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GLOBAL BEHAVIOR OF SOLUTIONS TO A SECOND ORDER DELAY DIFFERENTIAL EQUATION

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Abstract. In this paper, we study the asymptotic behavior of eventually positive (negative) and oscillatory solutions to the following nonhomogeneous second order delay differential equation

$$\begin{cases} u''(t) = p(t)u(t-\tau) + f(t), \\ u(t) = \phi(t), \quad t \in [0,\tau), \end{cases}$$

where p and f are suitable real functions.

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1. INTRODUCTION

Qualitative theory of delay differential, difference and functional equations is the subject of many recent publications. Specially, asymptotic behavior and oscillation of delay differential and difference equations of first and second order has been studied by many authors. We refer the reader to the interesting book by Gopalsamy [1] and to the recent articles [2–5]. In this paper, we consider the following second order delay differential equation

$$\begin{cases} u''(t) = p(t)u(t-\tau) + f(t), \quad t \geq \tau \\ u(t) = \phi(t), \quad t \in [0,\tau) \end{cases} \tag{1.1}$$

where $\tau > 0$ is a constant delay, and $p : [\tau,+\infty) \to \mathbb{R}^+$, $f : [\tau,+\infty) \to \mathbb{R}$, and $\phi : [0,\tau) \to \mathbb{R}$. In the special case $f(t) \equiv 0$ and $p(t) \equiv p > 0$, the equation (1.1) has the characteristic equation $\lambda^2 - pe^{-\lambda\tau} = 0$. This equation has negative roots for suitable constants p and τ. For example, if $\tau = 10$ and $p = \frac{1}{1000}$, then the equation $g(\lambda) = 1000\lambda^2 - e^{-10\lambda}$ has two negative roots $-\frac{1}{10} < \lambda_1 < 0$ and $-1 < \lambda_2 < -\frac{1}{10}$. Because $g(0) = -1 < 0$, $g(\frac{-1}{10}) = 10 - e > 0$ and $g(-1) = 1000 - e^{10} < 0$. Therefore $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are two bounded solutions of (1.1) and convergent to 0 as $t \to +\infty$. In this paper, our motivation is the study of the asymptotic behavior of solutions to (1.1) in the case when f and p are nonconstant under the following suitable assumptions:

$$\int_{\tau}^{+\infty} tp(t)dt = +\infty, \tag{1.2}$$

$$\int_{\tau}^{+\infty} t|f(t)|dt < +\infty, \tag{1.3}$$

$$\limsup_{t\to+\infty} \int_{t-\tau}^{t} ds \int_{\tau}^{s} p(r)dr < 1, \tag{1.4}$$

$$\lim_{t\to+\infty} \frac{f(t)}{p(t)} = 0, \text{ where } p(t) > 0, \text{ for sufficiently large } t > 0. \tag{1.5}$$

Throughout the paper we assume that p, f, and ϕ are continuous. By a solution of equation (1.1), we mean a continuous function $u : [0,+\infty) \to \mathbb{R}$ which is twice continuously differentiable on $[\tau,+\infty)$ and satisfies the equation (1.1) for all $t \geq \tau$.

2. Main Results

First, we prove two lemmas.

Lemma 1. *Let $g : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and bounded from above, then* $\liminf_{t\to+\infty} g'(t) \leq 0$.

Proof. Suppose to the contrary, $\liminf_{t\to+\infty} g'(t) > \lambda > 0$. Then there exists $t_0 > 0$ such that for each $t \geq t_0$, $g'(t) > \lambda$. Integrating from t_0 to T, we have $g(T) - g(t_0) \geq \lambda(T - t_0)$. We get a contradiction by letting $T \to +\infty$. □

Lemma 2. *Suppose that $g : \mathbb{R}^+ \to \mathbb{R}$ is bounded from above and twice continuously differentiable such that*

$$g''(t) \geq -h(t), \quad \forall t \geq 0, \tag{2.1}$$

where $h : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $\int_0^{+\infty} th(t)dt < +\infty$. Then there exists $\lim_{t\to+\infty} g(t)$ *and* $\limsup_{t\to+\infty} g'(t) \leq 0$.

Proof. Integrating (2.1) from $t = S$ to $t = T$, where $S < T$, we get

$$g'(S) \leq g'(T) + \int_S^T h(t)dt.$$

Taking lim inf as $T \to +\infty$, by Lemma 1, we get

$$g'(S) \leq \int_S^{+\infty} h(t)dt. \tag{2.2}$$

Taking lim sup as $S \to +\infty$, we derive that: $\limsup_{t\to+\infty} g'(t) \le 0$. Now, integrating (2.2) from $S = T_1$ to $S = T_2$, where $T_1 < T_2$, and applying Fubini's theorem, we obtain

$$g(T_2) \le g(T_1) + \int_{T_1}^{T_2} dS \int_S^{+\infty} h(t)dt \le g(T_1) + \int_{T_1}^{+\infty} dS \int_S^{+\infty} h(t)dt$$
$$= g(T_1) + \int_{T_1}^{+\infty} dt \int_{T_1}^{t} h(t)dS = g(T_1) + \int_{T_1}^{+\infty} (t - T_1)h(t)dt$$
$$\le g(T_1) + \int_{T_1}^{+\infty} th(t)dt.$$

Now, taking lim sup as $T_2 \to +\infty$ and lim inf as $T_1 \to +\infty$, the proof is complete. □

Theorem 1. *Suppose that u is a solution to (1.1).*

(1) *If (1.2) and (1.3) are satisfied, then every eventually positive or eventually negative bounded solution to (1.1) converges to 0 as $t \to +\infty$.*

(2) *If (1.4) and (1.5) are satisfied, then every oscillatory solution to (1.1) converges to 0 as $t \to +\infty$.*

Proof. (1) Assume that $u(t)$ is eventually positive and bounded solution of (1.1). The same proof works for eventually negative solution of (1.1). Then for large t, $u''(t) \ge f(t) = f^+(t) - f^-(t) \ge -f^-(t)$, where $f^+(t) = max\{f(t), 0\}$ and $f^-(t) = max\{-f(t), 0\}$. By Lemma 2, there exists $\lim_{t\to+\infty} u(t) = l$ and $\limsup_{t\to+\infty} u'(t) \le 0$. Suppose that $l > 0$, then there exists $t_0 > \tau$ such that for each $t > t_0$, $u(t-\tau) > \frac{l}{2}$. Integrating (1.1) from $t > t_0$ to T, we get

$$u'(T) - u'(t) \ge \frac{l}{2}\int_t^T p(s)ds - \int_t^T f^-(s)ds.$$

Taking lim sup as $T \to +\infty$, we obtain

$$-u'(t) \ge \frac{l}{2}\int_t^{+\infty} p(s)ds - \int_t^{+\infty} f^-(s)ds, \quad \text{for } t > t_0.$$

If $\int_\tau^\infty p(s)ds = +\infty$, it is a contradiction; because $\int_\tau^{+\infty} f^-(t)dt < +\infty$. Otherwise, integrating from $t = t_0$ to $t = T$, we get

$$-u(T) + u(t_0) \ge \frac{l}{2}\int_{t_0}^T dt \int_t^{+\infty} p(s)ds - \int_{t_0}^T dt \int_t^{+\infty} f^-(s)ds.$$

Letting $T \to +\infty$, and by (1.2), (1.3) and Fubini's theorem, we get

$$-l + u(t_0) \ge \frac{l}{2}\int_{t_0}^{+\infty} dt \int_t^{+\infty} p(s)ds - \int_{t_0}^{+\infty} dt \int_t^{+\infty} f^-(s)ds$$

$$= \frac{l}{2}\int_{t_0}^{+\infty}(t-t_0)p(t)dt - \int_{t_0}^{+\infty}(t-t_0)f^{-}(t)dt = +\infty.$$

This is a contradiction. Then $l = 0$.
(2) Suppose that $u(t)$ oscillates. Then, there exists a sequence t_n of extreme points of $u(t)$ such that $t_n \to +\infty$ as $n \to +\infty$. Let $\mu > 0$ such that $\limsup_{t\to+\infty}\int_{t-\tau}^{t} ds \int_{\tau}^{s} p(r)dr < \mu < 1$ and let $0 < \delta < \frac{1-\mu}{1+\mu}$. The sequence $\{t_n\}$ of extreme points of u, has a subsequence $s_j = t_{n_j}$ such that:
1) $u(s_{2j}) \leq 0$, $u(s_{2j+1}) \geq 0$, for all $j \geq 1$.
2) s_{2j} is a minimum point of u in the interval $[s_{2j-1}, s_{2j+1}]$ and s_{2j+1} is a maximum point of u in the interval $[s_{2j}, s_{2j+2}]$, for all $j \geq 1$.
It is enough to prove, $u(s_j) \to 0$ as $j \to +\infty$. By (1.1), we have

$$u''(s_{2j}) = p(s_{2j})u(s_{2j}-\tau) + f(s_{2j}) \Rightarrow u(s_{2j}-\tau) \geq \frac{-f(s_{2j})}{p(s_{2j})} \tag{2.3}$$

$$u''(s_{2j+1}) = p(s_{2j+1})u(s_{2j+1}-\tau) + f(s_{2j+1})$$
$$\Rightarrow u(s_{2j+1}-\tau) \leq \frac{-f(s_{2j+1})}{p(s_{2j+1})}. \tag{2.4}$$

Give a subsequence s_{j_i} of s_j such that

$$s_{j_1} > \tau, s_{j_{i+1}} - s_{j_i} > \tau, \quad \frac{|f(t)|}{p(t)} < \delta^i, \quad \text{for each } t \geq s_{j_i},$$
$$\text{and} \quad \int_{s_n-\tau}^{s_n} ds \int_{\tau}^{s} p(r)dr < \mu < 1, \ \text{for each } n \geq j_1. \tag{2.5}$$

Integrating (1.1) on $[s_{j_i}, s]$, we get

$$u'(s) = u'(s) - u'(s_{j_i}) = \int_{s_{j_i}}^{s} p(r)u(r-\tau)dr + \int_{s_{j_i}}^{s} f(r)dr. \tag{2.6}$$

For large m such that $s_{2m} - \tau > s_{j_i}$, integrating (2.6) on $[s_{2m}-\tau, s_{2m}]$, and by (2.3), we obtain

$$u(s_{2m}) + \frac{f(s_{2m})}{p(s_{2m})} \geq u(s_{2m}) - u(s_{2m}-\tau) =$$
$$\int_{s_{2m}-\tau}^{s_{2m}} ds \int_{s_{j_i}}^{s} p(r)u(r-\tau)dr + \int_{s_{2m}-\tau}^{s_{2m}} ds \int_{s_{j_i}}^{s} f(r)dr. \tag{2.7}$$

Integrating (2.6) on $[s_{2m+1}-\tau, s_{2m+1}]$, and by (2.4), we get

$$u(s_{2m+1}) + \frac{f(s_{2m+1})}{p(s_{2m+1})} \leq u(s_{2m+1}) - u(s_{2m+1}-\tau) =$$

$$\int_{s_{2m+1}-\tau}^{s_{2m+1}} ds \int_{s_{j_i}}^{s} p(r)u(r-\tau)dr + \int_{s_{2m+1}-\tau}^{s_{2m+1}} ds \int_{s_{j_i}}^{s} f(r)dr. \quad (2.8)$$

(2.7) and (2.8) imply that for sufficiently large n such that $s_n - \tau \geq s_{j_i}$

$$|u(s_n)| \leq \int_{s_n-\tau}^{s_n} ds \int_{s_{j_i}}^{s} p(r)|u(r-\tau)|dr + \int_{s_n-\tau}^{s_n} ds \int_{s_{j_i}}^{s} |f(r)|dr + \frac{|f(s_n)|}{p(s_n)}. \quad (2.9)$$

By (2.5) for each $n \geq j_i$, we have

$$\int_{s_n-\tau}^{s_n} ds \int_{s_{j_i}}^{s} p(r)dr < \mu$$

and

$$\int_{s_n-\tau}^{s_n} ds \int_{s_{j_i}}^{s} |f(r)|dr + \frac{|f(s_n)|}{p(s_n)} < \mu\delta^i + \delta^i.$$

Consider $\mu + \delta^i + \mu\delta^i < \mu + \delta + \mu\delta < 1$. From (2.9), we deduce that

$$|u(s_{j_2})| \leq M\mu + \delta + \mu\delta \leq (M+1)(\mu + \delta + \mu\delta),$$

where $M = max_{0 \leq n \leq j_2}\{|u(s_n)|\}$. Suppose that for $j_2 \leq n \leq m$, we have

$$|u(s_n)| \leq (M+1)(\mu + \delta + \mu\delta),$$

then

$$|u(s_{m+1})|$$

$$\leq \int_{s_{m+1}-\tau}^{s_{m+1}} ds \int_{s_{j_1}}^{s} p(r)|u(r-\tau)|dr + \int_{s_{m+1}-\tau}^{s_{m+1}} ds \int_{s_{j_1}}^{s} |f(r)|dr + \frac{|f(s_{m+1})|}{p(s_{m+1})}$$

$$\leq max\{(M+1), |u(s_{m+1})|\}\mu + \delta + \mu\delta.$$

If $|u(s_{m+1})| > M+1$, then

$$|u(s_{m+1})| \leq |u(s_{m+1})|\mu + \delta + \mu\delta.$$

Therefore

$$|u(s_{m+1})| \leq \frac{\delta + \mu\delta}{1-\mu}.$$

This follows that $1 > \frac{\delta+\mu\delta}{1-\mu} > M+1$. This is a contradiction. Therefore $|u(s_{m+1})| \leq M+1$. This implies that

$$|u(s_{m+1})| \leq (M+1)\mu + \delta + \mu\delta \leq (M+1)(\mu + \delta + \mu\delta).$$

Therefore, for all $n \geq j_2$, we get

$$|u(s_n)| \leq (M+1)(\mu + \delta + \mu\delta).$$

Now, by induction suppose that

$$|u(s_n)| \leq (\mu + \delta + \mu\delta)^k (M+1), \quad \text{for } n \geq j_{2k}. \quad (2.10)$$

By (2.9) for $n \geq j_{2k+2}$, we get

$$|u(s_n)| \leq \int_{s_n-\tau}^{s_n} ds \int_{s_{j_{2k+1}}}^{s} p(r)|u(r-\tau)|dr + \int_{s_n-\tau}^{s_n} ds \int_{s_{j_{2k+1}}}^{s} |f(r)|dr + \frac{|f(s_n)|}{p(s_n)}. \tag{2.11}$$

If $r \geq s_{j_{2k+1}}$, then $r - \tau \geq s_{j_{2k+1}} - \tau \geq s_{j_{2k}}$. By the hypothesis of induction and choosing the sequence s_n, we get

$$|u(r-\tau)| \leq (M+1)(\mu+\delta+\mu\delta)^k.$$

Now (2.11) implies that

$$\begin{aligned} |u(s_n)| &\leq (M+1)(\mu+\delta+\mu\delta)^k\mu + \delta^{2k+1} + \mu\delta^{2k+1} \\ &\leq (M+1)(\mu+\delta+\mu\delta)^k\mu + \delta^{k+1} + \mu\delta^{k+1} \leq (M+1)(\mu+\delta+\mu\delta)^{k+1}, \end{aligned} \tag{2.12}$$

for $n \geq j_{2k+2}$. This prove (2.10). The theorem is proved by letting $k \to +\infty$ in (2.10). □

Corollary 1. *If the conditions (1.2), (1.3), (1.4) and (1.5) are satisfied, then every bounded solution to (1.1) converges to* 0 *as* $t \to +\infty$.

Now, we give some examples.

Example 1. Give $\tau = 1$, $\phi(t) = \frac{1}{t+1}$, $p(t) = \begin{cases} \frac{1}{t^2-1}, & t > 1 \\ 1, & t = 1, \end{cases}$

and $f(t) = \frac{t^2-4t-1}{t(t-1)(t+1)^3}$. The assumptions of Theorem 2.3 are satisfied, then every bounded and eventually positive (eventually negative) solution of (1.1) converges to 0 as $t \to +\infty$. As well as every its oscillatory solution converges to 0. For example, $u(t) = \frac{1}{t+1}$ is a bounded solution of (1.1) with the above conditions which converges to 0.

Example 2. $u(t) \equiv 1$ is a bounded and positive solution of

$$\begin{cases} u''(t) = p(t)u(t-\tau) - p(t), & t \geq \tau \\ u(t) = 1, & 0 \leq t \leq \tau \end{cases} \tag{2.13}$$

where $p : [\tau, +\infty) \to \mathbb{R}^+$. But neither [(1.2) and (1.3)] nor (1.5) are satisfied and $u(t)$ is not convergent to 0.

Example 3. $u(t) = sint$ is an oscillatory solution of

$$\begin{cases} u''(t) = u(t-\pi), & t \geq \pi \\ u(t) = sint, & 0 \leq t \leq \pi. \end{cases} \tag{2.14}$$

But (1.4) are not satisfied and $u(t)$ is not convergent to 0.

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