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FIXED POINT THEOREMS FOR SOME GENERALIZED NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we first introduce the class of generalized nonexpansive mappings in Banach spaces. This class contains both the classes of nonexpansive and α-nonexpansive mappings. In addition, we obtain some fixed point and coincidence point theorems for generalized nonexpansive mappings in uniformly convex Banach spaces. Our results extend some well-known results in literature.

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1. Introduction and preliminaries.

Recently, Aoyama and Kohsaka [2] introduced the class of α-nonexpansive mappings in Banach spaces and obtained a fixed point theorem for α-nonexpansive mappings in uniformly convex Banach spaces. The class of α-nonexpansive mappings contains the class of nonexpansive mappings and is related to the classes of firmly nonexpansive mappings and λ-hybrid mappings in Banach spaces, for more information on firmly nonexpansive mappings and λ-hybrid mappings see [3], [4], [5], [8], [1] and references therein.

In this paper, we introduce the class of generalized nonexpansive mappings in Banach spaces. This class contains the class of α-nonexpansive mappings. In addition, we obtain some fixed point and coincidence point theorems for generalized nonexpansive mappings in uniformly convex and p-uniformly convex Banach spaces. Our fixed point theorems generalize some of the results obtained in [2].

In the rest of this section, we recall some definitions and facts which will be used in the next section.

Throughout this paper, every Banach space is real. Let E be a Banach space and let C be a nonempty subset of E. We denote the fixed point set of T by $F(T)$. For a Banach space E, the norm of E is denoted by $\|.\|$. Strong convergence of a sequence

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$\{x_n\}$ in E to $x \in E$ is denoted by $x_n \to x$. For a Banach space E, we denote the unit sphere and the closed unit ball centered at the origin of E by S_E and B_E , respectively. We also denote the closed ball with radius $r > 0$ centered at the origin of E by rB_E. Let E be a Banach space with dimension $E \geq 2$. The modulus of convexity of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) = \inf\{1 - \|\frac{x+y}{2}\| \ : \ \|x\| = 1, \|y\| = 1, \ \|x-y\| \geq \epsilon\}.$$

A Banach space E is said to be uniformly convex if for each $\epsilon \in (0,2]$, there exists $\delta > 0$ such that $\|\frac{(x+y)}{2}\| \leq 1-\delta$ whenever $x, y \in S_E$ and $\|x-y\| \geq \epsilon$. In other words, E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for each $\epsilon \in (0,2]$. Every uniformly convex Banach space is reflexive [9].
A Banach space E is called *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in (0,2]$. Notice that there is no p-uniformly convex Banach space for $p > 2$; see, for example [10].
In the sequel we will need the following lemmas.

Lemma 1 ([12]). *The Banach space E is uniformly convex if and only if $\|.\|^2$ is uniformly convex on bounded convex sets, i.e., for each $r > 0$ and $\epsilon \in (0,2r]$, there exists $\delta > 0$ such that*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\delta,$$

for all $t \in (0,1)$ and for all $x, y \in rB_E$ with $\|x-y\| \geq \epsilon$.

Lemma 2 ([11]). *Let $1 < p \leq 2$ be a given real number. Let E be a p-uniformly convex Banach space. Then, there exists a constant $d > 0$ such that*

$$\|tx + (1-t)y\|^p \leq t\|x\|^p + (1-t)\|y\|^p - (t^p(1-t) + t(1-t)^p)d\|x-y\|^p,$$

for all $t \in (0,1)$ and for all $x, y \in E$.

A function g of a nonempty subset C of a Banach space E into $\mathbb{R}$ is said to be coercive if $g(z_n) \to \infty$ whenever $\{z_n\}$ is a sequence in C such that $\|z_n\| \to \infty$. Let l^∞ denotes the Banach space of bounded real sequences with the supremum norm. It is known that there exists a bounded linear functional μ on l^∞ such that the following three conditions hold:

(1) If $\{t_n\} \in l^\infty$ and $t_n \geq 0$ for every $n \in \mathbb{N}$, then $\mu(\{t_n\}) \geq 0$;
(2) If $t_n = 1$ for every $n \in \mathbb{N}$, then $\mu(\{t_n\}) = 1$;
(3) $\mu(\{t_{n+1}\}) = \mu(\{t_n\})$ for all $\{t_n\} \in l^\infty$.

Such a functional μ is called a Banach limit and the value of μ at $\{t_n\} \in l^\infty$ is denoted by $\mu_n t_n$ [9]. Let μ be a Banach limit and let $\{t_n\} \in l^\infty$ be such that $\lim_{n\to\infty} t_n = t$, then the Banach limit of $\{t_n\}$ is also t. It is known that the reflexivity of the Banach space E implies the following.

Lemma 3 ([9]). *Let E be a reflexive Banach space, let C be a nonempty, closed, and convex subset of E, and let $g : C \to \mathbb{R}$ be a convex, continuous, and coercive function. Then there exists $u \in C$ such that $g(u) = \inf g(C)$.*

Definition 1 ([2]). Let E be a Banach space, let C be a nonempty subset of E, and let α be a real number such that $0 \le \alpha < 1$. A mapping $T : C \to E$ is said to be α-nonexpansive if

$$\|Tx - Ty\|^2 \le \alpha\|Tx - y\|^2 + \alpha\|Ty - x\|^2 + (1 - 2\alpha)\|x - y\|^2$$

for all $x, y \in C$.

The following is the main result of Aoyama and Kohsaka [2].

Theorem 1. *Let E be a uniformly convex Banach space, let C be a nonempty, closed and convex subset of E, and let $T : C \to C$ be an α-nonexpansive mapping for some real number α such that $\alpha < 1$. Then $F(T)$ is nonempty if and only if there exists $x \in C$ such that $\{T^n x\}$ is bounded.*

2. FIXED POINT THEORY

We first give the definition of generalized nonexpansive mappings.

Definition 2. Let E be a Banach space, and let C be a nonempty subset of E. Let $p > 1$, $\alpha_1 \ge 0,\ldots,\alpha_m \ge 0$ with $\Sigma_{i=1}^m \alpha_i = 1$, and let $a, b, c, d \in \mathbb{R}$ with $b < \alpha_1$ for $m = 1$ and $b \le \alpha_1$ for $m > 1$, $a + c > 0$ and $a + b + c \le 1$. A mapping $T : C \to C$ is said to be generalized nonexpansive if

$$\Sigma_{i=1}^m \alpha_i \|T^i x - T^i y\|^p \le a\|x - y\|^p + b\|Ty - x\|^p + c\|y - Tx\|^p + d\|x - Tx\|^p$$

for all $x, y \in C$.

In the following, we give an example of a generalized nonexpansive mapping which is not an α-nonexpansive mapping.

Example 1. Let $E = \mathbb{R}$, $C = [\sqrt{2}, \sqrt{3}]$ and let Q denotes the set of rational numbers. Let $T : C \to C$ be defined as

$$Tx = \begin{cases} \sqrt{2}, & x \in Q \\ \sqrt{3}. & x \notin Q \end{cases}$$

Then $T^2 x = \sqrt{3}$ for each $x \in C$ and so

$$|T^2 x - T^2 y|^2 = 0 \le |x - y|^2, \text{ for each } x, y \in [\sqrt{2}, \sqrt{3}].$$

Thus T is a generalized nonexpansive map. Now, we show that T is not α-nonexpansive. On the contrary, assume that there exists $0 \le \alpha < 1$ such that

$$|Tx - Ty|^2 \le \alpha|Tx - y|^2 + \alpha|Ty - x|^2 + (1 - 2\alpha)|x - y|^2, \text{ for each } x, y \in [\sqrt{2}, \sqrt{3}].$$

Let $x \in Q$ and $y \notin Q$ with $\sqrt{2} < x < y < \sqrt{3}$. Then from the above we would have

$$(\sqrt{3}-\sqrt{2})^2 \leq \alpha|\sqrt{2}-y|^2 + \alpha|\sqrt{3}-x|^2 + (1-2\alpha)|x-y|^2$$
$$< \alpha(\sqrt{3}-\sqrt{2})^2 + \alpha(\sqrt{3}-\sqrt{2})^2 + (1-2\alpha)(\sqrt{3}-\sqrt{2})^2 = (\sqrt{3}-\sqrt{2})^2,$$

a contradiction.

Now, we are ready to state our first main result.

Theorem 2. *Let E be a Banach space, let C be a nonempty, closed, and convex subset of E, and let $T : C \to C$ be a generalized nonexpansive mapping. Assume that E is uniformly convex if $p = 2$ and assume that E is p-uniformly convex for $1 < p < 2$. Then $\bigcup_{i=1}^m F(T^i) \neq \varnothing$ if there exists $x_0 \in C$ such that $\{T^n x_0\}$ is bounded and either $d = 0$ or $\lim_{n\to\infty} \|T^n x_0 - T^{n+1} x_0\| = 0$. Moreover, if $\bigcup_{i=1}^m F(T^i) \neq \varnothing$ then there exists $x_0 \in C$ such that $\{T^n x_0\}$ is bounded.*

Proof. Notice first that if $x_0 \in \bigcup_{i=1}^m F(T^i)$ then there exists $1 \leq j \leq m$ such that $T^j x_0 = x_0$ and so $\{T^n x_0 : n \in \mathbb{N}\} = \{T x_0, ..., T^j x_0\}$. Thus the sequence $\{T^n x_0\}$ is bounded. Now assume that there exists $x_0 \in C$ such that $\{T^n x_0\}$ is bounded. Let μ be a Banach limit and let $y \in C$ be given. For each bounded sequence $\{t_n\} \in l^\infty$ the value of μ at $\{t_n\} \in l^\infty$ is denoted by $\mu_n t_n$. Since T is generalized nonexpansive, we have

$$\Sigma_{i=1}^m \alpha_i \|T^{n+i} x_0 - T^i y\|^p$$
$$\leq a\|T^n x_0 - y\|^p + b\|Ty - T^n x_0\|^p$$
$$+ c\|y - T^{n+1} x_0\|^p + d\|T^n x_0 - T^{n+1} x_0\|^p,$$

for all $n \in \mathbb{N}$, where $p > 1$, $\alpha_i \geq 0$, $\Sigma_{i=1}^m \alpha_i = 1$, $b < \alpha_1$, $a + c > 0$ and $a + b + c \leq 1$. Since μ is a Banach limit, we have

$$\Sigma_{i=1}^m \alpha_i \mu_n \|T^{n+i} x_0 - T^i y\|^p$$
$$\leq a\mu_n\|T^n x_0 - y\|^p + b\mu_n\|Ty - T^n x_0\|^p$$
$$+ c\mu_n\|y - T^{n+1} x_0\|^p + d\mu_n\|T^n x_0 - T^{n+1} x_0\|^p.$$

Thus by our assumptions

$$\left(\frac{\alpha_1 - b}{a+c}\right)\mu_n\|T^n x_0 - Ty\|^p + \Sigma_{i=2}^m \frac{\alpha_i}{a+c}\mu_n\|T^n x_0 - T^i y\|^p \leq \mu_n\|T^n x_0 - y\|^p \tag{2.1}$$

Let $g : C \to \mathbb{R}$ be a function defined by $g(y) = \mu_n\|T^n x_0 - y\|^p$ for all $y \in C$. Now we assert that g is a convex, continuous, and coercive function. The convexity of g follows immediately from Lemmas 1 and 2. We show that g is continuous. Let $\{y_m\}$ be a sequence in C such that $y_m \to y$. Then by the mean value theorem, we have

$$|\|T^n x_0 - y_m\|^p - \|T^n x_0 - y\|^p| = |\|T^n x_0 - y_m\| - \|T^n x_0 - y\|||pc_{m,n}^{p-1}|,$$

for all $m,n \in \mathbb{N}$, where

$$\min\{\|T^n x_0 - y_m\|, \|T^n x_0 - y\|\} \leq c_{m,n} \leq \max\{\|T^n x_0 - y_m\|, \|T^n x_0 - y\|\}.$$

Hence

$$\begin{aligned} &|\|T^n x_0 - y_m\|^p - \|T^n x_0 - y\|^p| \\ &\quad \leq |\|T^n x_0 - y_m\| - \|T^n x_0 - y\|| p(\|T^n x_0 - y_m\| + \|T^n x_0 - y\|)^{p-1} \\ &\qquad \leq \|y_m - y\| \sup\{p(\|T^n x_0 - y_m\| + \|T^n x_0 - y\|)^{p-1} : m,n \in \mathbb{N}\}, \end{aligned}$$

for all $m,n \in \mathbb{N}$. This shows that the function $h : C \to l^\infty$ defined by

$$h(z) = \{\|T^n x_0 - z\|^p\}_n, \quad z \in C$$

is continuous. Thus $g = \mu \circ h$ is also continuous. We next show that g is coercive. If $\{z_m\}$ is a sequence in C such that $\|z_m\| \to \infty$, then we have

$$\|T^n x_0 - z_m\|^p \geq (|\|z_m\| - \|T^n x_0\||)^p$$

and hence $g(z_m) \to \infty$.
It follows from Lemma 3 that there exists $u \in C$ such that $g(u) = \inf g(C)$. Now, we prove that such a point u is unique. Suppose that there exist $u_1, u_2 \in C$ such that $u_1 \neq u_2$ and $g(u_1) = g(u_2) = \inf g(C)$. If $p = 2$ then from Lemma 1 for $\epsilon = \| u_1 - u_2 \| > 0$, we have $\delta > 0$ such that

$$\|\frac{1}{2}(T^n x_0 - u_1) + \frac{1}{2}(T^n x_0 - u_2)\|^2 \leq \frac{1}{2}\|T^n x_0 - u_1\|^2 + \frac{1}{2}\|T^n x_0 - u_2)\|^2 - \delta,$$

for all $n \in \mathbb{N}$. If $2 \neq p > 1$ then from Lemma 2, we get

$$\begin{aligned} &\|\frac{1}{2}(T^n x_0 - u_1) + \frac{1}{2}(T^n x_0 - u_2)\|^p \\ &\qquad \leq \frac{1}{2}\|T^n x_0 - u_1\|^p + \frac{1}{2}\|T^n x_0 - u_2)\|^p - (\frac{1}{2})^p d\|u_1 - u_2\|, \end{aligned}$$

for all $n \in \mathbb{N}$. The above inequalities imply that $g(\frac{u_1+u_2}{2}) < \inf g(C)$. On the other hand, since $\frac{u_1+u_2}{2} \in C$, we have $\inf g(C) \leq g(\frac{u_1+u_2}{2})$, a contradiction. Hence there exists a unique $u \in C$ such that $g(u) = \inf g(C)$. Now we show that there exists $j \in \{1,2,...,m\}$ such that $g(T^j u) \leq g(u)$. On the contrary, assume that $g(u) < g(T^i u)$, for each $1 \leq i \leq m$. Since by our assumptions $\frac{\alpha_1 - b}{a+c} + \Sigma_{i=2}^m \frac{\alpha_i}{a+c} \geq 1$ then, we get

$$g(u) < \frac{\alpha_1 - b}{a+c} g(Tu) + \Sigma_{i=2}^m \frac{\alpha_i}{a+c} g(T^i)$$

which contradicts (2.1). Hence there exists $j \in \{1,2,...,m\}$ such that $g(T^j u) \leq g(u)$. By the assumption on T, we also know that $T^j u \in C$, and so $T^j u = u$ for some $j \in \{1,2,...,m\}$. □

Theorem 2 immediately implies the following corollary.

Corollary 1. *Let E be a uniformly convex Banach space, and let C be a nonempty, closed, and convex subset of E. Let $T : C \to C$ be a mapping satisfying*

$$\|Tx - Ty\|^2 \le a\|x - y\|^2 + b\|Tx - y\|^2 + c\|x - Ty\|^2 + d\|x - Tx\|^2,$$

for all $x, y \in C$, where $b < 1$, $a + c > 0$ and $a + b + c \le 1$. Then $F(T)$ is nonempty if there exists $x_0 \in C$ such that $\{T^n x_0\}$ is bounded and either $d = 0$ or $\lim_{n\to\infty} \|T^n x_0 - T^{n+1} x_0\| = 0$.

The following corollary is a new coincident point result.

Corollary 2. *Let E be a uniformly convex Banach space, and let C be a nonempty, closed, bounded and convex subset of E. Let $T : C \to C$ and $S : C \to C$ be mappings such that $T(C) \subseteq S(C)$ and $S(C)$ is convex and closed. Assume that T and S satisfying*

$$\|Tx - Ty\|^2 \le (1 - 2\alpha)\|Sx - Sy\|^2 + \alpha\|Tx - Sy\|^2 + \alpha\|Sx - Ty\|^2,$$

for all $x, y \in C$, where $0 \le \alpha < 1$. Then T and S have a coincidence point, that is, there exists $u \in C$ such that $Tu = Su$.

Proof. We use the technique in [6]. There exists $D \subseteq C$ such that $S(D) = S(C)$ and $S : D \to C$ is one-to-one. Now, define a map $R : S(D) \to S(D)$ by $R(Sx) = Tx$. Since S is one-to-one on D and $T(C) \subseteq S(C)$, R is well-defined. Note that

$$\begin{aligned}\|R(Sx) - R(Sy)\|^2 &= \|Tx - Ty\|^2 \\ &\le (1 - 2\alpha)\|Sx - Sy\|^2 + \alpha\|Tx - Sy\|^2 + \alpha\|Sx - Ty\|^2 \\ &= (1 - 2\alpha)\|Sx - Sy\|^2 + \alpha\|R(Sx) - Sy\|^2 + \alpha\|Sx - R(Sy)\|^2\end{aligned}$$

for all $Sx, Sy \in S(D)$. Since $S(D) = S(C)$ is convex, closed and bounded, by using Corollary 1, R has a fixed point in $S(C)$, that is, there exists $u \in C$ such that $R(Su) = Su$, and so $Tu = Su$. □

Corollary 3. *Let $1 < p < 2$, E be a p-uniformly convex Banach space, and let C be a nonempty, closed, and convex subset of E. Let $T : C \to C$ be a mapping satisfying*

$$\|Tx - Ty\|^p \le (1 - 2\alpha)\|x - y\|^p + \alpha\|Tx - y\|^p + \alpha\|x - Ty\|^p,$$

for all $x, y \in C$, where $0 \le \alpha < 1$. Then $F(T)$ is nonempty if and only if there exists $x_0 \in C$ such that $\{T^n x_0\}$ is bounded.

Corollary 4. *Let $1 < p < 2$, E be a p-uniformly convex Banach space, and let C be a nonempty, closed, and convex subset of E. Let $T : C \to C$ be a mapping satisfying*

$$\|Tx - Ty\|^p \le (1 - 2\alpha)\|x - y\|^p + \alpha\|Tx - y\|^p + \alpha\|x - Ty\|^p,$$

for all $x, y \in C$, where $0 \le \alpha < 1$. Then $F(T)$ is nonempty if and only if there exists $x_0 \in C$ such that $\{T^n x_0\}$ is bounded.

By the same technique as in the proof of Corollary 2, we can deduce the following coincidence point result from Corollary 4. For some previous studies of coincident point theory, see [7].

Corollary 5. *Let $1 < p < 2$, E be a p-uniformly convex Banach space, and let C be a nonempty, closed, bounded and convex subset of E. Let $T : C \to C$ and $S : C \to C$ be mappings such that $T(C) \subseteq S(C)$ and $S(C)$ is convex and closed. Assume that T and S satisfying*

$$\|Tx - Ty\|^p \leq (1-2\alpha)\|Sx - Sy\|^p + \alpha\|Tx - Sy\|^p + \alpha\|Sx - Ty\|^p,$$

for all $x, y \in C$, where $0 \leq \alpha < 1$. Then T and S have a coincidence point, that is, there exists $u \in C$ such that $Tu = Su$.

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