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NEW KINDS OF MATRIX POLYNOMIALS

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Abstract. In this study, we introduce new matrix poynomials and derive their properties such as explict representations, recurrence relations and generating matrix functions.

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1. INTRODUCTION

Recently, mathematicians have interested in some properties of the orthogonal matrix polynomials. For example, Jódar and et. al have introduced and studied Laguerre matrix polynomials [8], Gegenbauer matrix polynomials [9], Hermite matrix polynomials [7], Chebyshev matrix polynomials [4] and Jacobi matrix polynomials [3]. Then the Konhauser matrix polynomials [11], the multivariable Humbert matrix polynomials [1] and the Bessel matrix polynomials [10] have been studied. Furthermore, one can find several papers concerning the orthogonal matrix polynomials (see [2, 12]). In this paper, we define new matrix polynomials and derive some of their properties.

Throughout this paper, for a matrix A in $\mathbb{C}^{r\times r}$, its spectrum $\sigma(A)$ denotes the set of all eigenvalues of A. Furthermore the identity matrix and the zero matrix of $\mathbb{C}^{r\times r}$ will be denoted by I and $\mathbf{0}$, respectively. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane and A is a matrix in $\mathbb{C}^{r\times r}$ with $\sigma(A)\subset\Omega$, then from the properties of the matrix functional calculus in [6], it follows that:

$$f(A)g(A)=g(A)f(A). \tag{1.1}$$

Let A be a matrix in $\mathbb{C}^{r\times r}$ satisfying $(-k)\notin\sigma(A)$ for $k\in\mathbb{Z}^+$ and λ be a complex number whose real part is positive. Then the Laguerre matrix polynomials $L_n^{(A,\lambda)}(x)$ are defined by [8]:

$$L_n^{(A,\lambda)}(x)=\sum_{k=0}^{n}\frac{(-1)^k}{k!\,(n-k)!}(A+I)_n\left[(A+I)_k\right]^{-1}(\lambda x)^k\ ,\ \ n\in\mathbb{N}. \tag{1.2}$$

Also Laguerre matrix polynomials have the following generating matrix function:

$$\sum_{n=0}^{\infty} L_n^{(A,\lambda)}(x)t^n = (1-t)^{-A-I} e^{-\frac{\lambda xt}{1-t}} \ ; \ x \in \mathbb{C},\ t \in \mathbb{C},\ |t| < 1. \tag{1.3}$$

2. Definitions and properties of new matrix polynomials

In this section, we define two new matrix polynomials $\left\{f_{n,m}^{(A,\lambda)}(x)\right\}_{n=0}^{\infty}$ and $\left\{g_{n,m}^{(A,\lambda)}(x)\right\}_{n=0}^{\infty}$ via following generating matrix functions

$$F^{(A,\lambda)}(x,t) = (1-t^m)^{-A} e^{-\frac{xt\lambda}{1-t^m}} = \sum_{n=0}^{\infty} f_{n,m}^{(A,\lambda)}(x)t^n \quad ; \quad |t| < 1 \tag{2.1}$$

$$G^{(A,\lambda)}(x,t) = (1+t^m)^{-A} e^{-\frac{xt\lambda}{1+t^m}} = \sum_{n=0}^{\infty} g_{n,m}^{(A,\lambda)}(x)t^n \quad ; \quad |t| < 1 \tag{2.2}$$

where A is a matrix in $\mathbb{C}^{r\times r}$, m is a positive natural number and λ is an arbitrary number, respectively. Here, (2.1) and (2.2) are matrix versions of polynomials given in [5].

Using (2.1) and (2.2), we obtain the following explicit representations for new matrix polynomials

$$f_{n,m}^{(A,\lambda)}(x) = \sum_{i=0}^{\left[\frac{n}{m}\right]} \frac{(-1)^{n-mi}(A+(n-mi)I)_i}{(n-mi)!\, i!}(\lambda x)^{n-mi}, \tag{2.3}$$

$$g_{n,m}^{(A,\lambda)}(x) = \sum_{i=0}^{\left[\frac{n}{m}\right]} \frac{(-1)^{n-(m-1)i}(A+(n-mi)I)_i}{(n-mi)!\, i!}(\lambda x)^{n-mi}$$

where [.] denotes the integer part. Taking $m = 1$ in (2.3), (2.3) reduces Laguerre matrix polynomials as follows:

$$\begin{aligned} f_{n,1}^{(A,\lambda)}(x) &= \sum_{i=0}^{n} \frac{(-1)^{n-i}(A+(n-i)I)_i}{(n-i)!\, i!}(\lambda x)^{n-i} \\ &= \sum_{i=0}^{n} \frac{(-1)^{i}\,(A)_n\,(A)_i^{-1}}{(n-i)!\, i!}(\lambda x)^{i} = L_n^{(A-I,\lambda)}(x) \end{aligned}$$

where

$$\alpha \notin \mathbb{Z}^- \cup \{0\} \text{ for } \forall \alpha \in \sigma(A) \text{ and } \lambda \text{ be a complex parameter with } Re(\lambda) > 0 \tag{2.4}$$

(see [8]).

Differentiating (2.1) with respect to x, the $f_{n,m}^{(A,\lambda)}(x)$ matrix polynomials satisfy recurrence relations

$$\frac{\partial f_{n,m}^{(A,\lambda)}(x)}{\partial x} = -\lambda f_{n-1,m}^{(A+I,\lambda)}(x) \tag{2.5}$$

and a generalization of the above equation

$$\frac{\partial^k f_{n,m}^{(A,\lambda)}(x)}{\partial x^k} = (-\lambda)^k f_{n-k,m}^{(A+kI,\lambda)}(x) \quad ; \quad n \geq k.$$

For the special case $m = 1$ in (2.5), it follows that

$$\frac{\partial L_n^{(A-I,\lambda)}(x)}{\partial x} = -\lambda L_{n-1}^{(A,\lambda)}(x)$$

where A and λ satisfy (2.4). Similarly, for $g_{n,m}^{(A,\lambda)}(x)$, we have

$$\frac{\partial g_{n,m}^{(A,\lambda)}(x)}{\partial x} = -\lambda g_{n-1,m}^{(A+I,\lambda)}(x)$$

$$\frac{\partial^k g_{n,m}^{(A,\lambda)}(x)}{\partial x^k} = (-\lambda)^k g_{n-k,m}^{(A+kI,\lambda)}(x) \quad ; \quad n \geq k.$$

On the other hand, differentiating (2.1) with respect to t, the $f_{n,m}^{(A,\lambda)}(x)$ matrix polynomials satisfy recurrence relation

$$\begin{aligned} n f_{n,m}^{(A,\lambda)}(x) - (n-m) f_{n-m,m}^{(A,\lambda)}(x) &= mA\left[f_{n-m,m}^{(A+I,\lambda)}(x) - f_{n-2m,m}^{(A+I,\lambda)}(x)\right] \\ &\quad - x\lambda\left[f_{n-1,m}^{(A+I,\lambda)}(x) + (m-1) f_{n-m-1,m}^{(A+I,\lambda)}(x)\right] \end{aligned} \tag{2.6}$$

where $n \geq 2m$. For the special case $m = 1$ in (2.6), it holds that

$$nL_n^{(A-I,\lambda)}(x) - (n-1)L_{n-1}^{(A-I,\lambda)}(x) = A\left[L_{n-1}^{(A,\lambda)}(x) - L_{n-2}^{(A,\lambda)}(x)\right] - x\lambda L_{n-1}^{(A,\lambda)}(x)$$

where $n \geq 2$, A and λ satisfy (2.4). For $f_{n,m}^{(A,\lambda)}(x)$, other recurrence relation via differentiating (2.1) with respect to t is

$$\begin{aligned} n f_{n,m}^{(A,\lambda)}(x) &= [m(A-2I) + 2nI]\, f_{n-m,m}^{(A,\lambda)}(x) - [m(A-2I) + nI]\, f_{n-2m,m}^{(A,\lambda)}(x) \\ &\quad - x\lambda\left[f_{n-1,m}^{(A,\lambda)}(x) + (m-1) f_{n-m-1,m}^{(A,\lambda)}(x)\right] \end{aligned} \tag{2.7}$$

where $n \geq 2m$. For $m = 1$, (2.7) reduces the recurrence relation satisfied by the Laguerre matrix polynomials as follows:

$$nL_n^{(A,\lambda)}(x) = [(A-I) + 2nI - x\lambda I]\, L_{n-1}^{(A,\lambda)}(x) - [(A-I) + nI]\, L_{n-2}^{(A,\lambda)}(x)\ ,\ n \geq 2$$

where A and λ satisfy (2.4). On the one hand, for $g_{n,m}^{(A,\lambda)}(x)$, we derive the following relations:

$$ng_{n,m}^{(A,\lambda)}(x) = -mAg_{n-m,m}^{(A+I,\lambda)}(x) - x\lambda g_{n-1,m}^{(A+2I,\lambda)}(x) + (m-1)x\lambda g_{n-m-1,m}^{(A+2I,\lambda)}(x)$$

where $n \geq m+1$ and

$$[2(n-m)I+mA]\,g_{n-m,m}^{(A,\lambda)}(x)+[(n-2m)I+mA]\,g_{n-2m,m}^{(A,\lambda)}(x)$$
$$=-x\lambda g_{n-1,m}^{(A,\lambda)}(x)+x\lambda(m-1)g_{n-m-1,m}^{(A,\lambda)}(x)-ng_{n,m}^{(A,\lambda)}(x)$$

where $n \geq 2m$. Now, we shall prove interesting relations for $f_{n,m}^{(A,\lambda)}(x)$ and $g_{n,m}^{(A,\lambda)}(x)$. Starting from (2.1), we have

$$F^{(A,\lambda)}(x,t)F^{(A,\lambda)}(y,t)=(1-t^m)^{-2A}e^{-\frac{(x+y)t\lambda}{1-t^m}}=\sum_{n=0}^{\infty}f_{n,m}^{(2A,\lambda)}(x+y)t^n. \quad (2.8)$$

Using (2.1) in the left-hand side of (2.8), it follows that

$$\sum_{i=0}^{n}f_{n-i,m}^{(A,\lambda)}(x)f_{i,m}^{(A,\lambda)}(y)=f_{n,m}^{(2A,\lambda)}(x+y). \quad (2.9)$$

For $m=1$, (2.9) reduces

$$\sum_{i=0}^{n}L_{n-i}^{(A-I,\lambda)}(x)L_{i}^{(A-I,\lambda)}(y)=L_{n}^{(2A-I,\lambda)}(x+y)$$

where A and λ satisfy (2.4) and $\mu \notin \mathbb{Z}^- \cup \{0\}$ for $\forall \mu \in \sigma(2A)$. In the same way, $g_{n,m}^{(A,\lambda)}(x)$ matrix polynomials satisfy

$$\sum_{i=0}^{n}g_{n-i,m}^{(A,\lambda)}(x)g_{i,m}^{(A,\lambda)}(y)=g_{n,m}^{(2A,\lambda)}(x+y).$$

Similarly, we have

$$\sum_{i=0}^{n}g_{n-i,m}^{(A,\lambda)}(x)g_{i,m}^{(B,\lambda)}(x)=g_{n,m}^{(A+B,\lambda)}(2x),$$
$$\sum_{i=0}^{n}f_{n-i,m}^{(A,\lambda)}(x)f_{i,m}^{(B,\lambda)}(x)=f_{n,m}^{(A+B,\lambda)}(2x) \quad (2.10)$$

where $AB=BA$. For $m=1$ in (2.10), we get

$$\sum_{i=0}^{n}L_{n-i}^{(A-I,\lambda)}(x)L_{i}^{(B-I,\lambda)}(x)=L_{n}^{(A+B-I,\lambda)}(2x)$$

where A and λ satisfy (2.4), $\nu \notin \mathbb{Z}^- \cup \{0\}$ for $\forall \nu \in \sigma(B)$ and $\eta \notin \mathbb{Z}^- \cup \{0\}$ for $\forall \eta \in \sigma(A+B)$. On the other hand, (2.8) can be written as

$$(1-t^m)^{-2A}e^{-\frac{xt\lambda}{1-t^m}}=e^{\frac{yt\lambda}{1-t^m}}\sum_{i=0}^{\infty}f_{i,m}^{(2A,\lambda)}(x+y)t^i$$

and then

$$\sum_{n=0}^{\infty} f_{n,m}^{(2A,\lambda)}(x)t^n = \left[\sum_{n=0}^{\infty} \frac{\left(\frac{yt\lambda}{1-t^m}\right)^n}{n!}\right]\left[\sum_{i=0}^{\infty} f_{i,m}^{(2A,\lambda)}(x+y)t^i\right]$$
$$= \left[\sum_{n=0}^{\infty} \frac{(yt\lambda)^n}{n!}\right]\left[\sum_{j=0}^{\infty} \frac{(n)_j}{j!} t^{mj}\right]\left[\sum_{i=0}^{\infty} f_{i,m}^{(2A,\lambda)}(x+y)t^i\right]$$
$$= \sum_{n=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{m}\right]}\sum_{i=0}^{n-mj} \frac{\lambda^{n-mj-i}y^{n-mj-i}}{(n-mj-i)!}\frac{(n-mj-i)_j}{j!} f_{i,m}^{(2A,\lambda)}(x+y)t^n.$$

Comparing the coefficients of t^n, we have

$$f_{n,m}^{(2A,\lambda)}(x) = \sum_{j=0}^{\left[\frac{n}{m}\right]}\sum_{i=0}^{n-mj} \frac{\lambda^{n-mj-i}y^{n-mj-i}}{(n-mj-i)!}\frac{(n-mj-i)_j}{j!} f_{i,m}^{(2A,\lambda)}(x+y). \quad (2.11)$$

If $m=1$ in (2.11), we obtain

$$L_n^{(2A-I,\lambda)}(x) = \sum_{j=0}^{n}\sum_{i=0}^{n-j} \frac{\lambda^{n-j-i}y^{n-j-i}}{(n-j-i)!}\frac{(n-j-i)_j}{j!} L_i^{(2A-I,\lambda)}(x+y)$$

where $\mu \notin \mathbb{Z}^- \cup \{0\}$ for $\forall \mu \in \sigma(2A)$. Similarly, for $g_{n,m}^{(A,\lambda)}(x)$, we get

$$g_{n,m}^{(2A,\lambda)}(x) = \sum_{j=0}^{\left[\frac{n}{m}\right]}\sum_{i=0}^{n-mj} \frac{(-1)^j\lambda^{n-mj-i}y^{n-mj-i}}{(n-mj-i)!}\frac{(n-mj-i)_j}{j!} g_{i,m}^{(2A,\lambda)}(x+y).$$

Now, let's give relations for the derivatives of new matrix polynomials. From (2.1), it follows that

$$\frac{\partial^s}{\partial x^s} F^{(A,\lambda)}(x,t) = (-1)^s t^s \lambda^s (1-t^m)^{-A-sI} e^{-\frac{xt\lambda}{1-t^m}}.$$

By the above equation and (2.1), we can write

$$\frac{\partial^s}{\partial x^s} F^{(A,\lambda)}(x,t)\frac{\partial^s}{\partial y^s} F^{(A,\lambda)}(y,t) = \sum_{n=0}^{\infty} \lambda^{2s} f_{n,m}^{(2A+2sI,\lambda)}(x+y)t^{n+2s},$$
$$\frac{\partial^s}{\partial x^s}\left[\sum_{n=0}^{\infty} f_{n,m}^{(A,\lambda)}(x)t^n\right]\frac{\partial^s}{\partial y^s}\left[\sum_{i=0}^{\infty} f_{i,m}^{(A,\lambda)}(y)t^i\right] = \sum_{n=0}^{\infty} \lambda^{2s} f_{n,m}^{(2A+2sI,\lambda)}(x+y)t^{n+2s},$$
$$\sum_{n=0}^{\infty}\sum_{i=0}^{n} D_x^s f_{n-i,m}^{(A,\lambda)}(x) D_y^s f_{i,m}^{(A,\lambda)}(y)t^n = \sum_{n=2s}^{\infty} \lambda^{2s} f_{n-2s,m}^{(2A+2sI,\lambda)}(x+y)t^n$$

where $D_x^s = \frac{d^s}{dx^s}$ and $D_y^s = \frac{d^s}{dy^s}$. Comparing the coefficients of t^n, we have

$$\sum_{i=0}^{n} D_x^s f_{n-i,m}^{(A,\lambda)}(x) D_y^s f_{i,m}^{(A,\lambda)}(y) = \lambda^{2s} f_{n-2s,m}^{(2A+2sI,\lambda)}(x+y) \quad ; \quad n \geq 2s. \tag{2.12}$$

If $m = 1$ in (2.12), (2.12) reduces

$$\sum_{i=0}^{n} D_x^s L_{n-i}^{(A-I,\lambda)}(x) D_y^s L_i^{(A-I,\lambda)}(y) = \lambda^{2s} L_{n-2s}^{(2A+(2s-1)I,\lambda)}(x+y) \quad ; \quad n \geq 2s$$

where A and λ satisfy (2.4) and $\mu \notin \mathbb{Z}^- \cup \{0\}$ for $\forall \mu \in \sigma(2A)$. In the same way, we get

$$\sum_{i=0}^{n} D_x^s g_{n-i,m}^{(A,\lambda)}(x) D_y^s g_{i,m}^{(A,\lambda)}(y) = \lambda^{2s} g_{n-2s,m}^{(2A+2sI,\lambda)}(x+y) \quad ; \quad n \geq 2s. \tag{2.13}$$

On the one hand, using (2.1) and (2.2), we can write

$$\frac{\partial^k}{\partial x^k} F^{(A,\lambda)}(x,t) \frac{\partial^k}{\partial x^k} G^{(A,\lambda)}(x,t) = \sum_{n=0}^{\infty} \lambda^{2k} f_{n,2m}^{(A+kI,\lambda)}(2x) t^{n+2k}.$$

Thus, it holds that

$$\sum_{i=0}^{n} D_x^k f_{n-i,m}^{(A,\lambda)}(x) D_x^k g_{i,m}^{(A,\lambda)}(x) = \lambda^{2k} f_{n-2k,2m}^{(A+kI,\lambda)}(2x) \quad ; \quad n \geq 2k. \tag{2.14}$$

The generalizations of (2.10) are as follows:

$$\sum_{i_1+i_2+...+i_k=n} f_{i_1,m}^{(A_1,\lambda)}(x_1)...f_{i_k,m}^{(A_k,\lambda)}(x_k) = f_{n,m}^{(A_1+...+A_k,\lambda)}(x_1+...+x_k),$$
$$\sum_{i_1+i_2+...+i_k=n} g_{i_1,m}^{(A_1,\lambda)}(x_1)...g_{i_k,m}^{(A_k,\lambda)}(x_k) = g_{n,m}^{(A_1+...+A_k,\lambda)}(x_1+...+x_k).$$

where the matrices $A_1, ..., A_k$ are assumed to be commutative. For $k = 0$ in (2.14), we have

$$\sum_{i=0}^{n} f_{n-i,m}^{(A,\lambda)}(x) g_{i,m}^{(A,\lambda)}(x) = f_{n,m}^{(A,\lambda)}(2x). \tag{2.15}$$

The generalization of (2.15) is

$$\sum_{s=0}^{n} \left\{ \sum_{i_1+i_2+...+i_k=n-s} f_{i_1,m}^{(A_1,\lambda)}(x_1)...f_{i_k,m}^{(A_k,\lambda)}(x_k) \right.$$
$$\left. \times \sum_{j_1+j_2+...+j_k=n} g_{j_1,m}^{(A_1,\lambda)}(x_1)...g_{j_k,m}^{(A_k,\lambda)}(x_k) \right\}$$

$$= \sum_{i_1+i_2+...+i_k=n} f_{i_1,2m}^{(A_1,\lambda)}(2x_1)...f_{i_k,2m}^{(A_k,\lambda)}(2x_k)$$

where the matrices $A_1, ..., A_k$ are assumed to be commutative.

3. Multilinear and multilateral generating matrix functions for $f_{n,m}^{(A,\lambda)}(x)$ and $g_{n,m}^{(A,\lambda)}(x)$

In this section, we derive several families of bilinear and bilateral generating matrix functions for the new matrix polynomials generated by (2.1) and (2.2). We first state our result.

Theorem 1. *Corresponding to a non-vanishing function $\Omega_\mu(y_1, ..., y_s)$ of s complex variables $y_1, ..., y_s$ ($s \in \mathbb{N}$) and of complex order μ, let*

$$\Lambda_{\mu,\nu}(y_1, ..., y_s; z) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(y_1, ..., y_s) z^k \;\; ; (a_k \neq 0, \; \mu, \nu \in \mathbb{C}) \tag{3.1}$$

and

$$\Theta_{n,p,\mu,\nu}(x; y_1, ..., y_s; \zeta) := \sum_{k=0}^{[n/p]} a_k \; f_{n-pk,m}^{(A,\lambda)}(x) \; \Omega_{\mu+\nu k}(y_1, ..., y_s) \zeta^k \tag{3.2}$$

where $n, p \in \mathbb{N}$ and (as usual) $[\alpha]$ represents the greatest integer in $\alpha \in \mathbb{R}$. Then we have

$$\sum_{n=0}^{\infty} \Theta_{n,p,\mu,\nu}\left(x; y_1, ..., y_s; \frac{\eta}{t^p}\right) t^n = (1-t^m)^{-A} e^{-\frac{xt\lambda}{1-t^m}} \Lambda_{\mu,\nu}(y_1, ..., y_s; \eta) \tag{3.3}$$

provided that each member of (3.3) exists for $|t| < 1$ and $Re(\lambda) > 0$.

Proof. For convenience, let S denote the first member of the assertion (3.3) of Theorem 1. Then, upon substituting for the polynomials $\Theta_{n,p,\mu,\nu}\left(x; y_1, ..., y_s; \frac{\eta}{t^p}\right)$ from the definition (3.2) into the left-hand side of (3.3), we obtain

$$S = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/p]} a_k \; f_{n-pk,m}^{(A,\lambda)}(x) \; \Omega_{\mu+\nu k}(y_1, ..., y_s) \eta^k t^{n-pk}. \tag{3.4}$$

Upon inverting the order of summation in (3.4), if we replace n by $n + pk$, we can write

$$S = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k \, f_{n,m}^{(A,\lambda)}(x) \; \Omega_{\mu+\nu k}(y_1, ..., y_s) \eta^k t^n$$

$$= \left(\sum_{n=0}^{\infty} f_{n,m}^{(A,\lambda)}(x) \; t^n \right) \left(\sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(y_1, ..., y_s) \eta^k \right)$$

$$= \left\{ (1-t^m)^{-A} e^{-\frac{xt\lambda}{1-t^m}} \right\} \Lambda_{\mu,\nu}(y_1, ..., y_s; \eta),$$

which completes the proof of Theorem 1. □

Corollary 1. *Corresponding to a non-vanishing function $\Omega_\mu(y_1, ..., y_s)$ of s complex variables $y_1, ..., y_s$ $(s \in \mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\nu}(y_1, ..., y_s; z) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(y_1, ..., y_s) z^k \ \ ; (a_k \neq 0, \ \mu, \nu \in \mathbb{C}) \tag{3.5}$$

and

$$\Theta_{n,p,\mu,\nu}(x; y_1, ..., y_s; \zeta) := \sum_{k=0}^{[n/p]} a_k \ g_{n-pk,m}^{(A,\lambda)}(x) \ \Omega_{\mu+\nu k}(y_1, ..., y_s) \zeta^k \tag{3.6}$$

where $n, p \in \mathbb{N}$ and $[\alpha]$ represents the greatest integer in $\alpha \in \mathbb{R}$. Then we have

$$\sum_{n=0}^{\infty} \Theta_{n,p,\mu,\nu}\left(x; y_1, ..., y_s; \frac{\eta}{t^p}\right) t^n = (1+t^m)^{-A} e^{-\frac{xt\lambda}{1+t^m}} \Lambda_{\mu,\nu}(y_1, ..., y_s; \eta) \tag{3.7}$$

provided that each member of (3.7) exists for $|t| < 1$ and $Re(\lambda) > 0$.

By expressing the multivariable function $\Omega_{\mu+\nu k}(y_1, ..., y_s)$ $(k \in \mathbb{N}_0, \ s \in \mathbb{N})$ in terms of simpler function of one and more variables, we can give further applications of Theorem 1. For example, if we set $s = 1$ and $\Omega_{\mu+\nu k}(y) = g_{\mu+\nu k,m}^{(B,\gamma)}(y)$ in Theorem 1, where $g_{k,m}^{(B,\gamma)}(y)$ is defined by (2.2), then we obtain the following result which provides a class of bilateral generating functions for the $f_{n,m}^{(A,\lambda)}(x)$ and $g_{n,m}^{(A,\lambda)}(x)$.

Corollary 2. *If $\Lambda_{\mu,\nu}(y; z) := \sum_{k=0}^{\infty} a_k g_{\mu+\nu k,m}^{(B,\gamma)}(y) z^k$ where $(a_k \neq 0, \ \mu, \nu \in \mathbb{N}_0)$; and*

$$\Theta_{n,p,\mu,\nu}(x; y; \zeta) := \sum_{k=0}^{[n/p]} a_k \ f_{n-pk,m}^{(A,\lambda)}(x) g_{\mu+\nu k,m}^{(B,\gamma)}(y) \zeta^k$$

where $n, p \in \mathbb{N}$, then we have

$$\sum_{n=0}^{\infty} \Theta_{n,p,\mu,\nu}\left(x; y; \frac{\eta}{t^p}\right) t^n = (1-t^m)^{-A} e^{-\frac{xt\lambda}{1-t^m}} \Lambda_{\mu,\nu}(y; \eta) \tag{3.8}$$

provided that each member of (3.8) exists.

Remark 1. Using the generating relation (2.2) for $g_{k,m}^{(B,\gamma)}(y)$ and taking $a_k = 1$, $\mu = 0$, $\nu = 1$, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} f_{n-pk,m}^{(A,\lambda)}(x) g_{k,m}^{(B,\gamma)}(y)\eta^k t^{n-pk}$$
$$= (1-t^m)^{-A} e^{-\frac{xt\lambda}{1-t^m}} (1+\eta^m)^{-B} e^{-\frac{y\eta\gamma}{1+\eta^m}}$$

where $|\eta| < 1$.

Choosing $s = 1$ and $\Omega_{\mu+\nu k}(y) = f_{\mu+\nu k,m}^{(B,\gamma)}(y)$, $(\mu, \nu \in \mathbb{N}_0)$, in Theorem 1, we obtain the following class of bilinear generating function for the $f_{n,m}^{(A,\lambda)}(x)$.

Corollary 3. *If* $\Lambda_{\mu,\nu}(y;z) := \sum_{k=0}^{\infty} a_k f_{\mu+\nu k,m}^{(B,\gamma)}(y) z^k$ *where* $(a_k \neq 0,\ \mu, \nu \in \mathbb{N}_0)$; *and*

$$\Theta_{n,p,\mu,\nu}(x;y;\zeta) := \sum_{k=0}^{[n/p]} a_k\ f_{n-pk,m}^{(A,\lambda)}(x) f_{\mu+\nu k,m}^{(B,\gamma)}(y)\zeta^k$$

where $n, p \in \mathbb{N}$, *then we have*

$$\sum_{n=0}^{\infty} \Theta_{n,p,\mu,\nu}\left(x; y; \frac{\eta}{t^p}\right) t^n = (1-t^m)^{-A} e^{-\frac{xt\lambda}{1-t^m}} \Lambda_{\mu,\nu}(y;\eta) \tag{3.9}$$

provided that each member of (3.9) *exists.*

Remark 2. Using Corollary 3 and taking $a_k = 1$, $\mu = 0$, $\nu = 1$, we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} f_{n-pk,m}^{(A,\lambda)}(x) f_{k,m}^{(B,\gamma)}(y)\eta^k t^{n-pk}$$
$$= (1-t^m)^{-A} e^{-\frac{xt\lambda}{1-t^m}} (1-\eta^m)^{-B} e^{-\frac{y\eta\gamma}{1-\eta^m}}$$

where $|\eta| < 1$.

Now, we obtain the following class of bilinear generating function for the $g_{n,m}^{(A,\lambda)}(x)$.

Corollary 4. *If*

$$\Lambda_{\mu,\nu}^{n,p}(x;y;z) := \sum_{k=0}^{[n/p]} a_k g_{n-pk,m}^{(A+B,\lambda)}(2x) g_{\mu+\nu k,m}^{(C,\gamma)}(y) z^k,$$

where $a_k \neq 0, n, p \in \mathbb{N},\ \mu, \nu \in \mathbb{N}_0,\ AB = BA$, *then we have*

$$\sum_{k=0}^{n}\sum_{l=0}^{[k/p]} a_l\, g_{n-k,m}^{(A,\lambda)}(x) g_{k-pl,m}^{(B,\lambda)}(x) g_{\mu+\nu l,m}^{(C,\gamma)}(y) z^l = \Lambda_{\mu,\nu}^{n,p}(x;y;z) \tag{3.10}$$

provided that each member of (3.10) *exists.*

Furthermore, for every suitable choice of the coefficients a_k $(k \in \mathbb{N}_0)$, if the multivariable function $\Omega_{\mu+\psi k}(y_1, ..., y_s)$, $(s \in \mathbb{N})$, is expressed as an appropriate product of several simpler functions, the assertions of Theorem 1 and Corollary 1 can be applied in order to derive various families of multilinear and multilateral generating functions for the $f_{n,m}^{(A,\lambda)}(x)$ and $g_{n,m}^{(A,\lambda)}(x)$.

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