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Abdelmalek Azizi and Mohammed Talbi

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GALOIS GROUP FOR SOME CLASS FIELD

ABDELMALEK AZIZI AND MOHAMMED TALBI

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Abstract. For some imaginary quartic cyclic fields $\mathbb{K}$, we will study the capitulation problem of the 2-class ideals of $\mathbb{K}$ and we will determine the structure of the Galois group of the second Hilbert 2-class field of $\mathbb{K}$ over $\mathbb{K}$.

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Keywords: cyclic quartic fields, class group, Hilbert class field

1. INTRODUCTION

Let $\mathbb{F}/\mathbb{L}$ a finite extension of number fields and $O_{\mathbb{L}}$ (resp. $O_{\mathbb{F}}$) the ring of integers of $\mathbb{L}$ (resp. of $\mathbb{F}$). We say, since Hilbert, that an ideal of $O_{\mathbb{L}}$ capitulates in $\mathbb{F}$, if it becomes principal by extension of scalars to $O_{\mathbb{F}}$, and, of course, when an ideal $\mathfrak{a}$ of $O_{\mathbb{L}}$ capitulates in $\mathbb{F}$, then the class $[\mathfrak{a}]$ of $\mathfrak{a}$ capitulates in $\mathbb{F}$ (ie. $[\mathfrak{a}O_{\mathbb{F}}]$ is trivial). Therefore, the study of the capitulation problem is precisely to describe the group of all classes of ideals of $\mathbb{L}$ which capitulate in $\mathbb{F}$, where $\mathbb{F}$ is an unramified abelian extension of $\mathbb{L}$.

Proposition 1 ([8]). *Let $\mathbb{G}$ a 2-group of finite order 2^m and $\mathbb{G}'$ its derived subgroup. Then $\mathbb{G}/\mathbb{G}'$ is of type $(2,2)$ if and only if $\mathbb{G}$ is isomorphic to one of 2-groups:*

$$Q_m = \langle\sigma,\tau\rangle \quad where \quad \sigma^{2^{m-2}} = \tau^2 = a, a^2 = 1, \tau^{-1}\sigma\tau = \sigma^{-1};$$
$$D_m = \langle\sigma,\tau\rangle \quad where \quad \sigma^{2^{m-1}} = \tau^2 = 1, \tau^{-1}\sigma\tau = \sigma^{-1};$$
$$S_m = \langle\sigma,\tau\rangle \quad where \quad \sigma^{2^{m-1}} = \tau^2 = 1, \tau^{-1}\sigma\tau = \sigma^{2^{m-2}-1};$$
$$(2,2) = \langle\sigma,\tau\rangle \quad where \quad \sigma^2 = \tau^2 = 1, \tau^{-1}\sigma\tau = \sigma.$$

Where Q_m is the quaternionic group, D_m the dihedral group, S_m the semi-dihedral group of order 2^m and (2.2) is an abelian group isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$.

Suppose that $\mathbb{G}$ is a 2-group of finite order 2^m such that $\mathbb{G}/\mathbb{G}'$ is of type $(2,2)$, then $\mathbb{G}$ is isomorphic to Q_m, D_m, S_m or $(2,2)$ defined in the Proposition 1. Let $\{\sigma,\tau\}$ generates $\mathbb{G}$ such that the relationships cited in the Proposition 1 are verified, by a simple calculation we can see that the derived subgroup $\mathbb{G}' = [\mathbb{G},\mathbb{G}] = \langle\sigma^2\rangle$

and $\mathbb{G}$ has three subgroups of index 2: $H_1 = \langle\sigma\rangle$, $H_2 = \langle\sigma^2, \tau\rangle$ and $H_3 = \langle\sigma^2, \sigma\tau\rangle$. Moreover if $\mathbb{G}$ is of type $(2,2)$, then the subgroups H_i are cyclic of order 2, if $\mathbb{G} \simeq Q_3$, then the subgroups H_i are cyclic of order 4 and if $\mathbb{G}$ is isomorphic to Q_m ($m > 3$), D_m or S_m, then H_1 is cyclic and H_2/H_2' and H_3/H_3' are of type $(2,2)$ where H_2' (resp. H_3') is the derived subgroup of H_2 (resp. H_3).

Throughout the remainder of this section we denote by $\mathbb{M}$ a number field, $\mathbb{M}_2^{(1)}$ the first 2-Hilbert class field of $\mathbb{M}$, $\mathbb{M}_2^{(2)}$ the second 2-Hilbert class field of $\mathbb{M}$, $C_{\mathbb{M}}$ the class group of $\mathbb{M}$, $C_{2,\mathbb{M}}$ the 2-part of $C_{\mathbb{M}}$, $\mathbb{G}$ the Galois group of $\mathbb{M}_2^{(2)}/\mathbb{M}$ and $\mathbb{G}'$ its derived sub-group. Then $\mathbb{G}' \simeq Gal(\mathbb{M}_2^{(2)}/\mathbb{M}_2^{(1)})$ and $\mathbb{G}/\mathbb{G}' \simeq Gal(\mathbb{M}_2^{(1)}/\mathbb{M})$, and we know from the class field theory that $Gal(\mathbb{M}_2^{(1)}/\mathbb{M}) \simeq C_{2,\mathbb{M}}$, so $\mathbb{G}/\mathbb{G}' \simeq C_{2,\mathbb{M}}$.

Definition 1. Let $\mathbb{F}$ be a cyclic unramified extension of $\mathbb{M}$ and j is the mapping of $C_{\mathbb{M}}$ into $C_{\mathbb{F}}$ that maps to the class of an ideal $\mathfrak{a}$ of $\mathbb{M}$ the class of the ideal generated by $\mathfrak{a}$ in $\mathbb{F}$. Then the extension $\mathbb{F}/\mathbb{M}$ is called:

- of type (A) if and only if $\#\ker j \cap N_{\mathbb{F}/\mathbb{M}}(C_{\mathbb{F}}) > 1$;
- of type (B) if and only if $\#\ker j \cap N_{\mathbb{F}/\mathbb{M}}(C_{\mathbb{F}}) = 1$.

Now suppose that $C_{2,\mathbb{M}} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then $\mathbb{G}/\mathbb{G}' \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and as $\mathbb{G}$ is a 2-finite group, then $\mathbb{G}'$ is cyclic. Thus the Hilbert 2-class field towers of $\mathbb{M}$ ends in $\mathbb{M}_2^{(2)}$.
In addition, we know that if $\mathbb{G}$ has order 2^m and $\mathbb{G}/\mathbb{G}' \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then $\mathbb{G}$ is isomorphic to Q_m, D_m, S_m or to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In all these cases, we have $\mathbb{G}' = \langle\sigma^2\rangle$ and the three subgroups of index 2 in $\mathbb{G}$ are: $H_1 = \langle\sigma\rangle$, $H_2 = \langle\sigma^2, \tau\rangle$ and $H_3 = \langle\sigma^2, \sigma\tau\rangle$, and if $\mathbb{G}' \neq 1$, then $\mathbb{M}_2^{(1)} \neq \mathbb{M}_2^{(2)}$ and $\langle\sigma^4\rangle$ is the only subgroup of $\mathbb{G}'$ of index 2.
Let $\mathbb{L}$ the subfield of $\mathbb{M}_2^{(2)}$ left fixed by $\langle\sigma^4\rangle$, $\mathbb{F}_i$ ($i = 1,2,3$) the subfield of $\mathbb{M}_2^{(2)}$ left fixed by H_i and j_i the mapping j defined for $\mathbb{F} = \mathbb{F}_i$.

Theorem 1 ([8]). *Assume that $\mathbb{G}/\mathbb{G}' \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, so we have*

(1) *If $\mathbb{M}_2^{(1)} = \mathbb{M}_2^{(2)}$, then the fields $\mathbb{F}_i$ are of type (A), $\#\ker j_i = 4$ for $i = 1,2,3$ and $\mathbb{G} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$;*
(2) *If $Gal(\mathbb{L}/\mathbb{M}) \simeq Q_3$, then the fields $\mathbb{F}_i$ are of type (A), $\#\ker j_i = 2$ for $i = 1,2,3$ and $\mathbb{G} \simeq Q_3$;*
(3) *If $Gal(\mathbb{L}/\mathbb{M}) \simeq D_3$, then the fields $\mathbb{F}_2$ and $\mathbb{F}_3$ are of type (B) and $\#\ker j_2 = \#\ker j_3 = 2$. Moreover, if $\mathbb{F}_1$ is of type (B) then $\#\ker j_1 = 2$ and $\mathbb{G} \simeq S_m$. If $\mathbb{F}_1$ is of type (A) and $\#\ker j_1 = 2$, then $\mathbb{G} \simeq Q_m$. Finally if $\mathbb{F}_1$ is of type (A) and $\#\ker j_1 = 4$, then $\mathbb{G} \simeq D_m$.*

Theorem 2 ([6]). *Let $\mathbb{L}/\mathbb{M}$ be an unramified cyclic extension of prime degree, then the number of classes that capitulate in $\mathbb{L}/\mathbb{M}$ is*

$$[\mathbb{L}:\mathbb{M}][E_{\mathbb{M}}:N_{\mathbb{L}/\mathbb{M}}(E_{\mathbb{L}})],$$

where $E_{\mathbb{M}}$ (resp. $E_{\mathbb{L}}$) is the group of units of $\mathbb{M}$ (resp. of $\mathbb{L}$).

For more details about the capitulation problem see [8, 11, 12].

2. Units of some number fields

Proposition 2 ([1]). *Let $\mathbb{K}_0$ a number field, abelian real and β an algebraic integer in $\mathbb{K}_0$, totally positive, without square factors. Assume that $\mathbb{F}=\mathbb{K}_0(\sqrt{-\beta})$ is a quadratic extension of $\mathbb{K}_0$, abelian over $\mathbb{Q}$ and $i=\sqrt{-1}\notin\mathbb{F}$. Let $\{\varepsilon_1,\varepsilon_2,\dots,\varepsilon_r\}$ be a fundamental system of units of $\mathbb{K}_0$. We choose, without limiting the generality, units ε_j positive. Then we have:*

(1) *If there is a unit of $\mathbb{K}_0$ such that $\varepsilon=\varepsilon_1^{j_1}\varepsilon_2^{j_2}\dots\varepsilon_{r-1}^{j_{r-1}}\varepsilon_r$ (close to a permutation), where the $j_k\in\{0,1\}$, such that $\beta\varepsilon$ is a square in $\mathbb{K}_0$, then $\{\varepsilon_1,\dots,\varepsilon_{r-1},\sqrt{-\varepsilon}\}$ is a fundamental system of units of $\mathbb{F}$;*
(2) *Otherwise $\{\varepsilon_1,\dots,\varepsilon_r\}$ is a fundamental system of units of $\mathbb{F}$.*

Corollary 1 ([2]). *Let $\mathbb{L}=\mathbb{Q}(\sqrt{-n\varepsilon\sqrt{d}})$ a cyclic extension of degree 4 over $\mathbb{Q}$, where ε is the fundamental unit of $\mathbb{Q}(\sqrt{d})$ with d a positive integer squarefree and n a positive integer, then $\{\varepsilon\}$ is a fundamental system of units of $\mathbb{L}$.*

Theorem 3 ([2]). *Let $\mathbb{K}_0=\mathbb{Q}(\sqrt{p},\sqrt{p'})$ where p and p' are two different primes such that $p\equiv p'\equiv 1 \bmod 4$, ε_1 (resp. ε_2, ε_3) the fundamental unit of $\mathbb{Q}(\sqrt{p})$ (resp. $\mathbb{Q}(\sqrt{p'})$, $\mathbb{Q}(\sqrt{pp'})$) and $\mathbb{F}=\mathbb{K}_0(\sqrt{-n\varepsilon_1\sqrt{p}})$ where n is a positive integer squarefree. Then we have:*

(1) *If ε_3 has norm 1, then $\{\varepsilon_1,\varepsilon_2,\sqrt{\varepsilon_3}\}$ is a fundamental system of units of $\mathbb{K}_0$ and of $\mathbb{F}$;*
(2) *Otherwise, $\{\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3},\varepsilon_2,\varepsilon_3\}$ is a fundamental system of units of $\mathbb{K}_0$ and of $\mathbb{F}$.*

Now, using the results of M. N. Gras [4], we'll define a fundamental system of units of real cyclic extension of degree 4 over $\mathbb{Q}$. Let $\mathbb{L}=\mathrm{k}(\sqrt{\varepsilon_0\sqrt{l}})$ where l is a prime number congruent to 1 modulo 8 and ε_0 is the fundamental unit of $\mathrm{k}=\mathbb{Q}(\sqrt{l})$, then $\mathbb{L}/\mathbb{Q}$ is a real cyclic extension of degree 4 with Galois group $H=\langle\gamma\rangle$ and quadratic subfield k. Since $\mathbb{L}$ has conductor $F_{\mathbb{L}}=l$, we have $\mathbb{L}\subset\mathbb{Q}^{(l)}$ and there exists a character χ' of $Gal(\mathbb{Q}^{(l)}/\mathbb{Q})\simeq(\mathbb{Z}/l\mathbb{Z})^*$ such that $\ker\chi'=Gal(\mathbb{Q}^{(l)}/\mathbb{L})$.
Let $\chi=\chi'+\chi'^{-1}$, then χ is a rational character of $\mathbb{Q}^{(l)}$ and $\mathbb{L}$ is fixed by the common kernel of χ' and χ'^{-1}. Let $E_{\mathbb{L}}$ be the group of units of $\mathbb{L}$, E_χ the group of χ-relative units of $\mathbb{L}$, $|E_{\mathbb{L}}|$ (resp. $|E_\chi|$) the group of absolute values of $E_{\mathbb{L}}$ (resp.

E_χ), $|E^{\mathbb{L}}| = |E_{\mathbb{L}}| \oplus |E_\chi|$, $Q = [|E_{\mathbb{L}}| : |E^{\mathbb{L}}|]$ and ε_χ a generator of E_χ. Then we have the following result:

Theorem 4 ([4]). *Let $\mathbb{L} = \mathrm{k}(\sqrt{\varepsilon_0\sqrt{l}})$ where l is a prime number congruent to 1 modulo 8 and ε_0 the fundamental unit of $\mathrm{k} = \mathbb{Q}(\sqrt{l})$, then:*

(1) *$Q = 2$;*
(2) *There exists ξ in $E_{\mathbb{L}}$ such that $\xi^2 = \pm\varepsilon_0\varepsilon_\chi^{1-\sigma}$ and $\{\xi, \xi^\sigma, \xi^{\sigma^2}\}$ is a fundamental system of units of $\mathbb{L}$.*

Remark 1. Since $\xi^2 = \pm\varepsilon_0\varepsilon_\chi^{1-\sigma}$, then:

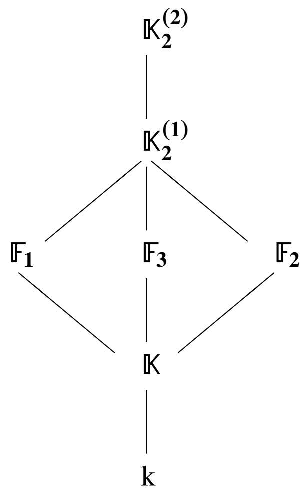
(1) $\xi^{1+\sigma} = \pm\varepsilon_\chi$;
(2) $\xi^{1+\sigma^2} = \pm\varepsilon_0$;
(3) $\xi^{1+\sigma+\sigma^2+\sigma^3} = \varepsilon_\chi^{1+\sigma^2} = \varepsilon_0^{1+\sigma}$;
(4) $N_{\mathrm{k}/\mathbb{Q}}(\varepsilon_0) = N_{\mathbb{L}/\mathrm{k}}(\varepsilon_\chi) = N_{\mathbb{L}/\mathbb{Q}}(\xi) = -1$.

Lemma 1. *With the same notation of Theorem 4, $\{\xi, \xi^\sigma, \xi^{\sigma^2}\}$ is a fundamental system of units of $\mathbb{F} = \mathbb{L}(\sqrt{-n})$ where n is an integer different from 1, relatively prime to l and square-free.*

Proof. By Proposition 2, to show that $\{\xi, \xi^\sigma, \xi^{\sigma^2}\}$ is a fundamental system of units of $\mathbb{F}$ it suffices to show that $n\mu$ is not a square in $\mathbb{L}$, for $\mu = \xi_1'^{\,j_1}\xi_2'^{\,j_2}\xi_3'$ where $\{\xi_1', \xi_2', \xi_3'\} = \{\xi, \xi^\sigma, \xi^{\sigma^2}\}$ and $j_1, j_2 \in \{0, 1\}$.
Indeed, if $\mu = \xi$, then if $n\xi = x^2$ in $\mathbb{L}$, so by calculating the norm in $\mathbb{L}/\mathrm{k}$, we find that $\xi^{1+\sigma^2} = \pm\varepsilon_0$ is a square in k, which is impossible.
If $\mu = \xi^\sigma$, then if $n\xi^\sigma = \pm n\frac{\varepsilon_\chi}{\xi} = x^2$ in $\mathbb{L}$, so by using the norm in $\mathbb{L}/\mathrm{k}$, we find that $\pm\frac{\varepsilon_0^{1+\sigma}}{\varepsilon_0} = \pm\varepsilon_0^{-1}$ is a square in k, which is absurd.
If $\mu = \xi^{\sigma^2}$, then if $n\xi^{\sigma^2} = \pm n\frac{\varepsilon_0}{\xi} = x^2$ in $\mathbb{L}$, so by calculating the norm in $\mathbb{L}/\mathrm{k}$, we find that $\pm\frac{\varepsilon_0^2}{\varepsilon_0} = \pm\varepsilon_0$ is a square in k, which is not the case.
If $\mu = \xi^{1+\sigma} = \pm\varepsilon_\chi$, then if $\pm n\varepsilon_\chi = x^2$ in $\mathbb{L}$, so by calculating the norm in $\mathbb{L}/\mathrm{k}$, we find that $\varepsilon_\chi^{1+\sigma^2} = \varepsilon_0^{1+\sigma} = -1$ is a square in k, which is absurd.
If $\mu = \xi^{1+\sigma^2} = \pm\varepsilon_0$, then $\pm n\varepsilon_0$ can not be a square in $\mathbb{L}$.
If $\mu = \xi^{\sigma+\sigma^2}$, then if $n\xi^{\sigma+\sigma^2} = x^2$ in $\mathbb{L}$, so by calculating the norm in $\mathbb{L}/\mathrm{k}$, we find that $\xi^{1+\sigma+\sigma^2+\sigma^3} = -1$ is a square in k, which is not the case.
If $\mu = \xi^{1+\sigma+\sigma^2}$, then if $n\xi^{1+\sigma+\sigma^2} = x^2$ in $\mathbb{L}$, so by calculating the norm in $\mathbb{L}/\mathrm{k}$, we find that $\xi^{1+\sigma+\sigma^2+\sigma^3}\xi^{1+\sigma^2} = \pm\varepsilon_0$ is a square in k, which is impossible, which completes the proof of the lemma. □

3. Capitulation of the 2-ideal class of $\mathbb{K}$ and structure of $\mathbb{G}_2$

In the following, let $\mathbb{K}=\mathrm{k}(\sqrt{-pq\varepsilon_0\sqrt{l}})$ where ε_0 is the fundamental unit of $\mathrm{k}=\mathbb{Q}(\sqrt{l})$ with l is a prime number congruent to 1 modulo 8, p and q two prime numbers such that $p\equiv -q\equiv 1 \bmod 4$ and $(\frac{p}{l})=(\frac{q}{l})=-1$, $\mathbb{K}_2^{(1)}$ the Hilbert 2-class field of $\mathbb{K}$, $\mathbb{K}_2^{(2)}$ the Hilbert 2-class field of $\mathbb{K}_2^{(1)}$, $\mathbb{G}_2$ the Galois group of $\mathbb{K}_2^{(2)}/\mathbb{K}$ and for an ideal $\mathfrak{a}$ of $\mathbb{K}$, we note $[\mathfrak{a}]$ the class of $\mathfrak{a}$. Then, by [3], $C_{2,\mathbb{K}}$, the 2-class group of $\mathbb{K}$, is of type $(2,2)$. Using the Results of [7] for the calculation of the genus fields of an extension of degree 2^s on $\mathbb{Q}$, we find that $\mathbb{K}_2^{(1)}=\mathbb{K}^{(*)}=\mathbb{K}(\sqrt{p},\sqrt{-q})$, where $\mathbb{K}^{(*)}$ is the genus field of $\mathbb{K}$, whose quadratic subfields over $\mathbb{K}$ are $\mathbb{F}_1=\mathbb{K}(\sqrt{p})$, $\mathbb{F}_2=\mathbb{K}(\sqrt{-q})$ and $\mathbb{F}_3=\mathbb{K}(\sqrt{-pq})$.
We study the capitulation problem of the 2-ideal classes of $\mathbb{K}$ in different sub-quadratic extension $\mathbb{F}_i/\mathbb{K}$ of $\mathbb{K}_2^{(1)}/\mathbb{K}$, and hence we determine the structure of $\mathbb{G}_2$.



Proposition 3. *Let $\mathbb{K}=\mathrm{k}(\sqrt{-pq\varepsilon_0\sqrt{l}})$, $\mathcal{P}$ the prime ideal of $\mathbb{K}$ above p and $\mathcal{Q}$ that above q. Then the classes $[\mathcal{P}]$, $[\mathcal{Q}]$ and $[\mathcal{P}\mathcal{Q}]$ are of order 2 in $\mathbb{K}$, $C_{2,\mathbb{K}}$ is generated by the classes $[\mathcal{P}]$ and $[\mathcal{Q}]$. Also $\mathcal{P}$ capitulates in $\mathbb{F}_1$, $\mathcal{Q}$ capitulates in $\mathbb{F}_2$ and $\mathcal{P}\mathcal{Q}$ capitulates in $\mathbb{F}_3$.*

Proof. The class $[\mathcal{P}]$ has order 2, indeed, since p is inert in $\mathrm{k}/\mathbb{Q}$ and p ramifies in $\mathbb{K}/\mathbb{Q}$, then $\mathcal{P}^2=(p)$. Assume that $\mathcal{P}=(\alpha)$ for some α in $\mathbb{K}$, which is equivalent to $(\alpha^2)=(p)$ in $\mathbb{K}$. So there is therefore a unit ε of K such that $p\varepsilon=\alpha^2$, but there exist a and b in k such that $\alpha=a+b\sqrt{-pq\varepsilon_0\sqrt{l}}$, thus $p\varepsilon=a^2-pq\varepsilon_0\sqrt{l}b^2+2ab\sqrt{-pq\varepsilon_0\sqrt{l}}$ and as $\{\varepsilon_0\}$ is a fundamental system of units of $\mathbb{K}$ (Corollary 1) and

$i=\sqrt{-1}\notin \mathbb{K}$, then $p\varepsilon\in \mathrm{k}$, therefore a or $b=0$. If $b=0$, then $p\varepsilon=a^2$, thus, if ε has norm 1 (the norm in $\mathrm{k}/\mathbb{Q}$), p will be norm in $\mathrm{k}/\mathbb{Q}$ which is not the case because $(\frac{p}{l})=-1$, if ε has norm -1 we find that -1 is a square in $\mathbb{Q}$ which is impossible. Similarly, if $a=0$ we find that $\pm l$ is a square in $\mathbb{Q}$, thus $[\mathcal{P}]$ has order 2 and similarly one shows that $[\mathcal{Q}]$ and $[\mathcal{P}Q]$ orders are 2, therefore $C_{2,\mathbb{K}}$ is generated by $[\mathcal{P}]$ and $[\mathcal{Q}]$. To show that $\mathcal{P}$ capitulates in $\mathbb{F}_1$, it suffices to see that $\sqrt{p}\in\mathbb{F}_1$ and $(\sqrt{p}^2)=(p)$ in $\mathbb{F}_1$, so $\mathcal{P}$ capitulated in $\mathbb{F}_1$ and even $\mathcal{Q}$ capitulated in $\mathbb{F}_2$ and $\mathcal{P}\mathcal{Q}$ capitulated in $\mathbb{F}_3$. □

Proposition 4 ([5]). *Let $\mathbb{M}$ a number field which contains the m-th roots of unity, $\mathbb{L}$ a finite extension of $\mathbb{M}$, $\alpha\in\mathbb{M}^*$ and $\beta\in\mathbb{L}^*$. We denote by P a prime ideal of $\mathbb{M}$, and $\mathcal{P}$ a prime ideal of $\mathbb{L}$ above P. Then*

$$\prod_{\mathcal{P}}\left(\frac{\beta,\alpha}{\mathcal{P}}\right)_m=\left(\frac{N_{\mathbb{L}/\mathbb{M}}(\beta),\alpha}{P}\right)_m,$$

where the product is taken for all prime ideals of $\mathbb{L}$ that are above P.

Theorem 5. *Let $\mathbb{K}=\mathrm{k}(\sqrt{-pq\varepsilon_0\sqrt{l}})$ where ε_0 is the fundamental unit of $\mathrm{k}=\mathbb{Q}(\sqrt{l})$ with l is a prime number congruent to 1 modulo 8, p and q be two prime numbers such that $p\equiv -q\equiv 1 \bmod 4$ and $(\frac{p}{l})=(\frac{q}{l})=-1$, $\mathbb{F}_1=\mathbb{K}(\sqrt{p})$, $\mathbb{F}_2=\mathbb{K}(\sqrt{-q})$ and $\mathbb{F}_3=\mathbb{K}(\sqrt{-pq})$. Then in each extension $\mathbb{F}_i$, $i\in\{1,2,3\}$, there exist exactly two classes of $C_{2,\mathbb{K}}$ which capitulate and the group $\mathbb{G}_2$ is quaternionic of order 2^m with $m>3$.*

Proof. Let ε_2 (resp. ε_3) the fundamental unit of $\mathbb{Q}(\sqrt{p})$ (resp. $\mathbb{Q}(\sqrt{lp})$), $\mathcal{P}$ the prime ideal of $\mathbb{K}$ above p, $\mathcal{Q}$ that above q, then, by Theorem 3, $\{\sqrt{\varepsilon_0\varepsilon_2\varepsilon_3},\varepsilon_2,\varepsilon_3\}$ is a fundamental system of units of $\mathbb{F}_1$, since $N_{\mathbb{F}_1/\mathbb{K}}(\sqrt{\varepsilon_0\varepsilon_2\varepsilon_3})=\pm\varepsilon_0$ and $N_{\mathbb{F}_1/\mathbb{K}}(\varepsilon_2)=N_{\mathbb{F}_1/\mathbb{K}}(\varepsilon_3)=-1$, then $N_{\mathbb{F}_1/\mathbb{K}}(E_{\mathbb{F}_1})=E_{\mathbb{K}}$, therefore, by Theorem 2, we found that only two classes of $C_{2,\mathbb{K}}$ capitulate in $\mathbb{F}_1$, namely $[\mathcal{P}]$ and its square. Also the extension $\mathbb{F}_1/\mathbb{K}$ is of type (B), indeed, let $\mathbb{K}'=\mathrm{k}(\sqrt{-q\varepsilon_0\sqrt{l}})$, then we have $\mathbb{K}\mathbb{K}'=\mathbb{F}_1$, since $N_{\mathbb{K}/\mathrm{k}}(\mathcal{P})=p$ and p is unramified in $\mathbb{K}'/\mathrm{k}$, then to show that $\mathcal{P}$ is inert in $\mathbb{F}_1/\mathbb{K}$, it suffices to show that p is inert in $\mathbb{K}'/\mathrm{k}$ (translation theorem) and for this we compute the norm residue symbol $\left(\frac{p,-q\varepsilon_0\sqrt{l}}{p}\right)$. Since $p\in\mathbb{Q}$ is inert in $\mathrm{k}/\mathbb{Q}$ and $-q\varepsilon_0\sqrt{l}\in\mathrm{k}$, therefore using the Proposition 4, we find

$$\left(\frac{p,-q\varepsilon_0\sqrt{l}}{p}\right)=\left(\frac{p,N_{\mathrm{k}/\mathbb{Q}}(-q\varepsilon_0\sqrt{l})}{p}\right)=\left(\frac{p,lq^2}{p}\right)=(\frac{l}{p})=-1,$$

so p is inert in $\mathbb{K}'/\mathrm{k}$, which gives that $\mathcal{P}$ is inert in $\mathbb{F}_1/\mathbb{K}$, then $\mathbb{F}_1/\mathbb{K}$ is of type (B). Similarly, we show that extension $\mathbb{F}_2/\mathbb{K}$ is of type (B), using Theorem 1, only two classes of $C_{2,\mathbb{K}}$ capitulate in $\mathbb{F}_2$, namely $[\mathcal{Q}]$ and its square.
Using the notation of Theorem 4, we have that $\{\xi,\xi^\sigma,\xi^{\sigma^2}\}$ is a fundamental system

of units of $\mathbb{F}_3$, since $N_{\mathbb{F}_3/\mathbb{K}}(\xi) = N_{\mathbb{F}_3/\mathbb{K}}(\xi^{\sigma^2}) = \pm\varepsilon_0$ and $N_{\mathbb{F}_3/\mathbb{K}}(\xi^{\sigma}) = \pm\varepsilon_0^{-1}$, then we have $N_{\mathbb{F}_3/\mathbb{K}}(E_{\mathbb{F}_3}) = E_{\mathbb{K}}$, using Theorem 2, we found that only two classes of $C_{2,\mathbb{K}}$ capitulate in $\mathbb{F}_3$, namely $[\mathcal{P}\mathcal{Q}]$ and its square. Moreover, the extension $\mathbb{F}_3/\mathbb{K}$ is of type (A), indeed, let $\mathbb{L} = \mathrm{k}(\sqrt{\varepsilon_0\sqrt{l}})$, then we have $\mathbb{K}\mathbb{L} = \mathbb{F}_3$, $N_{\mathbb{K}/\mathrm{k}}(\mathcal{P}) = p$ and p is unramified in $\mathbb{L}/\mathrm{k}$, so to show that $\mathcal{P}$ is inert in $\mathbb{F}_3/\mathbb{K}$, it suffices to show that p is inert in $\mathbb{L}/\mathrm{k}$, for this, we compute the norm residue symbol $\left(\frac{p,\varepsilon_0\sqrt{l}}{p}\right)$. We have $p \in \mathbb{Q}$ is inert in $\mathrm{k}/\mathbb{Q}$ and $\varepsilon_0\sqrt{l} \in \mathrm{k}$, so by Proposition 4, we have $\left(\frac{p,\varepsilon_0\sqrt{l}}{p}\right) = \left(\frac{p,N_{\mathrm{k}/\mathbb{Q}}(\varepsilon_0\sqrt{l})}{p}\right) = \left(\frac{p,l}{p}\right) = (\frac{l}{p}) = -1$, therefore p is inert in $\mathbb{L}/\mathrm{k}$, what gives that $\mathcal{P}$ is inert in $\mathbb{F}_3/\mathbb{K}$, and similarly one shows that $\mathcal{Q}$ remains inert in $\mathbb{F}_3/\mathbb{K}$ and like $\mathcal{P}\mathcal{Q}$ capitulates in $\mathbb{F}_3/\mathbb{K}$, then by applying the Artin reciprocity law, we find that $\mathbb{F}_3/\mathbb{K}$ is of type (A), therefore, using Theorem 1, the group $\mathbb{G}_2$ is isomorphic to Q_m with $m > 3$. □

Example 1. Let $\mathbb{K} = \mathbb{Q}(\sqrt{-55(4+\sqrt{17})\sqrt{17}})$, $\mathbb{F}_1 = \mathbb{K}(\sqrt{5})$, $\mathbb{F}_2 = \mathbb{K}(\sqrt{-11})$ and $\mathbb{F}_3 = \mathbb{K}(\sqrt{-55})$. By Theorem 5, in each extension $\mathbb{F}_i$ there are exactly two classes of $C_{2,\mathbb{K}}$ which capitulate and $\mathbb{G}_2 \simeq Q_m$ with $m > 3$.

Corollary 2. *With the same notation of Theorem 5, we have* $\#\mathbb{G}_2 = 4h_2(\mathbb{K}_0)$ *where* $\mathbb{K}_0 = \mathbb{Q}(\sqrt{l}, \sqrt{-pq})$ *and* $h_2(\mathbb{K}_0)$ *its* 2*-class number.*

Proof. We have $\mathbb{K}_2^{(1)}/\mathbb{F}_3$ is an unramified extension and the extension $\mathbb{F}_3/\mathbb{K}$ is of type (A), then, according to [8], $C_{2,\mathbb{F}_3}$ is cyclic, thus $\mathbb{F}_3$ and $\mathbb{K}_2^{(1)}$ has the same Hilbert 2-class field which is $\mathbb{K}_2^{(2)}$, therefore $\#\mathbb{G}_2 = 2h_2(\mathbb{F}_3)$. Moreover $\mathbb{F}_3/\mathrm{k}$ is a biquadratic normal extension with Galois group isomorphic to $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ and quadratic subfields $\mathbb{K}$, $\mathbb{K}_0$ and $\mathbb{L} = \mathrm{k}(\sqrt{\varepsilon_0\sqrt{l}})$, so, by using [9], page 247, we find that

$$h_2(\mathbb{F}_3) = \frac{1}{2}q(\mathbb{F}_3/\mathrm{k})h_2(\mathbb{K})h_2(\mathbb{K}_0)h_2(\mathbb{L}),$$

where $q(\mathbb{F}_3/\mathrm{k})$ is the unit index of $\mathbb{F}_3/\mathrm{k}$ and $h_2(\mathbb{F})$ is the 2-class number of a number field $\mathbb{F}$. We have $h_2(\mathbb{K}) = 4$, according to [13] we have $h_2(\mathbb{L}) = 1$, by Corollary 1 we have $\{\varepsilon_0\}$ is a fundamental system of units of $\mathbb{K}$, using the Proposition 2 we show that $\{\varepsilon_0\}$ is a fundamental system of units of $\mathbb{K}_0$, according to Lemma 1, we have $\{\xi, \xi^{\sigma}, \xi^{\sigma^2}\}$ is a fundamental system of units of $\mathbb{L}$ and $\mathbb{F}_3$, which gives that $q(\mathbb{F}_3/\mathrm{k}) = 1$, thus $h_2(\mathbb{F}_3) = 2h_2(\mathbb{K}_0)$, therefore $\#\mathbb{G}_2 = 4h_2(\mathbb{K}_0)$. □

Corollary 3. *Let* $\mathbb{K} = \mathrm{k}(\sqrt{-pq\varepsilon_0\sqrt{l}})$ *where* ε_0 *is the fundamental unit of* $\mathrm{k} = \mathbb{Q}(\sqrt{l})$ *with* l *a prime number congruent to* 1 *modulo* 8*,* p *and* q *be two primes such that* $p \equiv -q \equiv 1 \bmod 4$ *and* $(\frac{p}{l}) = (\frac{q}{l}) = (\frac{p}{q}) = -1$*, then* $\mathbb{G}_2 \simeq Q_4$.

Proof. According to the Theorem 5, $\mathbb{G}_2$ is quaternionic of order 2^m with $m > 3$. Let $\mathbb{K}_0 = \mathbb{Q}(\sqrt{l}, \sqrt{-pq})$, since $(\frac{p}{l}) = (\frac{q}{l}) = (\frac{p}{q}) = -1$, then, according to [10], we have $h_2(\mathbb{K}_0) = 4$, so, using Corollary 2, we find that $\mathbb{G}_2 \simeq Q_4$. □

Example 2. Let $\mathbb{K} = \mathbb{Q}(\sqrt{-15(4+\sqrt{17})\sqrt{17}})$, $\mathbb{F}_1 = \mathbb{K}(\sqrt{5})$, $\mathbb{F}_2 = \mathbb{K}(\sqrt{-3})$ and $\mathbb{F}_3 = \mathbb{K}(\sqrt{-15})$. According to the Theorem 5, in each extension $\mathbb{F}_i$ there exist exactly two classes of $C_{2,\mathbb{K}}$ which capitulate and the group $\mathbb{G}_2$ is quaternionic of order 2^m with $m > 3$. Furthermore, since $(\frac{5}{3}) = -1$, then, according to the Corollary 3, $\mathbb{G}_2 \simeq Q_4$.

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Authors' addresses

Abdelmalek Azizi

Mohamed First University, Department of Mathematics and Computer Sciences, Faculty of Sciences, Oujda, Morocco

E-mail address: abdelmalekazizi@yahoo.fr

Mohammed Talbi
Regional Center of Education and Training, Oujda, Morocco
E-mail address: talbimm@yahoo.fr