon_0 n/q_2}\left(\int_{R^n}|f_1(x)|^{q_1}dx\right)^{1/q_1}\\
&\le C2^{\epsilon_0\alpha}||f\chi_{B_{\epsilon_0}}||_{L^{q_1}}\le C||f||_{\dot{K}^{\alpha,\infty}_{q_1}}.
\end{aligned}
$$

For J_2, similar to the estimates of Theorem 1, we obtain using Hölder's inequality and recalling that $\max(-n/q_2-1/2,-n/q_2-\gamma)<\alpha\le -n/q_2$, $1/q_2=1/q_1-(\delta+m\beta)/n$,

$$
\begin{aligned}
&|\mu^{\vec{b}}_{\Omega,\delta}(f_2)(x)-\mu^{\vec{b}}_{\Omega,\delta}(f_2)(x_0)|\\
&\le|\mu_{\Omega,\delta}(\prod_{j=1}^m(b_j-(b_j)_B)(f_2)(x)-\mu_{\Omega,\delta}(\prod_{j=1}^m(b_j-(b_j)_B)(f_2)(x_0)|\\
&\quad+|\prod_{j=1}^m(b_j(x)-(b_j)_B)||\mu_{\Omega,\delta}(f_2)(x)-\mu_{\Omega,\delta}(f_2)(x_0)|=W_1(x)+W_2(x).
\end{aligned}
$$

For $W_1(x)$, similar to the proof of $S_4(x)$ in Theorem 1, set $1/v_1+\cdots+1/v_m+1/q_1=1$, by the Minkowski's inequality, we have

$$W_1(x)\le V_1+V_2+V_3.$$

For V_1, we have

$$
\begin{aligned}
V_1&\le C\sum_{k=1}^\infty\int_{B_{\epsilon_0+k}}\left|\prod_{j=1}^m(b_j(y)-(b_j)_Q)\right|\frac{|Q|^{1/2n}|f(y)|}{|x_0-y|^{n+1/2-\delta}}dy\\
&\le C\sum_{k=1}^\infty\int_{B_{\epsilon_0+k}}\frac{2^{\epsilon_0/2}}{2^{(\epsilon_0+k)(n+1/2-\delta)}}|\prod_{j=1}^m(b_j(y)-(b_j)_B)||f(y)|dy
\end{aligned}
$$

$$\le C\sum_{k=1}^{\infty}2^{k(\delta+m\beta-n/q_1-\alpha-1/2)}2^{\epsilon_0(\delta+m\beta-n/q_1-\alpha)}||\vec{b}||_{\dot{\wedge}_\beta}2^{(\epsilon_0+k)\alpha}||f\chi_{\epsilon_0+k}||_{L^{q_1}}$$
$$\le C2^{\epsilon_0(-n/q_2-\alpha)}||\vec{b}||_{\dot{\wedge}_\beta}||f||_{\dot{K}^{\alpha,\infty}_{q_1}}.$$

Similarly, we have $V_2\le C2^{\epsilon_0(-n/q_2-\alpha)}||\vec{b}||_{\dot{\wedge}_\beta}||f||_{\dot{K}^{\alpha,\infty}_{q_1}}$.
For V_3, we obtain

$$V_3\le C\sum_{k=1}^{\infty}\int_{B_{\epsilon_0+k}}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_Q)\right|\left(\frac{|Q|^{1/n}}{|x_0-y|^{n+1-\delta}}+\frac{|Q|^{\gamma/n}}{|x_0-y|^{n+\gamma-\delta}}\right)|f(y)|dy$$
$$\le C\sum_{k=1}^{\infty}\int_{B_{\epsilon_0+k}}\left(\frac{2^{\epsilon_0}}{2^{(\epsilon_0+k)(n+1-\delta)}}+\frac{2^{\epsilon_0\gamma}}{2^{(\epsilon_0+k)(n+\gamma-\delta)}}\right)|\prod_{j=1}^{m}(b_j(y)-(b_j)_B)||f(y)|dy$$
$$\le C\sum_{k=1}^{\infty}(2^{k(m\beta+n-n/q_1-n+\delta-1)}+2^{k(m\beta+n-n/q_1-n+\delta-\gamma)})$$
$$\times 2^{\epsilon_0(m\beta+n-n/q_1-n+\delta)}||\vec{b}||_{\dot{\wedge}_\beta}||f\chi_{\epsilon_0+k}||_{L^{q_1}}$$
$$\le C2^{\epsilon_0(-n/q_2-\alpha)}||\vec{b}||_{\dot{\wedge}_\beta}||f||_{\dot{K}^{\alpha,\infty}_{q_1}}.$$

For $W_2(x)$, we have,

$$||F_{t,\delta}(f_2)(x)-F_{t,\delta}(f_2)(x_0)||$$
$$\le\left(\int_0^\infty\left[\int_{|x-y|\le t,|x_0-y|>t}\frac{|\Omega(x-y)||f_2(y)|}{|x-y|^{n-1-\delta}}dy\right]^2\frac{dt}{t^3}\right)^{1/2}$$
$$+\left(\int_0^\infty\left[\int_{|x-y|>t,|x_0-y|\le t}\frac{|\Omega(x_0-y)||f_2(y)|}{|x_0-y|^{n-1-\delta}}dy\right]^2\frac{dt}{t^3}\right)^{1/2}$$
$$+\left(\int_0^\infty\left[\int_{|x-y|\le t,|x_0-y|\le t}\left|\frac{|\Omega(x-y)|}{|x-y|^{n-1-\delta}}-\frac{|\Omega(x_0-y)|}{|x_0-y|^{n-1-\delta}}\right||f_2(y)|dy\right]^2\frac{dt}{t^3}\right)^{1/2}$$
$$=V_1'+V_2'+V_3'.$$

For V_3', similar to the proof of I_3 in Theorem 1, we get

$$V_3'\le C\sum_{k=1}^{\infty}\int_{B_{\epsilon_0+k}}\left(\frac{|Q|^{1/n}}{|x_0-y|^{n+1-\delta}}+\frac{|Q|^{\gamma/n}}{|x_0-y|^{n+\gamma-\delta}}\right)|f(y)|dy$$
$$\le C\sum_{k=1}^{\infty}\int_{B_{\epsilon_0+k}}\left(\frac{2^{\epsilon_0}}{2^{(\epsilon_0+k)(n+1-\delta)}}+\frac{2^{\epsilon_0\gamma}}{2^{(\epsilon_0+k)(n+\gamma-\delta)}}\right)2^{(\epsilon_0+k)(n-n/q_1)}$$
$$\left(\int_{B_{\epsilon_0+k}}|f(y)|^{q_1}dy\right)^{1/q_1}$$

$$\leq C\sum_{k=1}^{\infty}(2^{k(\delta-n/q_1-\alpha-1)}+2^{k(\delta-n/q_1-\alpha-\gamma)})2^{\epsilon_0(\delta-n/q_1-\alpha)}2^{(\epsilon_0+k)\alpha}||f\chi_{\epsilon_0+k}||_{L^{q_1}}$$
$$\leq C2^{\epsilon_0(\delta-\alpha-n/q_1)}||f||_{\dot{K}_{q_1}^{\alpha,\infty}}.$$

The proofs of V_1', V_2' are similar to that of V_3' , we omit the details. Thus, by Hölder's inequality with $1/v_1+\cdots+1/v_m=1$, we have

$$2^{\epsilon_0(\alpha+n/q_2)}\left(\frac{1}{2^{\epsilon_0 n}}\int_{|x|<2^{\epsilon_0}}|W_2(x)|^{q_2}dx\right)^{1/q_2}$$
$$\leq C2^{\epsilon_0(\alpha+n/q_2)}2^{\epsilon_0(\delta-\alpha-n/q_1)}\left(\frac{1}{2^{\epsilon_0 n}}\int_{|x|<2^{\epsilon_0}}|\prod_{j=1}^{m}(b_j(x)-(b_j)_B)|^{q_2}dx\right)^{1/q_2}||f||_{\dot{K}_{q_1}^{\alpha,\infty}}$$
$$\leq C2^{\epsilon_0(\alpha+n/q_2)}2^{\epsilon_0(\delta-\alpha-n/q_1)}2^{\epsilon_0(m\beta+n/q_2(1/v_1+\cdots+1/v_m))}$$
$$\times\left(\frac{1}{2^{\epsilon_0 n}}\frac{1}{|B|^{m\beta/n}}\prod_{j=1}^{m}\left(\frac{1}{|B|}\int_{|x|<2^{\epsilon_0}}|(b_j(x)-(b_j)_B)|^{q_2v_j}dx\right)^{1/q_2v_j}\right)||f||_{\dot{K}_{q_1}^{\alpha,\infty}}$$
$$\leq C||\vec{b}||_{\dot{\wedge}_\beta}||f||_{\dot{K}_{q_1}^{\alpha,\infty}}.$$

Thus
$$J_2\leq C||\vec{b}||_{\dot{\wedge}_\beta}||f||_{\dot{K}_{q_1}^{\alpha,\infty}}.$$

This completes the proof of Theorem 2. □

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