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ASYMPTOTIC PROPERTIES OF EIGENVALUES AND EIGENFUNCTIONS OF A STURM-LIOUVILLE PROBLEM WITH DISCONTINUOUS WEIGHT FUNCTION

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Abstract. In this paper, we extend some spectral properties of regular Sturm-Liouville problems to those which consist of a Sturm-Liouville equation with discontinuous weight at two interior points together with spectral parameter-dependent boundary conditions. By modifying some techniques of [C. T. Fulton, Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions, Proc. Roy. Soc. Edinburgh Sect. A 77 (1977) 293-308; O. Sh. Mukhtarov and M. Kadakal, Some spectral properties of one Sturm-Liouville type problem with discontinuous weight, Siberian Mathematical Journal, 46 (2005) 681-694], we give an operator-theoretic formulation for the considered problem and obtain asymptotic formulas for the eigenvalues and eigenfunctions.

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Keywords: Sturm-Liouville problem, eigenparameter, transmission conditions, asymptotics of eigenvalues and eigenfunctions

1. INTRODUCTION

Sturmian theory is one of the most extensively developing fields in theoretical and applied mathematics The literature is voluminous and we refer to [1–25]. Particularly, there has been an increasing interest in the spectral analysis of boundary-value problems with eigenvalue-dependent boundary conditions [1–3, 5–10, 12–14, 16, 17, 19–22, 24, 25].

In this paper following [12] we consider the boundary value problem for the differential equation

$$\tau u := -u'' + q(x)u = \lambda\omega(x)u \tag{1.1}$$

for $x \in [-1, h_1) \cup (h_1, h_2) \cup (h_2, 1]$ (i.e., x belongs to $[-1, 1]$ but the two inner points $x = h_1$ and $x = h_2$), where $q(x)$ is a real valued function, continuous in $[-1, h_1)$, (h_1, h_2) and $(h_2, 1]$ with the finite limits $q(\pm h_1) = \lim_{x\to\pm h_1}$, $q(\pm h_2) = \lim_{x\to\pm h_2}$; $\omega(x)$ is a discontinuous weight function such that $\omega(x) = \omega_1^2$ for $x \in [-1, h_1)$, $\omega(x) = \omega_2^2$ for $x \in (h_1, h_2)$ and $\omega(x) = \omega_3^2$ for $x \in (h_2, 1]$, $\omega > 0$ together with the standart boundary condition at $x = -1$

$$L_1 u := \cos\alpha u(-1) + \sin\alpha u'(-1) = 0, \tag{1.2}$$

the spectral parameter dependent boundary condition at $x = 1$

$$L_2 u := \lambda\left(\beta_1' u(1) - \beta_2' u'(1)\right) + \left(\beta_1 u(1) - \beta_2 u'(1)\right) = 0, \tag{1.3}$$

and the four transmission conditions at the points of discontinuity $x = h_1$ and $x = h_2$

$$L_3 u := \gamma_1 u(h_1 - 0) - \delta_1 u(h_1 + 0) = 0, \tag{1.4}$$

$$L_4 u := \gamma_2 u'(h_1 - 0) - \delta_2 u'(h_1 + 0) = 0, \tag{1.5}$$

$$L_5 u := \gamma_3 u(h_2 - 0) - \delta_3 u(h_2 + 0) = 0, \tag{1.6}$$

$$L_6 u := \gamma_4 u'(h_2 - 0) - \delta_4 u'(h_2 + 0) = 0, \tag{1.7}$$

in the Hilbert space $L_2(-1, h_1) \oplus L_2(h_1, h_2) \oplus L_2(h_2, 1)$ where $\lambda \in \mathbb{C}$ is a complex spectral parameter; and all coefficients of the boundary and transmission conditions are real constants. We assume naturally that $|\alpha_1| + |\alpha_2| \neq 0$, $\left|\beta_1'\right| + \left|\beta_2'\right| \neq 0$ and $|\beta_1| + |\beta_2| \neq 0$. Moreover, we will assume that $\rho := \beta_1'\beta_2 - \beta_1\beta_2' > 0$. A Sturm-Liouville problem with eigenparameter contained in the boundary condition arise upon separation of variables in the one-dimensional wave and heat equations for a varied assortment of physical problems, e.g. in the diffusion of water vapour through a porous membrane and several electric circuit problems involving long cables. (for example, see [3, 13]), vibrating string problems when the string loaded additionally with point masses (for example, see [18]), and a thermal conduction problem for a thin laminated plate (for example, see [23]).

2. Operator-Theoretic Formulation of the Problem

In this section, we introduce a special inner product in the Hilbert space $(L_2(-1, h_1) \oplus L_2(h_1, h_2) \oplus L_2(h_2, 1)) \oplus \mathbb{C}$ and define a linear operator A in it so that the problem (1.1)-(1.7) can be interpreted as the eigenvalue problem for A. To this end, we define a new Hilbert space inner product on

$$H := (L_2(-1, h_1) \oplus L_2(h_1, h_2) \oplus L_2(h_2, 1)) \oplus \mathbb{C}$$

by

$$\langle F, G\rangle_H = \omega_1^2 \int_{-1}^{h_1} f(x)\overline{g(x)}dx + \omega_2^2 \frac{\delta_1\delta_2}{\gamma_1\gamma_2} \int_{h_1}^{h_2} f(x)\overline{g(x)}dx + \omega_3^2 \frac{\delta_1\delta_2\delta_3\delta_4}{\gamma_1\gamma_2\gamma_3\gamma_4} \int_{h_2}^{1} f(x)\overline{g(x)}dx + \frac{\delta_1\delta_2\delta_3\delta_4}{\rho\gamma_1\gamma_2\gamma_3\gamma_4} f_1\overline{g_1}$$

for $F = \begin{pmatrix} f(x) \\ f_1 \end{pmatrix}$ and $G = \begin{pmatrix} g(x) \\ g_1 \end{pmatrix} \in H$. For convenience we will use the notations

$$R_1(u) := \beta_1 u(1) - \beta_2 u'(1), \quad R_1'(u) := \beta_1' u(1) - \beta_2' u'(1).$$

In this Hilbert space we construct the operator $A : H \to H$ with domain

$$D(A) = \left\{ F = \begin{pmatrix} f(x) \\ f_1 \end{pmatrix} \mid f(x), f'(x) \text{ are absolutely continuous in } [1,h_1]\cup[h_1,h_2]\cup[h_2,1]; \right.$$
$$\text{has finite limits } f(h_1 \pm 0),\ f(h_2 \pm 0), f'(h_1 \pm 0), f'(h_2 \pm 0);$$
$$\tau f \in L_2(-1,h_1) \oplus L_2(h_1,h_2) \oplus L_2(h_2,1);$$
$$\left. L_1 f = L_3 f = L_4 f = L_5 f = L_6 f = 0, f_1 = R_1'(f) \right\} \tag{2.1}$$

which acts by the rule

$$AF = \begin{pmatrix} \frac{1}{\omega(x)}[-f'' + q(x)f] \\ -R_1(f) \end{pmatrix} \quad \text{with } F = \begin{pmatrix} f(x) \\ R_1'(f) \end{pmatrix} \in D(A). \tag{2.2}$$

Thus we can pose the boundary-value-transmission problem (1.1)-(1.7) in H as

$$AU = \lambda U, \quad U := \begin{pmatrix} u(x) \\ R_1'(u) \end{pmatrix} \in D(A). \tag{2.3}$$

It is readily verified that the eigenvalues of A coincide with those of the problem (1.1)-(1.7).

Theorem 1. *The operator A is symmetric.*

Proof. Let $F = \begin{pmatrix} f(x) \\ R_1'(f) \end{pmatrix}$ and $G = \begin{pmatrix} g(x) \\ R_1'(g) \end{pmatrix}$ be arbitrary elements of $D(A)$. Twice integrating by parts we find

$$\begin{aligned} \langle AF, G\rangle_H - \langle F, AG\rangle_H &= W(f,\overline{g};h_1-0) - W(f,\overline{g};-1) \\ &+ \frac{\delta_1\delta_2}{\gamma_1\gamma_2}\left(W(f,\overline{g};h_2-0) - W(f,\overline{g};h_1+0)\right) \\ &+ \frac{\delta_1\delta_2\delta_3\delta_4}{\gamma_1\gamma_2\gamma_3\gamma_4}\left(W(f,\overline{g};1) - W(f,\overline{g};h_2+0)\right) \\ &+ \frac{\delta_1\delta_2\delta_3\delta_4}{\rho\gamma_1\gamma_2\gamma_3\gamma_4}\left(R_1'(f)R_1(\overline{g}) - R_1(f)R_1'(\overline{g})\right) \end{aligned} \tag{2.4}$$

where, as usual, $W(f,g;x)$ denotes the Wronskian of f and g; i.e.,

$$W(f,g;x) := f(x)g'(x) - f'(x)g(x).$$

Since $F, G \in D(A)$, the first components of these elements, i.e. f and g satisfy the boundary condition (1.2). From this fact we easily see that

$$W(f,\overline{g};-1) = 0, \tag{2.5}$$

since $\cos\alpha$ and $\sin\alpha$ are real. Further, as f and g also satisfy both transmission conditions, we obtain

$$W(f,\overline{g};h_1-0)=\frac{\delta_1\delta_2}{\gamma_1\gamma_2}W(f,\overline{g};h_1+0) \tag{2.6}$$

$$W(f,\overline{g};h_2-0)=\frac{\delta_1\delta_2\delta_3\delta_4}{\gamma_1\gamma_2\gamma_3\gamma_4}W(f,\overline{g};h_2+0) \tag{2.7}$$

Moreover, the direct calculations give

$$R_1'(f)R_1(\overline{g})-R_1(f)R_1'(\overline{g})=-\rho W(f,\overline{g};1) \tag{2.8}$$

Now, inserting (2.5)-(2.8) in (2.4), we have

$$\langle AF,G\rangle_H=\langle F,AG\rangle_H \qquad (F,G\in D(A))$$

and so A is symmetric. □

Recalling that the eigenvalues of (1.1)-(1.7) coincide with the eigenvalues of A, we have the next corollary:

Corollary 1. *All eigenvalues of (1.1)-(1.7) are real.*

Since all eigenvalues are real it is enough to study only the real-valued eigenfunctions. Therefore we can now assume that all eigenfunctions of (1.1)-(1.7) are real-valued.

3. Asymptotic Formulas for Eigenvalues and Fundamental Solutions

Let us define fundamental solutions

$$\phi(x,\lambda)=\begin{cases}\phi_1(x,\lambda), & x\in[-1,h_1),\\ \phi_2(x,\lambda), & x\in(h_1,h_2),\\ \phi_3(x,\lambda), & x\in(h_2,1]\end{cases}$$

and

$$\chi(x,\lambda)=\begin{cases}\chi_1(x,\lambda), & x\in[-1,h_1),\\ \chi_2(x,\lambda), & x\in(h_1,h_2),\\ \chi_3(x,\lambda), & x\in(h_2,1]\end{cases}$$

of (1.1) by the following procedure. We first consider the next initial-value problem:

$$-u''+q(x)u=\lambda\omega_1^2u,\ x\in[-1,h_1] \tag{3.1}$$

$$u(-1)=\sin\alpha, \tag{3.2}$$

$$u'(-1)=-\cos\alpha \tag{3.3}$$

By virtue of ([19], Theorem 1.5) the problem (3.1)-(3.3) has a unique solution $u=\phi_1(x,\lambda)$ which is an entire function of $\lambda\in\mathbb{C}$ for each fixed $x\in[-1,h_1]$. Similarly,

$$-u''+q(x)u=\lambda\omega_2^2u,\quad x\in[h_1,h_2] \tag{3.4}$$

$$u(h_1)=\frac{\gamma_1}{\delta_1}\phi_1(h_1,\lambda), \tag{3.5}$$

$$u'(h_1)=\frac{\gamma_2}{\delta_2}\phi_1'(h_1,\lambda), \tag{3.6}$$

has a unique solution $u=\phi_2(x,\lambda)$ which is an entire function of $\lambda\in\mathbb{C}$ for each fixed $x\in[h_1,h_2]$. Continuing in this manner

$$-u''+q(x)u=\lambda\omega_3^2u,\quad x\in[h_2,1] \tag{3.7}$$

$$u(h_2)=\frac{\gamma_3}{\delta_3}\phi_2(h_2,\lambda), \tag{3.8}$$

$$u'(h_2)=\frac{\gamma_4}{\delta_4}\phi_2'(h_2,\lambda), \tag{3.9}$$

has a unique solution $u=\phi_3(x,\lambda)$ which is an entire function of $\lambda\in\mathbb{C}$ for each fixed $x\in[h_2,1]$. Slightly modifying the method of ([19], Theorem 1.5) we can prove that the initial-value problem

$$-u''+q(x)u=\lambda\omega_3^2u,\ x\in[h_2,1] \tag{3.10}$$

$$u(1)=\beta_2'\lambda+\beta_2, \tag{3.11}$$

$$u'(1)=\beta_1'\lambda+\beta_1 \tag{3.12}$$

(3.10)-(3.13) has a unique solution $u=\chi_3(x,\lambda)$ which is an entire function of spectral parameter $\lambda\in\mathbb{C}$ for each fixed $x\in[h_2,1]$. Similarly,

$$-u''+q(x)u=\lambda\omega_2^2u,\quad x\in[h_1,h_2] \tag{3.13}$$

$$u(h_2)=\frac{\delta_3}{\gamma_3}\chi_3(h_2,\lambda), \tag{3.14}$$

$$u'(h_2)=\frac{\delta_4}{\gamma_4}\chi_3'(h_2,\lambda), \tag{3.15}$$

has a unique solution $u=\chi_2(x,\lambda)$ which is an entire function of $\lambda\in\mathbb{C}$ for each fixed $x\in[h_1,h_2]$. Continuing in this manner

$$-u''+q(x)u=\lambda\omega_3^2u,\quad x\in[-1,h_1] \tag{3.16}$$

$$u(h_1)=\frac{\delta_1}{\gamma_1}\chi_2(h_1,\lambda), \tag{3.17}$$

$$u'(h_1)=\frac{\delta_2}{\gamma_2}\chi_2'(h_1,\lambda), \tag{3.18}$$

has a unique solution $u=\chi_1(x,\lambda)$ which is an entire function of $\lambda\in\mathbb{C}$ for each fixed $x\in[-1,h_1]$.

By virtue of (3.2) and (3.3) the solution $\phi(x,\lambda)$ satisfies the first boundary condition (1.2). Moreover, by (3.5), (3.6), (3.8) and (3.9), $\phi(x,\lambda)$ satisfies also transmission conditions (1.4)-(1.7). Similarly, by (3.11), (3.12), (3.14), (3.15), (3.17) and (3.18) the other solution $\chi(x,\lambda)$ satisfies the second boundary condition (1.3) and transmission conditions (1.4)-(1.7). It is well-known from the theory of ordinary differential equations that each of the Wronskians $\Delta_1(\lambda)=W(\phi_1(x,\lambda),\chi_1(x,\lambda))$, $\Delta_2(\lambda)=W(\phi_2(x,\lambda),\chi_2(x,\lambda))$ and $\Delta_3(\lambda)=W(\phi_3(x,\lambda),\chi_3(x,\lambda))$ are independent of x in $[-1,h_1]$, $[h_1,h_2]$ and $[h_2,1]$ respectively.

Lemma 1. *The equality* $\Delta_1(\lambda)=\frac{\delta_1\delta_2}{\gamma_1\gamma_2}\Delta_2(\lambda)=\frac{\delta_1\delta_2\delta_3\delta_4}{\gamma_1\gamma_2\gamma_3\gamma_4}\Delta_3(\lambda)$ *holds for each* $\lambda\in\mathbb{C}$.

Proof. Since the above Wronskians are independent of x, using (3.8), (3.9), (3.11), (3.12), (3.14), (3.15), (3.17) and (3.18) we find

$$\begin{aligned}\Delta_1(\lambda)&=\phi_1(h_1,\lambda)\chi_1'(h_1,\lambda)-\phi_1'(h_1,\lambda)\chi_1(h_1,\lambda)\\&=\left(\frac{\delta_1}{\gamma_1}\phi_2(h_1,\lambda)\right)\left(\frac{\delta_2}{\gamma_2}\chi_2'(h_1,\lambda)\right)-\left(\frac{\delta_2}{\gamma_2}\phi_2'(h_1,\lambda)\right)\left(\frac{\delta_1}{\gamma_1}\chi_2(h_1,\lambda)\right)\\&=\frac{\delta_1\delta_2}{\gamma_1\gamma_2}\Delta_2(\lambda)=\left(\frac{\delta_1\delta_3}{\gamma_1\gamma_3}\phi_3(h_2,\lambda)\right)\left(\frac{\delta_2\delta_4}{\gamma_2\gamma_4}\chi_3'(h_2,\lambda)\right)\\&\quad-\left(\frac{\delta_2\delta_4}{\gamma_2\gamma_4}\phi_3'(h_2,\lambda)\right)\left(\frac{\delta_1\delta_3}{\gamma_1\gamma_3}\chi_3(h_2,\lambda)\right)=\frac{\delta_1\delta_2\delta_3\delta_4}{\gamma_1\gamma_2\gamma_3\gamma_4}\Delta_3(\lambda).\end{aligned}$$

□

Corollary 2. *The zeros of* $\Delta_1(\lambda)$, $\Delta_2(\lambda)$ *and* $\Delta_3(\lambda)$ *coincide.*

In view of Lemma 3.1 we denote $\Delta_1(\lambda)$, $\frac{\delta_1\delta_2}{\gamma_1\gamma_2}\Delta_2(\lambda)$ and $\frac{\delta_1\delta_2\delta_3\delta_4}{\gamma_1\gamma_2\gamma_3\gamma_4}\Delta_3(\lambda)$ by $\Delta(\lambda)$. Recalling the definitions of $\phi_i(x,\lambda)$ and $\chi_i(x,\lambda)$, we can state the next corollary.

Corollary 3. *The function* $\Delta(\lambda)$ *is an entire function.*

Theorem 2. *The eigenvalues of (1.1)-(1.7) are the roots of* $\Delta(\lambda)=0$.

Proof. Let $\Delta(\lambda_0)=0$. Then $W(\phi_1(x,\lambda_0),\chi_1(x,\lambda_0))=0$ for all $x\in[-1,h_1]$. Consequently, the functions $\phi_1(x,\lambda_0)$ and $\chi_1(x,\lambda_0)$ are linearly dependent, i.e. $\chi_1(x,\lambda_0)=k\phi_1(x,\lambda_0)$, $x\in[-1,h_1]$, for some $k\neq 0$. By (3.2) and (3.3), from this equality, we have

$$\cos\alpha\chi(-1,\lambda_0)+\sin\alpha\chi'(-1,\lambda_0)=\cos\alpha\chi_1(-1,\lambda_0)+\sin\alpha\chi_1'(-1,\lambda_0)$$

$$=k\left(\cos\alpha\phi_1(-1,\lambda_0)+\sin\alpha\phi_1'(-1,\lambda_0)\right)=k(\cos\alpha\sin\alpha+\sin\alpha(-\cos\alpha))=0,$$

and so $\chi(x,\lambda_0)$ satisfies the first boundary condition (1.2). Recalling that the solution $\chi(x,\lambda_0)$ also satisfies the other boundary condition (1.3) and transmission conditions (1.4)-(1.7). We conclude that $\chi(x,\lambda_0)$ is an eigenfunction of (1.1)-(1.7); i.e., λ_0 is an eigenvalue. Thus, each zero of $\Delta(\lambda)$ is an eigenvalue. Now let λ_0 be an eigenvalue and let $u_0(x)$ be an eigenfunction with this eigenvalue. Suppose that $\Delta(\lambda_0)\neq 0$. Whence $W(\phi_1(x,\lambda_0),\chi_1(x,\lambda_0))\neq 0$, $W(\phi_2(x,\lambda_0),\chi_2(x,\lambda_0))\neq 0$ and $W(\phi_3(x,\lambda_0),\chi_3(x,\lambda_0))\neq 0$. From this, by virtue of the well-known properties of Wronskians, it follows that each of the pairs $\phi_1(x,\lambda_0)$, $\chi_1(x,\lambda_0)$; $\phi_2(x,\lambda_0)$, $\chi_2(x,\lambda_0)$ and $\phi_3(x,\lambda_0)$, $\chi_3(x,\lambda_0)$ is linearly independent. Therefore, the solution $u_0(x)$ of (1.1) may be represented as

$$u_0(x)=\begin{cases} c_1\phi_1(x,\lambda_0)+c_2\chi_1(x,\lambda_0), & x\in[-1,h_1), \\ c_3\phi_2(x,\lambda_0)+c_4\chi_2(x,\lambda_0), & x\in(h_1,h_2), \\ c_5\phi_3(x,\lambda_0)+c_6\chi_3(x,\lambda_0), & x\in(h_2,1], \end{cases}$$

where at least one of the coefficients c_i $\left(i=\overline{1,6}\right)$ is not zero. Considering the true equalities

$$L_\upsilon(u_0(x))=0,\quad \upsilon=\overline{1,6}, \tag{3.19}$$

as the homogenous system of linear equations in the variables c_i $\left(i=\overline{1,6}\right)$ and taking (3.5), (3.6), (3.8), (3.9), (3.14), (3.15), (3.17) and (3.18) into account, we see that the determinant of this system is equal to $-\frac{(\delta_1\delta_2\delta_3\delta_4)^2}{\gamma_1\gamma_2\gamma_3\gamma_4}\Delta^4(\lambda_0)$ and so it does not vanish by assumption. Consequently the system (3.19) has the only trivial solution $c_i=0$ $\left(i=\overline{1,6}\right)$. This is a contradiction. And the proof is complete. □

Theorem 3. *Let $\lambda=\mu^2$ and $\operatorname{Im}\mu=t$. Then the following asymptotic equalities hold as $|\lambda|\to\infty$:*

(1) In case $\sin\alpha\neq 0$

$$\phi_1^{(k)}(x,\lambda)=\sin\alpha\frac{d^k}{dx^k}\cos[\mu\omega_1(x+1)]+O\left(\frac{1}{|\mu|^{1-k}}\exp(|t|\omega_1(x+1))\right), \tag{3.20}$$

$$\begin{aligned}\phi_2^{(k)}(x,\lambda)&=\frac{\gamma_1}{\delta_1}\sin\alpha\frac{d^k}{dx^k}\cos[\mu(\omega_2x+\omega_1h_1+\omega_1)]\\&\quad+O\left(\frac{1}{|\mu|^{1-k}}\exp(|t|(\omega_2x+\omega_1h_1+\omega_1))\right),\end{aligned} \tag{3.21}$$

$$\begin{aligned}\phi_3^{(k)}(x,\lambda)&=\frac{\gamma_1\gamma_3}{\delta_1\delta_3}\sin\alpha\frac{d^k}{dx^k}\cos[\mu(\omega_3x+\omega_2h_2+\omega_1)]\\&\quad+O\left(\frac{1}{|\mu|^{1-k}}\exp(|t|(\omega_3x+\omega_2h_2+\omega_1))\right).\end{aligned} \tag{3.22}$$

(2) In case $\sin\alpha = 0$

$$\phi_1^{(k)}(x,\lambda) = \frac{-1}{\mu\omega_1}\cos\alpha \frac{d^k}{dx^k}\sin\left[\mu\omega_1(x+1)\right] + O\left(\frac{1}{|\mu|^{2-k}}\exp\left(|t|\,\omega_1(x+1)\right)\right), \tag{3.23}$$

$$\begin{aligned}\phi_2^{(k)}(x,\lambda) = &-\frac{\gamma_1}{\mu\delta_1}\cos\alpha\frac{d^k}{dx^k}\sin\left[\mu\left(\omega_2 x + \omega_1 h_1 + \omega_1\right)\right] \\ &+ O\left(\frac{1}{|\mu|^{2-k}}\exp\left(|t|\left(\omega_2 x + \omega_1 h_1 + \omega_1\right)\right)\right),\end{aligned} \tag{3.24}$$

$$\begin{aligned}\phi_3^{(k)}(x,\lambda) = &-\frac{\gamma_1\gamma_3}{\mu\delta_1\delta_3}\cos\alpha\frac{d^k}{dx^k}\sin\left[\mu\left(\omega_3 x + \omega_2 h_2 + \omega_1\right)\right] \\ &+ O\left(\frac{1}{|\mu|^{2-k}}\exp\left(|t|\left(\omega_3 x + \omega_2 h_2 + \omega_1\right)\right)\right).\end{aligned} \tag{3.25}$$

for $k = 0$ *and* $k = 1$*. Moreover, each of these asymptotic equalities holds uniformly for* x*.*

Proof. Asymptotic formulas for $\phi_1(x,\lambda)$ and $\phi_2(x,\lambda)$ are found in ([19], Lemma 1.7) and ([12], Theorem 3.2) respectively. But the formulas for $\phi_3(x,\lambda)$ need individual considerations, since this solution is defined by the initial condition with some special nonstandart form. The initial-value problem (3.7)-(3.9) can be transformed into the equivalent integral equation

$$\begin{aligned}u(x) = &\frac{\gamma_3}{\delta_3}\phi_2(h_2,\lambda)\cos\mu\omega_3 x + \frac{\gamma_4}{\mu\omega_3\delta_4}\phi_2'(h_2,\lambda)\sin\mu\omega_3 x \\ &+ \frac{\omega_3}{\mu}\int_{h_2}^{x}\sin\left[\mu\omega_3(x-y)\right]q(y)\,u(y)\,dy\end{aligned} \tag{3.26}$$

Let $\sin\alpha \neq 0$. Inserting (3.21) in (3.26) we have

$$\begin{aligned}\phi_3(x,\lambda) = &\frac{\gamma_1\gamma_3}{\delta_1\delta_3}\sin\alpha\cos\left[\mu\left(\omega_3 x + \omega_2 h_2 + \omega_1\right)\right] \\ &+ \frac{\omega_3}{\mu}\int_{h_2}^{x}\sin\left[\mu\omega_3(x-y)\right]q(y)\,\phi_3(y,\lambda)\,dy \\ &+ O\left(\frac{1}{|\mu|}\exp\left(|t|\left(\omega_3 x + \omega_2 h_2 + \omega_1\right)\right)\right).\end{aligned} \tag{3.27}$$

Multiplying this by $\exp\left(-|t|\left(\omega_3 x + \omega_2 h_2 + \omega_1\right)\right)$ and denoting

$$F(x,\lambda) = \exp\left(-|t|\left(\omega_3 x + \omega_2 h_2 + \omega_1\right)\right)\phi_3(x,\lambda),$$

we have the following integral equation

$$F(x,\lambda)=\frac{\gamma_1\gamma_3}{\delta_1\delta_3}\sin\alpha\exp\left(-|t|\left(\omega_3x+\omega_2h_2+\omega_1\right)\right)\cos\left[\mu\left(\omega_3x+\omega_2h_2+\omega_1\right)\right]$$
$$+\frac{\omega_3}{\mu}\int_{h_2}^{x}\sin\left[\mu\omega_3\left(x-y\right)\right]\exp\left(-|t|\,\omega_3\left(x-y\right)\right)q\left(y\right)F(y,\lambda)dy+O\left(\frac{1}{\mu}\right).$$

Putting $M(\lambda)=\max_{x\in[h_2,1]}|F(x,\lambda)|$, from the last equation we derive that

$$M(\lambda)\leq M_0\left(\left|\frac{\gamma_1\gamma_3}{\delta_1\delta_3}\right|+\frac{1}{\mu}\right)$$

for some $M_0>0$. Consequently, $M(\lambda)=O(1)$ as $|\lambda|\to\infty$, and so $\phi_3\left(x,\lambda\right)=O\left(\exp\left(|t|\left(\omega_3x+\omega_2h_2+\omega_1\right)\right)\right)$ as $|\lambda|\to\infty$. Inserting the integral term of (3.27) yields (3.22) for $k=0$. The case $k=1$ of (3.22) follows at once on differentiating (3.21) and making the same procedure as in the case $k=0$. The proof of (3.25) is similar to that of (3.22). □

Theorem 4. *Let $\lambda=\mu^2$, $\mu=\sigma+it$. Then the following asymptotic formulas hold for the eigenvalues of the boundary-value-transmission problem(1.1)-(1.7):*

Case 1: $\beta_2'\neq0$, $\sin\alpha\neq0$

$$\mu_n=\frac{\pi\left(n-1\right)}{\omega_3+\omega_2h_2+\omega_1}+O\left(\frac{1}{n}\right), \tag{3.28}$$

Case 2: $\beta_2'\neq0$, $\sin\alpha=0$

$$\mu_n=\frac{\pi\left(n-\frac{1}{2}\right)}{\omega_3+\omega_2h_2+\omega_1}+O\left(\frac{1}{n}\right), \tag{3.29}$$

Case 3: $\beta_2'=0$, $\sin\alpha\neq0$

$$\mu_n=\frac{\pi\left(n-\frac{1}{2}\right)}{\omega_3+\omega_2h_2+\omega_1}+O\left(\frac{1}{n}\right), \tag{3.30}$$

Case 4: $\beta_2'=0$, $\sin\alpha=0$

$$\mu_n=\frac{\pi n}{\omega_3+\omega_2h_2+\omega_1}+O\left(\frac{1}{n}\right), \tag{3.31}$$

Proof. Let us consider only the case 1. Putting $x=1$ in

$$\Delta_3\left(\lambda\right)=\phi_3\left(x,\lambda\right)\chi_3'\left(x,\lambda\right)-\phi_3'\left(x,\lambda\right)\chi_3\left(x,\lambda\right)$$

and inserting $\chi_3\left(1,\lambda\right)=\beta_2'\lambda+\beta_2$, $\chi_3'\left(1,\lambda\right)=\beta_1'\lambda+\beta_1$ we have the following representation for $\Delta_3\left(\lambda\right)$:

$$\Delta_3\left(\lambda\right)=\left(\beta_1'\lambda+\beta_1\right)\phi_3\left(1,\lambda\right)-\left(\beta_2'\lambda+\beta_2\right)\phi_3'\left(1,\lambda\right). \tag{3.32}$$

Putting $x=1$ in (3.22) and inserting the result in (3.32), we derive now that

$$\begin{aligned}\Delta_3(\lambda) &= \frac{\delta_2\delta_4}{\gamma_2\gamma_4}\omega_3\beta_2'(\sin\alpha)\mu^3\sin[\mu(\omega_3+\omega_2h_2+\omega_1)] \\ &\quad + O\left(|\mu|^2\exp(2|t|(\omega+\omega_2h_2+\omega_1))\right). \qquad (3.33)\end{aligned}$$

By applying the Rouché Theorem, it follows that $\Delta_3(\lambda)$ has the same number of zeros inside the contour as the leading term in (3.33). Hence, if $\lambda_0<\lambda_1<\lambda_2...$ are the zeros of $\Delta_3(\lambda)$ and $\mu_n^2=\lambda_n$, we have

$$\frac{\pi(n-1)}{\omega_3+\omega_2h_2+\omega_1}+\delta_n \qquad (3.34)$$

for sufficiently large n, where $|\delta_n|<\frac{\pi}{4(\omega_3+\omega_2h_2+\omega_1)}$ for sufficiently large n. By putting in (3.33) we have $\delta_n=O\left(\frac{1}{n}\right)$, and the proof is completed in Case 1. The proofs for the other cases are similar. □

Theorem 5. *The following asymptotic formulas hold for the eigenfunctions*

$$\phi_{\lambda_n}(x)=\begin{cases}\phi_1(x,\lambda_n), & x\in[-1,h_1),\\ \phi_2(x,\lambda_n), & x\in(h_1,h_2),\\ \phi_3(x,\lambda_n), & x\in(h_2,1]\end{cases}$$

of (1.1)-(1.7):

Case 1: $\beta_2'\neq 0$, $\sin\alpha\neq 0$

$$\phi_{\lambda_n}(x)=\begin{cases}\sin\alpha\cos\left[\frac{\omega_1\pi(n-1)(x+1)}{\omega_2+\omega_1}\right]+O\left(\frac{1}{n}\right), & x\in[-1,h_1),\\ \frac{\gamma_1}{\delta_1}\sin\alpha\cos\left[\frac{(\omega_2x+\omega_1h_1+\omega_1)\pi(n-1)}{\omega_2+\omega_1h_1+\omega_1}\right]+O\left(\frac{1}{n}\right), & x\in(h_1,h_2),\\ \frac{\gamma_1\gamma_3}{\delta_1\delta_3}\sin\alpha\cos\left[\frac{(\omega_3x+\omega_2h_2+\omega_1)\pi(n-1)}{\omega_3+\omega_2h_2+\omega_1}\right]+O\left(\frac{1}{n}\right), & x\in(h_2,1].\end{cases}$$

Case 2: $\beta_2'\neq 0$, $\sin\alpha=0$

$$\phi_{\lambda_n}(x)$$

$$=\begin{cases}-\frac{\omega_1+\omega_2}{\omega_1}\frac{\cos\alpha}{\pi\left(n-\frac{1}{2}\right)}\sin\left[\frac{\omega_1\pi\left(n-\frac{1}{2}\right)(x+1)}{\omega_2+\omega_1}\right]+O\left(\frac{1}{n^2}\right), & x\in[-1,h_1),\\ \frac{-\gamma_1}{\delta_1}\frac{\omega_1+\omega_2}{\omega_1}\frac{\cos\alpha}{\pi\left(n-\frac{1}{2}\right)}\sin\left[\frac{(\omega_2x+\omega_1h_1+\omega_1)\pi\left(n-\frac{1}{2}\right)}{\omega_2+\omega_1h_1+\omega_1}\right]+O\left(\frac{1}{n^2}\right), & x\in(h_1,h_2),\\ \frac{-\gamma_1\gamma_3}{\delta_1\delta_3}\frac{\omega_1+\omega_2}{\omega_1}\frac{\cos\alpha}{\pi\left(n-\frac{1}{2}\right)}\sin\left[\frac{(\omega_3x+\omega_2h_2+\omega_1)\pi\left(n-\frac{1}{2}\right)}{\omega_3+\omega_2h_2+\omega_1}\right]+O\left(\frac{1}{n^2}\right), & x\in(h_2,1].\end{cases}$$

Case 3: $\beta_2' = 0$, $\sin\alpha \neq 0$

$$\phi_{\lambda_n}(x) = \begin{cases} \sin\alpha\cos\left[\frac{\omega_1\pi\left(n-\frac{1}{2}\right)(x+1)}{\omega_2+\omega_1}\right] + O\left(\frac{1}{n}\right), \; x \in [-1, h_1), \\ \frac{\gamma_1}{\delta_1}\sin\alpha\cos\left[\frac{(\omega_2 x+\omega_1 h_1+\omega_1)\pi\left(n-\frac{1}{2}\right)}{\omega_2+\omega_1 h_1+\omega_1}\right] + O\left(\frac{1}{n}\right), \; x \in (h_1, h_2), \\ \frac{\gamma_1\gamma_3}{\delta_1\delta_3}\sin\alpha\cos\left[\frac{(\omega_3 x+\omega_2 h_2+\omega_1)\pi\left(n-\frac{1}{2}\right)}{\omega_3+\omega_2 h_2+\omega_1}\right] + O\left(\frac{1}{n}\right), \; x \in (h_2, 1]. \end{cases}$$

Case 4: $\beta_2' = 0$, $\sin\alpha = 0$

$$\phi_{\lambda_n}(x) = \begin{cases} -\frac{\omega_1+\omega_2}{\omega_1}\frac{\cos\alpha}{\pi n}\sin\left[\frac{\omega_1\pi n(x+1)}{\omega_2+\omega_1}\right] + O\left(\frac{1}{n^2}\right), \; x \in [-1, h_1), \\ \frac{-\gamma_1}{\delta_1}\frac{\omega_1+\omega_2}{\omega_1}\frac{\cos\alpha}{\pi n}\sin\left[\frac{(\omega_2 x+\omega_1 h_1+\omega_1)\pi n}{\omega_2+\omega_1 h_1+\omega_1}\right] + O\left(\frac{1}{n^2}\right), \; x \in (h_1, h_2), \\ \frac{-\gamma_1\gamma_3}{\delta_1\delta_3}\frac{\omega_1+\omega_2}{\omega_1}\frac{\cos\alpha}{\pi n}\sin\left[\frac{(\omega_3 x+\omega_2 h_2+\omega_1)\pi n}{\omega_3+\omega_2 h_2+\omega_1}\right] + O\left(\frac{1}{n^2}\right), \; x \in (h_2, 1]. \end{cases}$$

All these asymptotic formulas hold uniformly for x.

Proof. Let us consider only the Case 1. Inserting (3.22) in the integral term of (3.27), we easily see that

$$\int_{h_2}^{x} \sin[\mu\omega_3(x-y)]\, q(y)\, \phi_3(y,\lambda)\, dy = O\left(\exp\left(|t|\left(\omega_3 x + \omega_2 h_2 + \omega_1\right)\right)\right).$$

Inserting in (3.20) yields

$$\begin{aligned} \phi_3(x,\lambda) = {} & \frac{\gamma_1\gamma_3}{\delta_1\delta_3}\sin\alpha\cos\left[\mu\left(\omega_3 x + \omega_2 h_2 + \omega_1\right)\right] \\ & + O\left(\frac{1}{|\mu|}\exp|t|\left(\omega_3 x + \omega_2 h_2 + \omega_1\right)\right). \end{aligned} \tag{3.35}$$

We already know that all eigenvalues are real. Furthermore, putting $\lambda = -H$, $H > 0$ in (3.33) we infer that $\omega(-H) \to \infty$ as $H \to +\infty$, and so $\omega(-H) \neq 0$ for sufficiently large $R > 0$. Consequently, the set of eigenvalues is bounded below. Letting $\sqrt{\lambda_n} = \mu_n$ in (3.35) we now obtain

$$\phi_3(x,\lambda_n) = \frac{\gamma_1\gamma_3}{\delta_1\delta_3}\sin\alpha\cos\left[\mu_n\left(\omega_3 x + \omega_2 h_2 + \omega_1\right)\right] + O\left(\frac{1}{\mu_n}\right)$$

since $t_n = lm\mu_n$ for sufficiently large n. After some calculation, we easily see that

$$\cos\left[\mu_n\left(\omega_3 x + \omega_2 h_2 + \omega_1\right)\right] = \cos\left[\frac{\left(\omega_3 x + \omega_2 h_2 + \omega_1\right)\pi(n-1)}{\omega_3 + \omega_2 h_2 + \omega_1}\right] + O\left(\frac{1}{n}\right).$$

Consequently,

$$\phi_3(x,\lambda_n) = \frac{\gamma_1\gamma_3}{\delta_1\delta_3}\sin\alpha\cos\left[\frac{\left(\omega_3 x + \omega_2 h_2 + \omega_1\right)\pi(n-1)}{\omega_3 + \omega_2 h_2 + \omega_1}\right] + O\left(\frac{1}{n}\right).$$

In a similar method, we can deduce that

$$\phi_2(x,\lambda_n) = \frac{\gamma_1}{\delta_1}\sin\alpha\cos\left[\frac{(\omega_2 x + \omega_1 h_1 + \omega_1)\pi(n-1)}{\omega_2 + \omega_1 h_1 + \omega_1}\right] + O\left(\frac{1}{n}\right),$$

and

$$\phi_1(x,\lambda_n) = \sin\alpha\cos\left[\frac{\omega_1 \pi(n-1)(x+1)}{\omega_2 + \omega_1}\right] + O\left(\frac{1}{n}\right).$$

Thus the proof of the theorem completed in Case 1. The proofs for the other cases are similar. □

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